Introduction

We all are familiar with the concept of Counting. It is the most basic mathematical activity we do quite regularly, yet it can be complex too. We can feel the complexity one can face while counting, with the help of the following situation:

When we go to an ice – cream parlour, we get several options on the menu card. Suppose on a particular day, there are 3 flavours available – chocolate, vanilla and strawberry, and there are two different toppings available – berry and cherry. We want to know of different one – scoop ice creams with single topping available.

The one obvious way is to make a list of all 6 possibilities.

The second and the non –obvious way is to multiply the number of flavours and the number of toppings, $i.e.,\ 3 \times 2 = 6$ possible ice creams.

This non – obvious way of counting can be summarised in simple words as "To count without counting".

In the present chapter, we shall be studying some counting techniques which will help us in determining the number of different ways of arranging or selecting objects.

Fundamental Principle of Counting

Multiplication Rule:

If an operation can be performed in m different ways and if corresponding to each of these there are n different ways of performing another operation then both the operations can be performed in $m \times n$ different ways.

Or, If a work is done only when all of several works are done, then the number of ways of doing the work is the product of the number of ways of doing all the works separately.

The multiplication principle can be generalised for any finite number of events.

We have the following working rule to understand 'when' to use the above principle:

When to use it

If the order is important and the repetition is allowed.

Example: In a class, there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?

Sol: Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, and (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

Addition Rule:

If there are two operations such that they can be performed independently in m and n ways respectively, then either of the two operations can be performed in $(m + n)$ ways.

Or, If a work is done, when one of several works is done, then the number of ways of doing the work is the sum of doing all the works separately.

This rule of counting is also true for any finite number of works.

Example: In a class, there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways can the teacher make this selection?

Sol: Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, or (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of addition, the required number of ways is $10 + 8 = 18$.

Example: There are 5 routes from A to B and 3 routes from place B to place C. Find how many different routes are there from A to C via B.

Sol: For going from A to C via B, we have 5 different routes of going from A to B and 3 different routes for going from B to C.

Therefore by the fundamental principle of counting, the total number of routes from A to C via $B = 5 \times 3 = 15$.

Example: A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door?

Sol: A person can enter the room through any one of the six doors. So, there are 6 ways of entering into the room. After entering into the room, the person can come out through any one of the remaining 5 doors. So, he can come out through a different door in 5 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$.

Example: Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of two flags one below the other?

Sol: The total number of signals is equal to the number of ways of filling two vacant places in succession by four flags of different colours. The upper vacant place can be filled in 4 different ways by any one of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining different flags.

Hence by the fundamental principle of multiplication, the required number of signals is $4 \times 3 = 12$.

Example: How many words (with or without meaning) of three distinct letters of the English alphabets are there?

Sol: Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabets, the first place can be filled by any one of these letters in 26

ways. Now, the second place can be filled up by any one of the remaining 25 letters in 25 ways. After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of ways $= 26 \times 25 \times 24 = 15600$.

Example: Find the total number of ways of answering 5 objective type questions, each question having 4 choices.

Sol: Since each question can be answered in 4 ways, so, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 1024$.

Example: How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

Sol: Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three-digit numbers such that every digit is either 2 or 9.

Thus each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$.

Example: In a monthly test, a teacher decides that there will be one question from each of the three exercises of the textbook. There is 10, 9 and 15 question in the exercises. In how many ways can the three questions be selected?

Sol: Number of ways of choosing 1^{st} is 10

Number of ways of choosing 2^{nd} is 9

Number of ways of choosing 3^{rd} is 15

So, by the fundamental principle of counting the number ways, the three questions that can be selected is $10 \times 9 \times 15 = 1350$.

Example: Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word $ROSE$, when

(i) the repetition of the letters is not allowed. (ii) the repetition of the letters is allowed.

Sol: (i) The total number of words is the same as the number of ways of filling 4 vacant places by 4 letters. The first place can be filled in 4 different ways by anyone of the 4 letters R, O, S, E. Since the repetition of letters is not allowed, so, the second place can be filled by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled by the remaining 2 letters in 2 different ways; following which the fourth place can be filled in by the remaining one letter in one way. Thus, by the fundamental principle of counting, the required number of ways $= 4 \times 3 \times 2 \times 1 = 24$.

Hence, the required number of words is 24.

(ii) If the repetition of the letters is allowed, then each of the 4 vacant places can be filled in succession in 4 different ways.

Hence, the required number of words= $4 \times 4 \times 4 \times 4 = 256$.

Example: How many numbers are there between 100 and 1000 such that 7 is in the unit's place?

Sol: Every number between 100 and 1000 is a three-digit number. So, we have to form $3 - digit$ numbers with 7 at the unit's place by using the digits 0, 1, 2, …,9. Here, the repetition of digits is allowed. The hundred's place can be filled with any of the digits 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways. Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, the unit's place can be filled in only one way. Hence, by the fundamental principle of counting the total number of the number between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

Example: There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?

Sol: Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$.

Example: How many three-digit numbers more than 600 can be formed by using the digits $2, 3, 4, 6, 7$.

Sol: Clearly, the repetition of digits is allowed. Since a three-digit number greater 600 will have 6 or 7 at hundred's place, so, hundred's place can be filled in 2 ways. Each of the ten's and one's place can be filled in 5 ways.

Hence, the total number of required numbers $= 2 \times 5 \times 5 = 50$.

Example: In how many ways can five people be seated in a car with two people in the front seat and three in the rear, if two particular persons out of the five cannot drive?

Sol: For the driver's seat we can select any of the three persons who can drive while the rest of the seats can be occupied by the remaining people. The number of choices for the driver's seat $= 3$.

The number of choices for the second front seat $= 4$.

The number of choices for the first rear seat $= 3$.

The number of choices for the second rear seat $= 2$.

The number of choices for the third rear seat $= 1$.

By using the multiplication rule of FPC,

the total number of seating arrangements $= 3 \times 4 \times 3 \times 2 \times 1 = 72$.

Example: In a railway compartment, 6 seats are vacant on a bench. In how many ways can three passengers sit on them?

Sol: Number of ways for the 1st passenger to occupy seat $= 6$.

Number of ways for the 2nd passenger to occupy seat $= 6 - 1 = 5$

Number of ways for the 3rd passengers to occupy seat$= 5 - 1 = 4$

So, by the fundamental principle of counting, the total number of ways of all the three to occupy seats is $6 \times 5 \times 4 = 120$.

Factorial Notation

The product of first n natural numbers is denoted by $\lfloor n$ or $n!$ and is read as ' factorial n' or 'n factorial'.

Thus $n! = 1.\ 2.\ 3.\ \ldots.\ (n-1)\, n = n(n-1)(n-2) \ldots 3.\ 2.\ 1$

We have $5! = 5.\ 4.\ 3.\ 2.\ 1 = 120$

$4! = 4.\ 3.\ 2.\ 1$

Remember: $0!$ Is the product of integers from 1 to zero, which does not make any sense.

We define $0! = 1$.

- We have $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$ etc.

Example: Compute $\frac{8!}{4!}$

Sol: $\frac{8!}{4!} = \frac{8\times7\times6\times5\times4!}{4!} = 8 \times 7 \times 6 \times 5 = 1680$

Example: Convert the following product into factorials: $2.\ 4.\ 6.\ 8.\ 10$

Sol: We have, $2.\ 4.\ 6.\ 8.\ 10 = (2 \times 1)\ .\ (2 \times 2).\ (2 \times 3).\ (2 \times 4).\ (2 \times 5)$

$= 2^5\ (1.\ 2.\ 3.\ 4.\ 5) = 2^5\ .\ 5!$

Example: Compute $\frac{12!-10!}{9!}$.

Sol: Consider, $\frac{12!-10!}{9!} = \frac{12\times11\times10\times9!-10\times9!}{9!} = \frac{10\times9!(132-1)}{9!} = 10 \times 131 = 1310$

Example: Find the LCM of 6 !, 7! and 8!.

Sol: We have, $7! = 7 \times 6!$ And $8! = 8 \times 7 \times 6!$

So, the LCM of 6, ! 7! And 8! $= LCM(6\,!,\ 7 \times 6\,!,\ 8 \times 7 \times 6!) = 8 \times 7 \times 6! = 8!$

Example: If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Sol: We have $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$

$\Rightarrow x = \frac{10!}{8!} + \frac{10!}{9!} = \frac{10\times9\times8!}{8!} + \frac{10\times9!}{9!}$

$= 90 + 10 = 100.$

Example: If $(n+1)! = 12\ (n-1)\ !$, find n.

Sol: We have $(n+1)! = 12\ (n-1)!$

$\Rightarrow\ (n+1)\ n\ (n-1)! = 12\ (n-1)!$

$\Rightarrow (n+1)\ n = 12 = 4 \times 3$

$\Rightarrow n = 3$

Example: Prove that $2.\ 6.\ 10.\ \ldots n\ factors = \frac{(2n)!}{n!}$

Sol: $RHS = \frac{(2n)!}{n!} = \frac{1\,.2.\,3\,..\,(2n-1)(2n)}{n!}$

$= \frac{[\,2.\,4.\,\ldots(2n)][\,1.\,3,\,5.\,\ldots(2n-1)]}{n!}$

$= \frac{2^n[1.\,2.\,\ldots n][\,1.\,3.\,5.\,\ldots.\,(2n-1)]}{n!}$

$= \frac{2^n\ n![\,1.\,3.\,5.\,\ldots(2n-1)]}{n!}$

$= 2^n[1.\ 3.\ 5.\ \ldots(2n-1)]$

$= 2.\ 6.\ 10.\ \ldots\ (4n-2)$

$= 2.\ 6.\ 10.\ \ldots n\ factors = LHS$

Permutations

Each of the different arrangements that can be made by taking some or all objects of a finite set of distinct objects at a time, is called a permutation.

For example, if there are three objects say $A,\ B,\ and\ C$ then the permutations of these objects taking two at a time are $AB,\ BA,\ AC,\ CA,\ BC,\ CB$ and the permutations of these three objects taking all at a time is $ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA$.

Here, in each case number of permutations is 6.

Remember: It should be noted that in permutations the order of arrangement is important. Because, when the order is changed, then different permutation is obtained.

Notation

The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is denoted by n_{P_r} or $P(n,\ r)$.

Note: The number of permutations of n objects taking r at a time is equal to the number of ways in which r places can be filled up by n objects.

Some Theorems

Theorem – 1: Let n and r be positive integers such that $1 \leq r \leq n$.Then the number of permutations of n different objects taken r at a time is given by

$$n_{P_r} = n(n-1)(n-2)\ldots\left(n-(r-1)\right) = \frac{n!}{(n-r)!}$$

Theorem – 2: The number of permutations of n distinct objects, taken all at a time $n_{P_n} = n!$.

Example: Prove that $0! = 1$.

Sol: We have, $n_{P_n} = n(n-1)(n-2)\ \ldots(n-n+1) = n!$

Also, $n_{P_n} = \frac{n!}{0!}$

$\Rightarrow n! = \frac{n!}{0!}$

$\Rightarrow 0! = \frac{n!}{n!} = 1$

Example: Find the value of $P(15,\ 2)$.

Sol: $P(15,\ 2) = 15_{P_2} = \frac{15!}{(15-2)!} = \frac{15!}{13!} = \frac{15\times14\times13!}{13!} = 15 \times 14 = 210$

Example: If $P(n\ ,\ 4) = 20 \times P(n,\ 2)$, then find n.

Sol: We have, $P(n,\ 4) = 20 \times P(n,\ 2)$

$\Rightarrow \frac{n!}{(n-4)!} = 20\ \times \frac{n!}{(n-2)!}$

$\Rightarrow (n-2)! = 20 \times (n-4)!$

$\Rightarrow (n-2)(n-3)(n-4)! = 20 \times (n-4)!$

$\Rightarrow (n-2)(n-3) = 20 = 5 \times 4$

$\Rightarrow n - 2 = 5 \Rightarrow n = 7$

Example: If $11_{P_r} =\ 12_{P_{r-1}}$, find r.

Sol: Given that $11_{P_r} =\ 12_{P_{r-1}}$

$\Rightarrow \frac{11!}{(11-r)!} = \frac{12!}{(12-r+1)!}$

$\Rightarrow \frac{11!}{(11-r)!} = \frac{12!}{(13-r)!}$

$\Rightarrow \frac{11!}{(11-r)!} = \frac{12\times11!}{(13-r)(12-r)(11-r)!}$

$\Rightarrow \frac{12}{(13-r)(12-r)} = 1$

$\Rightarrow (13-r)(12-r) = 12 = 4 \times 3$

$\Rightarrow 13 - r = 4\ \Rightarrow r = 9$

Example: If $n-1_{P_3} : n_{P_4} = 1:9$, find n.

Sol: Given that $\frac{n-1_{P_3}}{n_{P_4}} = \frac{1}{9} \Rightarrow 9\ (n-1)_{P_3} =\ n_{P_4}$

$\Rightarrow 9\ \frac{(n-1)!}{(n-4)!} = \frac{n!}{(n-4)!} \Rightarrow 9(n-1)! = n!$

$\Rightarrow 9\ (n-1)! = n(n-1)! \Rightarrow n = 9$

Example: If $2n+1_{P_{n-1}} : 2n-1_{P_n} = 3:5$, find n.

Sol: We have, $\frac{2n+1_{P_{n-1}}}{2n-1_{P_n}} = \frac{3}{5}\ \Rightarrow 5\ (2n+1)_{P_{(n-1)}} = 3\ (2n-1)_{P_n}$

$\Rightarrow 5\ \frac{(2n+1)!}{(n+2)!} = 3\ \frac{(2n-1)!}{(n-1)!} \Rightarrow\ \ 5\ \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)(n)(n-1)!} = 3\ \frac{(2n-1)!}{(n-1)!}$

$\Rightarrow 5\ \frac{(2n+1)\times2}{(n+2)(n+1)} = 3$

$\Rightarrow 10(2n+1) = 3(n+2)(n+1)$

$\Rightarrow 20n + 10 = 3n^2 + 9n + 6$

$\Rightarrow 3n^2 - 11n - 4 = 0$

$\Rightarrow (3n + 1)(n - 4) = 0$

$\Rightarrow n = -\frac{1}{3}$ or $n = 4$

Since $n = -\frac{1}{3}$ is not possible, so, $n = 4$.

Simple Applications on Permutations

Example: In how many ways two different rings can be worn in four fingers with at most one in each finger?

Sol: The required number of ways is the same as the number of arrangements of 4 different things taken two at a time.

So, the required number of ways $= 4_{P_2} = \frac{4!}{2!} = 12.$

Example: Seven athletes are participating in a race. In how many ways can the first three prizes be won?

Sol: The total number of ways in which the first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So, the required number of ways $= 7_{P_3} = \frac{7!}{4!} = 210.$

Example: In how many ways can 6 persons stand in a queue?

Sol: The number of ways in which 6 persons can stand in a queue is the same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= 6_{P_6} = 6! = 720.$

Example: How many different signals can be made by 5 flags from 8 flags of different colours?

Sol: The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, the required number of signals $= 8_{P_5} = \frac{8!}{3!} = 6720.$

Example: How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed?

Sol: Every number lying between 100 and 1000 is a three-digit number. Therefore, we have to find the number of permutations of five digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of numbers $= 5_{P_3} = \frac{5!}{2!} = 60.$

Permutations with Repetitions

Theorem – 3: The number of permutations of n different objects taken r at a time, when each may be repeated any number of times in each arrangement, is n^r.

Example: In how many ways, 3 prizes can be given to 7 boys when each boy is eligible for any of the prizes?

Sol: Here, three prizes can be given to the same boy but two boys cannot get the same prize. Thus boys can be repeated.

So, here we have $n = 7$ and $r = 3$.

Hence, the required number of ways $= 7^3 = 343.$

Example: Find the number of 5-digit telephone numbers having at least one of their digits repeated.

Sol: Using the digits 0, 1, 2, … , 9, the number of 5-digit telephone numbers which can be formed is 10^5.

The number of 5-digit telephone numbers, which have none of the digits repeated $= 10_{P_5}$.

Hence, the required number of telephone numbers having at least one of their digits repeated

$= 10^5 - 10_{P_5} = 100000 - 30240 = 69760$

Permutations when all objects are not distinct

Theorem – 4: The number of permutations of n objects, where p objects are of the same kind or identical and others are distinct, is given by $\frac{n!}{p!}$

Theorem – 5: The number of permutations of n objects, where p_1 objects are of one kind, p_2 objects are of the second kind, …, p_k are of kth kind and the rest if any, are of a different kind is

$$\frac{n!}{p_1!\,p_2!\ldots p_k!}.$$

Example: In how many different ways the letters of the word "MATHEMATICS" be arranged?

Sol: The word " MATHEMATICS" contains 11 letters, out of which 2 are M, 2 are A, 2 are T, one H, one E, one I, one C, and one S.

The number of different ways in which the letters can be arranged is $\frac{11!}{2!2!2!} = 4989600.$

Example: In how many ways, 5 flags in which 3 are red, 1 is white, and 1 is blue, be arranged on staff, one below the other, if flags of one colour are not distinguishable?

Sol: Since, out of 5 flags, 3 are of the same kind (red) and others are different, so, the required number of ways $= \frac{5!}{3!1!1!} = 20.$

Example: Find the number of permutations of the letters of the word " INDEPENDENCE".

Sol: Here, we have 12 letters of which 3 are N's, 4 are E's, 2 are D's, and rest are different.

So, the required number of permutations $= \frac{12!}{3!4!2!} = 1663200$

Restricted Permutations

Here, we shall discuss permutations of objects under certain conditions $i.e.$, permutation when certain objects occur together when the position of particular objects are fixed when a particular object occurs in every arrangement etc.

Theorem – 6: The number of permutations of n objects taken r at a time, when a particular object is taken in each arrangement, is $r\left(n-1_{P_{r-1}}\right)$.

Theorem – 7: The number of permutations of n objects taken r at a time, when a particular object is never taken in each arrangement, is $n-1_{P_r}$.

Example: It is required to seat 8 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Sol: In all 12 persons are to be seated in a row and the row of 12 positions there are exactly 6 even places viz 2nd, 4th,6th, 8th, and 12th. It is given that 4 women are to occupy 4 places out of these six even places. This can be done in 6_{P_4} ways. The remaining 8 positions can be filled by the 8 men in 8_{P_8} ways. Therefore, the number of seating arrangements $= 6_{P_4} \times 8_{P_8} = 360 \times 40320 = 14515200.$

Example: How many four-digit numbers are there with distinct digits?

Sol: The total number of arrangements of 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is 10_{P_4}.

But, these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four-digit numbers. When 0 is fixed at thousand's place, we have to arrange the remaining 9 digits by taking 3 at a time. The number of such arrangements is 9_{P_3}.

So, the total number of numbers having 0 at thousand's place $= 9_{P_3}$.

Hence, the total number of four-digit numbers $= 10_{P_4} - 9_{P_3} = 5040 - 504 = 4536.$

Example: How many 4 – letter words, with or without meaning, can be formed using all the letters of the word "LOGARITHMS" if repetition of letters is not allowed?

Sol: There are 10 letters in the word "LOGARITHMS".

So, the number of 4- letter word = Number of arrangements of 10 letters, taken 4 at a time

$= 10_{P_4} = 5040.$

Example: In how many ways can the letters of the word PENCIL, be arranged so that

(i) N is always next to E? (ii) N and E are always together?

Sol: (i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in $5_{P_5} = 5! = 120$ ways. Hence, the total number of ways in which N is always next to E is 120.

(ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in $5_{P_5} = 5!$ Ways. But, E and N can be put together in 2! ways($i.e.,$ EN , NE).

Hence, the total number of ways $= 5! \times 2! = 240.$

Example: In how many ways can 6 boys and 7 girls be arranged in a row so that the girls always placed together?

Sol: Consider the 7 girls as one girl. Altogether there are 7 persons ($i.e.,$ 6 $boys\ and$ 1 $group\ of\ girls$) and these 7 persons can be arranged in a row in 7! ways. But, in every arrangement, these 7 girls can interchange their positions still staying together in 7! Ways. Thus, the total number of ways $= 7! \times 7! = (5040)^2$.

Example: How many different words can be formed with the letters of the word EQUATION so that

(i) Do the words begin with E?

(ii) the words begin with E and end with N?

(iii) the words begin and end with a consonant?

Sol: The given word contains 8 letters out of which 5 are vowels and 3 consonants.

(i) Since all words must begin with E, so, we fix E in the first place. Now, the remaining 7 letters can be arranged in $7_{P_7} = 7! = 5040$ ways.

So, the total number of words $= 5040.$

(ii)Since all words must begin with E and end with N, so, we fix E at the first place and N at the last place.

Now, the remaining 6 letters can be arranged in $6_{P_6} = 6! = 720$ ways.

Hence, the required number of words $= 720.$

(iii) There are 3 consonants and all words should begin and end with a consonant. So, first and last places can be filled with 3 consonants in $3_{P_2} = 6$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in $6_{P_6} = 6! = 720$ ways.

Hence, the required number of words $= 6 \times 720 = 4320$.

Example: In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?

Sol: The number of arrangements in which the best and the worst papers never come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the best and worst come together.

The total number of arrangements of 9 papers = $9_{P_9} = 9!$

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in $8_{P_8} = 8!$ ways.

But the best and worst papers can be put together in 2! ways.

So, the number of permutations in which the best and the worst papers can be put together = $(2! \times 8!)$.

Hence, the number of ways in which the best and the worst papers never come together

$= 9! - 2! \times 8! = 282240$.

Example: How many words can be formed from the letters of the word "DAUGHTER" so that

(i) the vowels always come together?

(ii) the vowels never come together?

Sol: There are 8 letters in the word " DAUGHTER", including 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R).

(i) Considering three vowels as one letter, we have 6 letters which can be arranged in $6_{P_6} = 6!$ ways.

But, corresponding each way of these arrangements, the vowels A, U, E can be put together in 3! ways.

Hence, the required number of words $= 6! \times 3! = 4320$.

(ii) The total number of words formed by using all the eight letters of the word "DAUGHTER" is $8_{P_8} = 8! = 40320$.

So, the total number of words in which vowels are never together

= Total number of words − Number of words in which vowels are always together

$= 40320 - 4320 = 36000$

Example: If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?

Sol: In the dictionary, the words at each stage are arranged in alphabetical order. Starting with the letter A, and arranging the other four letters GAIN, we obtain $4! = 24$ words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with G, and arranging the other four letters A, A, I, N in different ways, we obtain $\frac{4!}{2!} = 12$ words.

Thus, there are 12 words, which start with G.

Now, we start with I. The remaining 4 letters A, G, A, N can be arranged in $\frac{4!}{2!} = 12$ ways.

So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words. The 49th word is NAAGI and hence the 50th word is NAAIG.

Example: If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word 'LATE'.

Sol: Starting with the letter A and arranging the other three 3 letters L, T, E, we will obtain $3! = 6$ words.

There are 6 words which start with letter A.

Then, starting with letter E and arranging other 3 letters A, L, T, we will obtain $3! = 6$ words.

There are 6 words which start with letter E. Now, we have constructed 12 words. The 13th word will start with L. The 13th word is LAET.

Thus the 14th word is LATE.

Example: Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

Sol: The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time

$=$ Number of the arrangement of 4 digits, taken all at a time $= 4_{P_4} = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers.

Each of the digits 2, 3, 4, 5 occurs in $3! = 6$ times in the unit's place.

So, the total for the digits in the unit's place in all the numbers $= (2 + 3 + 4 + 5) \times 3! = 84$.

Since each of the digits 2, 3, 4, 5 occurs 3! times in any one of the remaining places. So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers $(2 + 3 + 4 + 5) \times 3! = 84$.

Hence, the sum of all the numbers $= 84(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

Combinations

The word combination means selection.

Each of the different selection, which can be made by taking some or all of several different objects at a time irrespective of their arrangements is called a combination.

Example: List the different combinations formed of three letters A, B, C taken two at a time.

Sol: The different combinations are AB, AC, BC.

Difference between Permutations and Combinations

The process of selecting objects is called combination and that of arranging objects is called a permutation.

In a combination, the ordering of the selected objects is immaterial whereas, in a permutation, the ordering is essential.

If we have 4 objects $A, B, C,$ and D, the possible selection (or a combination) and arrangements (or permutation) of 3 objects out of 4 are given below. This will help you to understand clearly the difference between permutations and combinations.

Selection ⟶ Combination	Arrangement ⟶ Permutation
ABC	ABC, ACB, BAC, BCA, CAB, CBA
ABD	ABD, ADB, BAD, BDA, DAB, DBA
ACD	ACD, ADC, CAD, CDA, DAC, DCA
BCD	BCD, BDC, CBD, CDB, DBC, DCB
Total 4 combinations	24 permutations

Notation

The number of combinations of n distinct objects taken r at a time is generally denoted by n_{C_r} or $C(n, r)$ or $\binom{n}{r}$ or $C_{n,r}$.

Theorem:

The number of all combinations of n distinct objects, taken r at a time is given by $n_{C_r} = \frac{n\,!}{(n-r)!}$

Proof: Let $n_{C_r} = x$.

Each one of these x combinations contains r thing and these r things can be arranged among themselves in $r!$ ways. Hence, one combination gives $r!$ permutations.

So, x combinations will give rise to $(x\,.\,r!)$ permutations. But the number of permutations of n things taken r at a time is $\frac{n!}{(n-r)!}$.

$\therefore x\,.\,r\,! = \frac{n!}{(n-r)!} \Rightarrow x = \frac{n!}{r!(n-r)!}$

$\Rightarrow n_{C_r} = \frac{n!}{r!(n-r)!}$.

Remember: n_{C_r} is defined only when n and r are non-negative integers such that $0 \le r \le n$.

Some particular cases:

1. When $r = 0$, then $n_{C_0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$.
2. When $r = n$, then $n_{C_n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$

   Thus $n_{C_0} = n_{C_n}$
3. We have, $n_{C_r} = \frac{n_{P_r}}{r!}$
4. We have, $n_{C_r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$

Example: Evaluate the following.

$(i)\ 7_{C_4} = \frac{7!}{4!\,3!} = 35$

$(ii)\ 8_{C_5} = \frac{8!}{5!\,3!} = 56$

$(iii)\ 100_{C_{99}} = \frac{100!}{99!\ 1!} = 100$

$(iv)\ 35_{C_{35}} = 1$

Example: If $n_{P_r} = 1680$ and $n_{C_r} = 70$, then find n and r.

Sol: We have $n_{P_r} = r!\ \times\ n_{C_r}$

$\Rightarrow 1680 = r! \times 70$

$\Rightarrow\ r! = \frac{1680}{70} = 24 = 4!$

$\Rightarrow r = 4$

Again $n_{P_r} = 1680$

$\Rightarrow \ n_{P_4} = 1680 \Rightarrow \frac{n!}{(n-4)!} = 1680 \Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = 1680$

$\Rightarrow n(n-1)(n-2)(n-3) = 8 \times 7 \times 6 \times 5$

$\Rightarrow n = 8.$

Example: If $n + 2_{C_8} : n - 2_{P_4} = 57:16$, then find the value of n.

Sol: We have, $n + 2_{C_8} : n - 2_{P_4} = 57:16$

$\Rightarrow \frac{n+2_{C_8}}{n-2_{P_4}} = \frac{57}{16} \Rightarrow 16 \times n + 2_{C_8} = 57 \times n - 2_{P_4}$

$\Rightarrow 16 \times \frac{(n+2)!}{8!(n-6)!} = 57 \times \frac{(n-2)!}{(n-6)!}$

$\Rightarrow \frac{(n+2)!}{(n-2)!} = \frac{57\times 8!}{16} \Rightarrow \frac{(n+2)(n+1)(n)(n-1)(n-2)!}{(n-2)!} = \frac{19\times3\times8\times7\times6\times5\times4\times3\times2\times1}{16}$

$\Rightarrow (n+2)(n-1)(n)(n-1) = 21 \times 20 \times 19 \times 18$

On comparing both sides, we get $n = 19$.

Example: If $n_{P_r} = n_{P_{r+1}}$ and $n_{C_r} = n_{C_{r-1}}$, then find n and r.

Sol: Given that $n_{P_r} = n_{P_{r+1}}$

$\Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \Rightarrow \frac{n!}{(n-r)(n-r-1)!} = \frac{n!}{(n-r-1)!}$

$\Rightarrow n - r = 1 \Rightarrow n = r + 1 \ldots (1)$

Also, since $n_{C_r} = n_{C_{r-1}}$, so, $\frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r+1)!}$

$\Rightarrow \frac{n!}{r(r-1)!(n-r)!} = \frac{n!}{(r-1)!(n-r+1)(n-r)!}$

$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n - r + 1 = r \ \Rightarrow n = 2r - 1 \ldots (2)$

From (1) and (2), we get $r + 1 = 2r - 1 \ \Rightarrow r = 2$

Hence, $n = 3$.

Example: If $\alpha = m_{C_2}$, then find the value of α_{C_2}.

Sol: Given, $\alpha = m_{C_2} = \frac{m(m-1)}{2}$

Now, $\alpha_{C_2} = \frac{\alpha(\alpha-1)}{2} = \frac{\frac{m(m-1)}{2}\left[\frac{m(m-1)}{2}-1\right]}{2}$

$= \frac{\frac{m(m-1)}{2}\left[\frac{m^2-m-2}{2}\right]}{2} = \frac{m(m-1)(m+1)(m-2)}{8} = \frac{(m-2)(m-1)(m)(m+1)}{8}$

Some properties

1. For $0 \le r \le n$, we have $n_{C_r} = n_{C_{n-r}}$.

This property is used to simplify $n_{C_r}$when r is large.

The combinations $n_{C_r} and\ n_{C_{r-r}}$ are called complementary combinations.

If $n_{C_r} = n_{C_s}$then either $r = s$ or $r + s = n$.

Example: If $n_{C_{10}} = n_{C_{12}}$, then find the value of 23_{C_n}.

Sol: Given that $n_{C_{10}} = n_{C_{12}}$. So $n = 10 + 12 = 22$.

Now, $23_{C_n} = 23_{C_{22}} = 23_{C_1} = 23$.

2. Let n and r be non -negative integers such that $r \le n$. Then $n_{C_r} + n_{C_{r-1}} = n + 1_{C_r}$.

Proof: We have, $n_{C_r} + n_{C_{r-1}} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$

$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$

$= \frac{n!}{(r-1)!(n-r)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} = \frac{n!}{(r-1)!(n-r)!}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$

$= \frac{n!(n+1)}{(r-1)!(n-r)!r\,(n-r+1)} = \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}$

$= \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = n + 1_{C_r}$

Note: This property is known as Pascal's Rule.

Example: If $8_{C_r} - 7_{C_3} = 7_{C_2}$, find r.

Sol: We have $8_{C_r} - 7_{C_3} = 7_{C_2}$

$\Rightarrow\ 8_{C_r} = 7_{C_3} + 7_{C_2} \Rightarrow\ 8_{C_r} = 8_{C_3}$

$\Rightarrow r = 3$ or $r + 3 = 8$

$\Rightarrow r = 3$ or $r = 5$.

3. Let n and r be non-negative integers such that $1 \le r \le n$. Then $n_{C_r} = \frac{n}{r}\,.n - 1_{C_{r-1}}$.

Example: Find the value of 10_{C_3}.

Sol: $10_{C_3} = \frac{10}{3} \times 9_{C_2} = \frac{10}{3} \times \frac{9}{2} \times 8_{C_1} = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 7_{C_0} = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120.$

4. If $1 \leq r \leq n$, then $\frac{n_{C_{r-1}}}{n-1_{C_{r-1}}} = \frac{n}{n-r+1}$.

Proof: We have $\frac{n_{C_{r-1}}}{n-1_{C_{r-1}}} = \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{(n-1)!}{(r-1)!(n-r)!}} = \frac{n!}{(n-r+1)!} \times \frac{(n-r)!}{(n-1)!}$

$= \frac{n(n-1)!}{(n-r+1)(n-r)!} \times \frac{(n-r)!}{(n-1)!} = \frac{n}{n-r+1}$

5. If $1 \leq r \leq n$, then $\frac{n_{C_r}}{n_{C_{r-1}}} = \frac{n-r+1}{r}$.
6. The number of ways of selecting at least one object from n distinct objects is
   $n_{C_1} + n_{C_2} + \cdots + n_{C_n} = 2^n - 1$
   $i.e., n_{C_0} + n_{C_1} + n_{C_2} + \cdots + n_{C_n} = 2^n$
7. If n is an even natural number, then the greatest of the values $n_{C_0}, n_{C_1}, n_{C_2}, \ldots, n_{C_n}$ is $n_{C_{\frac{n}{2}}}$.
8. If n is an odd natural number, then the greatest of the values $n_{C_0}, n_{C_1}, n_{C_2}, \ldots, n_{C_n}$ is $n_{C_{\frac{n+1}{2}}}$.

Example: If $n_{C_{r-1}} = 36, n_{C_r} = 84$ and $n_{C_{r+1}} = 126$, then find r.

Sol: We have, $\frac{n_{C_r}}{n_{C_{r-1}}} = \frac{84}{36}$

$\Rightarrow \frac{n-r+1}{r} = \frac{7}{3} \Rightarrow 3n - 3r + 3 = 7r$

$\Rightarrow 3n = 10r - 3 \Rightarrow n = \frac{10r-3}{3} \ldots (1)$

Also, $\frac{n_{C_{r+1}}}{n_{C_r}} = \frac{126}{84}$

$\Rightarrow \frac{n-r}{r+1} = \frac{3}{2} \Rightarrow 2n - 2r = 3r + 3$

$\Rightarrow 2n = 5r + 3 \Rightarrow n = \frac{5r+3}{2} \ldots (2)$

From (1) and (2), we get $\frac{10r-3}{3} = \frac{5r+3}{2}$

$\Rightarrow 20r - 6 = 15r + 9 \Rightarrow 5r = 15 \Rightarrow r = 3$

Practical Problems on Combinations

Example: In how many ways, can 5 sportsmen be selected from a group of 10?

Sol: Required number of ways = $10_{C_5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$

Example: If there are 15 persons in a party and if each two of them shake hands with each other, then how many hand-shakes happen at the party?

Sol: When two person shakes hands, then it is counted as one hand-shake, not two.

So, the total number of hand-shakes is same as the number of ways of selecting 2 persons among 15 persons = $15_{C_2} = \frac{15\times14}{2\times1} = 105.$

Example: A question paper has two parts, part A and part B, each containing 10 questions. If a student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?

Sol: There are 10 questions in part A out of which 8 questions can be chosen in 10_{C_8} ways.

Similarly, 5 questions can be chosen from part B containing 10 questions in 10_{C_5} ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= 10_{C_8} \times 10_{C_5} = \frac{10!}{8!\ \ 2!} \times \frac{10!}{5!\,5!} = 11340.$$

Example: In how many ways, can a cricket team of 11 players be selected out of 16 players,

(i) if two particular players are always be included?

(ii) if two particular players are always be excluded?

Sol: (i) since, two particular players are always included, so, we have to select 9 players out of 14 players. This can be done in 14_{C_9} ways.

Hence, the required number of ways = $14_{C_9} = 2002.$

(ii) Since two particular players are always excluded, so, we have to select 11 players out of 14 players. This can be done in $14_{C_{11}}$ ways.

Hence, the required number of ways = $14_{C_{11}} = 364.$

Example: A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Sol: There are 5 persons (2 men and 3 women). To constitute a committee of 3 persons we need to select three persons out of given 5 persons. This can be done in 5_{C_3} ways.

So, the committee can be formed in $5_{C_3} = 10$ ways.

Now, 1 man can be selected from 2 men in $^2C_1$ ways and 2 women can be selected from 3 women in $^3C_2$ ways.

Therefore, required number of committees $= {}^{2}C_1 \times {}^{3}C_2 = 2 \times 3 = 6.$

Example: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit?

(ii) Do four cards belong to four different suits?

(iii) four cards are face cards?

(iv) two are red cards and two are black cards?

(v) cards are of the same colour?

Sol: Four cards can be chosen from 52 playing cards in ${}^{52}C_4$ ways.

Now, $52_{C_4} = \frac{52!}{4!48!} = 270725.$

Hence, the required number of ways $= 270725.$

(i) There are four suits (diamond, spade, club, and heart) of 13 cards each. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamond cards, ${}^{13}C_4$ ways of choosing 4 spade cards, ${}^{13}C_4$ of choosing 4 club cards, and ${}^{13}C_4$ of choosing 4 heart cards.

$\therefore$ Required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4 \times {}^{13}C_4 = 4 \times \frac{13!}{4!9!} = 2860.$

(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards, one card can be drawn in ${}^{13}C_1$ ways. Similarly, there are ${}^{13}C_1$ ways of choosing one club card, ${}^{13}C_1$ ways of choosing one spade card and ${}^{13}C_1$ ways of choosing one heart card.

$\therefore$ Number of ways of selecting one card from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4.$

(iii) There are 12 face cards out of which 4 cards can be chosen in ${}^{12}C_4$ ways.

$\therefore$ Required number of ways $= {}^{12}C_4 = \frac{12!}{4!8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, 2 red cards can be chosen in ${}^{26}C_2$ ways and 2 black cards can be chosen in ${}^{26}C_2$ ways. Hence, 2 red and 2 black cards can be chosen in ${}^{26}C_2 \times {}^{26}C_2 = 105625$ ways.

(v) Out of 26 red cards, 4 red cards can be chosen in ${}^{26}C_4$ ways. Similarly, 4 black cards can be chosen in ${}^{26}C_4$ ways.

Hence, 4 red and 4 black cards can be chosen in ${}^{26}C_4 + {}^{26}C_4 = 29900$ ways.

Example: A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when

(i) at least two ladies are included?

(ii) at most two ladies are included?

Sol: (i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways:

I . Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in $^4C_2 \times {}^6C_3$ ways.

II. Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in $^4C_3 \times {}^6C_2$ ways.

III. Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in $^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, so the total number of ways of forming the committee $=$ $^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1 = 120 + 60 + 6 = 186.$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways:

I. Selecting 5 gents out of 6. This can be done in $^6C_5$ ways.

II. Selecting 4 gents out of 6 and one lady out of 4. This can be done in $^6C_4 \times {}^4C_1$ ways.

III. Selecting 3 gents out of 6 and 2 ladies out of 4. This can be done in $^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so the total number of ways of forming the committee $=$ $^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186.$

Example: How many diagonals are there in a polygon with n sides?

Sol: A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon.

Number of line segments obtained by joining the vertices of an n sided polygon taken two at a time $=$ Number of ways of selecting 2 out of $n = {}^nC_2 = \frac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$.

Mixed Problems on Permutations and Combinations

Example: Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Sol: Three consonants out of 7 and 2 vowels out of 4 can be chosen in $^7C_3 \times {}^4C_2$ ways. Thus, there are $^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels. Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

Hence, the required number of words $= {}^7C_3 \times {}^4C_2 \times 5! = 25200$.

Example: How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word $INVOLUTE$?

Sol: In the word $INVOLUTE$, there are 4 vowels, namely, I, O, E, U, and 4 consonants, namely, N, V, L and T.

The number of ways of selecting 3 vowels out of 4 $= {}^4C_3 = 4$.

The number of ways of selecting 2 consonants out of 4 $= {}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5! ways. Therefore, the required number of different words is $24 \times 5! = 2880$.

Example: How many words with or without meaning, can be formed using all the letters of the word $EQUATION$ at a time so that vowels and consonants occur together?

Sol: There are 5 vowels and 3 consonants in the word $EQUATION$. All vowels can be put together in 5! ways and all consonants can be put together in 3! ways. Considering all vowels as one letter and all consonants as one letter, vowels and consonants can be arranged in 2! ways.

Therefore, vowels and consonants can be put together in $5! \times 3! \times 2! = 1440$ ways.

Example: In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Sol: Since boys are to be separated, so let us first seat 5 girls. This can be done in 5! ways. For each such arrangement, three boys can be seated only at the cross marked places.

$$\times G \times G \times G \times G \times G \times$$

There are 6 crossed marked places and 3 boys can be seated in $^6C_3 \times 3!$ ways.

Hence, by the fundamental principle of counting, the total number of ways is $5! \times {}^6C_3 \times 3! = 14400$.