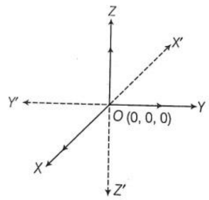
Chapter- 12

Introduction to 3 – D Geometry

Introduction:

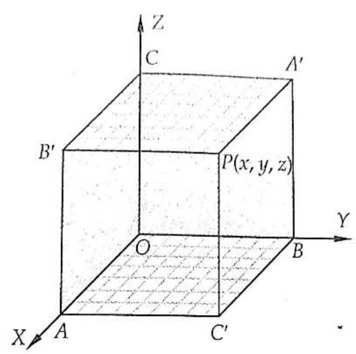
We introduce a coordinate system in three–dimensional space by choosing three mutually perpendicular axes as a frame of reference. The orientation of the reference system will be right-handed in the sense that if you stand at the origin with your right arm along the positive $x - axis$ and your left arm along the positive $y - axis$, your head will then point in the direction of positive $z - axis$.

Coordinates of a Point in Space



Let $X'OX$, Y′OY and $Z'OZ$ be three mutually perpendicular lines intersecting at O such that two of them $i.e.$, $Y'OY$ and $Z'OZ$ lies in the plane of the paper and the third $X'OX$ is perpendicular to the plane of the paper and is projecting out from the plane of the paper.

Let O be the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ be $x - axis$, $y - axis$, and $z - axis$ respectively. These three lines are also called the rectangular axes of coordinates. The planes containing the lines $X'OX$, $Y'OY$ and $Z'OZ$ in pairs determine three mutually perpendicular planes XOY, YOZ, and ZOX or simply XY, YZ, and ZX which are called rectangular coordinates planes.



Let P be a point in space. Through P draw three planes parallel to the coordinate planes to meet the axes in A, B, and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken in this order determined by the point P are called the coordinates of the point P, written as (x, y, z). Here x, y, z are positive or negative according to as they are measured along with positive or negative directions of the coordinate axes.

Conversely, given an ordered triad $(x,\ y,\ z)$of real numbers, we can always find the point whose coordinates are $(x,\ y,\ z)$ in the following manner.

(i) Measure $OA,\ OB,\ OC$ along $x-axis,\ y-axis$, and $z-axis$ respectively.

(ii) Through points $A,\ B,\ C$ draw planes parallel to the coordinate planes $YOZ,\ ZOX$, and XOY respectively. The point of intersection of these planes is the required point P.

To give another explanation about the coordinate of a point P we draw three planes through P parallel to the coordinate planes. These three planes determine a rectangular parallelepiped which has three pairs of rectangular faces, $iz.\ PB'AC',\ OCA'B$; $PA'BC',\ OAB'C$; $PA'CB',\ OAC'B$. Then we have

$x = OA = CB' = PA' =$ Perpendicular distance from P on the YOZ plane

$y = OB = A'C = PB' =$ Perpendicular distance from P on the ZOX plane

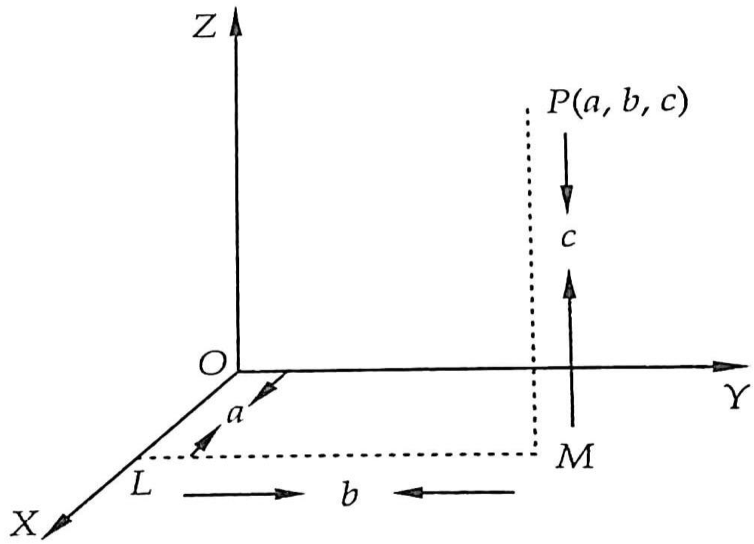
$z = OC = A'B = PC' =$ Perpendicular distance from P on the XOY plane

Thus, the coordinates of point P are the perpendicular distances from P on the three mutually rectangular coordinate planes $YOZ,\ ZOX$, and XOY respectively.

Alternatively, to find the coordinates of a point P in space, we first draw perpendicular PM on the $xy-plane$ with M as the foot of this perpendicular. Now from the point M, we draw perpendicular ML on $x-axis$ with L as the foot of this perpendicular. If $OL = a,\ LM = b$and $PM = c$, then we say that $a,\ b$, and c are $x,\ y$, and z coordinates, respectively, of the point$P$ in space. In such a case, we say that the point P has coordinates $(a,\ b,\ c)$.

Conversely, if we are given the coordinates $(a,\ b,\ c)$ of a point P and we have located the point, then first fix the point L on $x-axis$ such that $OL = a$. Now, find a point M on perpendicular to $x-axis$ at point L such that $LM = b$. We can say that M has coordinates $(a,\ b)$ in $xy-plane$. Having reached the point M, we draw the perpendicular on $xy-plane$ at point M and locate a point$P$ on this perpendicular such that $PM = c$. The point P so obtained has the coordinates $(a,\ b,\ c)$.

Thus, there one–to–one correspondence between the points in space and the ordered triplets $(x,\ y,\ z)$ of real numbers.



Sign of Coordinates of a Point

The three axes $X'OX$, $Y'OY$ and $Z'Odivid$es the space into eight compartments known as octants. The octant having OX, OY, and OZ as its edges are denoted by $OXYZ$. Similarly, the other octants are denoted by $OX'YZ$, $OXY'Z$, $OX'Y'Z$, $OXYZ'$, $OX'YZ'$, $OXY'Z'$, $OX'Y'Z'$. The signs of the coordinates of a point depend upon the octant in which it lies.

The following table shows the signs of coordinates of points in various octants.

Octants / Coordinates	I OXYZ	II OX'YZ	III OX'Y'Z	IV OXY'Z	V OXYZ'	VI OX'YZ'	VII OX'Y'Z'	VIII OXY'Z'
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

If a point P lies in $xy - plane$, then $z - coordinate$ of P is zero. Therefore, the coordinates of a point on $xy - plane$ are of the form $(x,\ y,\ 0)$ and we may take the equation of $xy - plnae$as $z = 0$. Similarly, the coordinates of any point in yz and $zx - planes$ are of the forms $(0,\ y,\ z)$ and $(x,\ 0,\ z)$ respectively and their equations may be taken as $x = 0$ and $y = 0$ respectively.

If a point lies on the $x - axis$, then its y and $z - coordinates$ are both zero. Therefore, the coordinates of an on $x - axis$ are of the form $(x,\ 0,\ 0)$ and we may take the equation of $x - axis$as $y = 0,\ z = 0$. Similarly, the coordinates of a point on y and $z - axes$ are of the form $(0,\ y,\ 0)$ and $(0,\ 0,\ z)$ respectively and their equations may be taken as $x = 0,\ z = 0$ and $x = 0,\ y = 0$ respectively.

Distance between two Points

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

IisO be the origin and $P(x,\ y,\ z)$ is a point in space, then

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

The distance of any point $P(x,\ y,\ z)$ from the $x - axis$ is $\sqrt{y^2 + z^2}$

Similarly, the distance of P from $y - axis$ is $\sqrt{x^2 + z^2}$ and the distance of P from $z - axis$is $\sqrt{x^2 + y^2}$.

Example: Find the distance between the points $P(1,\ -3,\ 4)$and $Q(-4,\ 1,2)$.

Sol: The distance PQ between the points $P(1,\ -3,\ 4)$ and $Q(-4,\ 1,\ 2)$ is

$$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2} = \sqrt{45} = 3\sqrt{5}\ units.$$

Example: Show that the points $P(-2,\ 3,\ 5)$, $Q(1,\ 2,\ 3)$, and $R(7,\ 0,\ -1)$ are collinear.

Sol: We know that the points are said to be collinear if they lie on a line.

Now $PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2} = \sqrt{14}$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2} = 2\sqrt{14}$$

$$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2} = 3\sqrt{14}$$

Thus $PQ + QR = PR$.

Hence, P, Q, and R are collinear.

Example Are the points $A(3,\ 6,\ 9)$, $B(10,\ 20,\ 30)$ and $C(25,\ -41,\ 5)$, the vertices of a right-angled triangle?

Sol: By, the distance formula, we have $AB^2 = (10-3)^2 + (20-6)^2 + (30-9)^2 = 686$

$$BC^2 = (25-10)^2 + (-41-20)^2 + (5-30)^2 = 4571$$

$$CA^2 = (3-25)^2 + (6+41)^2 + (9-5)^2 = 2709$$

We find that $CA^2 + AB^2 \neq BC^2$.

Hence, the triangle ABC is not a right-angled triangle.

Example: Find the equation of the set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points $(3,\ 4,\ 5)$ and $(-1,\ 3,\ -7)$ respectively.

Sol: Let the coordinates of point Pbe $(x,\ y,\ z)$.

Here $PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$

$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$, we have

$(x-3)^2 + (y-4)^2 + (z-5)^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

$\Rightarrow 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109.$

Example: Determine the point in $xy - plane$ which is equidistant from three points $A(2,\ 0,\ 3)$, $B(0,\ 3,\ 2)$and $C(0,\ 0,\ 1)$.

Sol: We know that $z - coordinate$ of every point on $xy - plane$ is zero. So, let $P(x,\ y,\ 0)$ be a point on $xy - plane$ such that $PA = PB = PC$.

Now, $PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow\ (x-2)^2 + (y-0)^2 + (0-3)^2 = (x-0)^2 + (y-3)^2 + (0-2)^2$

$\Rightarrow\ 4x - 6y = 0 \Rightarrow 2x - 3y = 0 \ldots (1)$

Again, $PB = PC \Rightarrow PB^2 = PC^2$

$\Rightarrow\ (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$

$\Rightarrow\ -6y + 12 = 0 \Rightarrow y = 2$.

Putting $y = 2$in (1), we get $x = 3$

Hence, the required point has the coordinates $(3,\ 2,\ 0)$

Example: Show that the points $A(1,\ 2,\ 3)$, $B(-1,\ -2,\ -1)$, $C(2,\ 3,\ 2)$ and $D(4,\ 7,\ 6)$ are the vertices of a parallelogram $ABCD$, but it is not a rectangle.

Sol: To show $ABCD$ is a parallelogram we need to show the opposite sides are equal.

Now, $AB = \sqrt{(-1-1)^2 + (-2-2)^2 + (-1-3)^2} = 6$

$BC = \sqrt{(2+1)^2 + (3+2)^2 + (2+1)^2} = \sqrt{43}$

$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2} = 6$

$DA = \sqrt{(1-4)^2 + (2-7)^2 + (3-6)^2} = \sqrt{43}$

Since $AB = CD$and $BC = DA$, so, $ABCD$ is a parallelogram.

Now, it is required to prove that $ABCD$ is not a rectangle. For this, we show that diagonals AC and BD are unequal.

We have $AC = \sqrt{(2-1)^2 + (3-2)^2 + (2-3)^2} = \sqrt{3}$

$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2} = \sqrt{155}$

Since, $AC \neq BD$, so, $ABCD$ is not a rectangle.

Example: Find the equation of the set of the points P such that the distances from the points $A(3,\ 4,\ -5)$ and $B(-2,\ 1,\ 4)$ are equal.

Sol: If $P(x,\ y,\ z)$ be any point such that $PA = PB$

$\Rightarrow\ PA^2 = PB^2$

$\Rightarrow\ (x-3)^2 + (y-4)^2 + (z+5)^2 = (x+2)^2 + (y-1)^2 + (z-4)^2$

$\Rightarrow\ 10x + 6y - 18z - 29 = 0.$

Section Formulae

Internal Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment P and Q such that it divides the join of P and Q internally in the ratio $m:n$. Then, the coordinates of R are $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

External Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment P and Q such that it divides the join of P and Q externally in the ratio $m:n$. Then, the coordinates of R are $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$

Notes:

- If R is the midpoint of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the coordinates of R are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
- The coordinates of the point R which divided PQ in the ratio $k{:}1$, are given as $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$
- If the vertices of a $\Delta\ ABC$ are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then the centroid of a ΔABC is $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Example: Find the coordinates of the point which divides the line segment joining the points $(1,\ -2,\ 3)$ and $(3,\ 4,\ -5)$ in the ratio $2:3(i)$ internally (ii) externally.

Sol: Let $P(x,\ y,\ z)$ be the point that divides the line segment joining $A(1,\ -2,\ 3)$ and $B(3,\ 4,\ -5)$ internally in the ratio $2:3$. Therefore

$$x = \frac{2\times 3 + 3\times 1}{2+3} = \frac{9}{5},\ \ y = \frac{2\times 4 + 3(-2)}{2+3} = \frac{2}{5}\ ,\ \ z = \frac{2(-5) + 3\times 3}{2+3} = -\frac{1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5},\ -\frac{1}{5}\right)$.

(ii) Let $P(x,\ y,\ z)$ be the point that divides the line segment joining $A(1,\ -2,\ 3)$ and $B(3,\ 4,\ -5)$ externally in the ratio $2:3$. Therefore

$$x = \frac{2\times 3 - 3\times 1}{2-3} = -3,\ \ y = \frac{2\times 4 - 3(-2)}{2-3} = -14,\ \ z = \frac{2(-5) - 3\times 3}{2-3} = 19$$

Therefore, the required point is $(-3,\ -14,\ 19)$

Example: Using, the section formula, prove that the three points $(-4,\ 6,\ 10)$, $(2,\ 4,\ 6)$ and $(14,\ 0,\ -2)$ are collinear.

Sol: Let $A(-4,\ 6,\ 10)$, $B(2,\ 4,\ 6)$ and $C(14,\ 0,\ -2)$ be the given points. Let the point P divide AB in the ratio $k:1$. Then coordinates of the point P are $\left(\frac{2k-4}{k+1},\ \frac{6k+6}{k+1},\ \frac{6k+10}{k+1}\right)$

Let us examine whether, for some value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1} = 14$, we get $k = -\frac{3}{2}$

When $k = -\frac{3}{2}$, then $\frac{4k+6}{k+1} = \frac{4\left(-\frac{3}{2}\right)+6}{-\frac{3}{2}+1} = 0$ and $\frac{6k+10}{k+1} = \frac{6\left(-\frac{3}{2}\right)+10}{-\frac{3}{2}+1} = -2$

Therefore, $C(14,\ 0,\ -2)$ is a point that divides AB externally in the ratio of $3:2$ and is the same as P. Hence $A,\ B,\ C$ are collinear.

Example: Find the ratio in which the line segment joining the points $(4,\ 8,\ 10)$ and $(6,\ 10,\ -8)$ is divided by the $yz\ -plane$.

Sol: Let $yz - plane$ divide the line segment joining $A(4,\ 8,\ 10)$ and $B(6,\ 10,\ -8)$ at $P(x,\ y,\ z)$ in the ratio $k:1$. Then the coordinates of $P$are $\left(\frac{4+6k}{k+1},\ \frac{8+10k}{k+1},\ \frac{10-8k}{k+1}\right)$.

Since P lies on the $yz - plane$, its $x - coordiante$ is zero. $i.e.,$ $\frac{4+6k}{k+1} = 0$

$$\Rightarrow\ k = -\frac{2}{3}$$

Therefore, $yz - plane$ divide AB externally in the ratio of $2:3$.

Example: The centroid of a triangle ABC is at the point $(1,\ 1,\ 1)$. If the coordinates of A and B are $(3,\ -5,\ 7)$ and $(-1,\ 7,\ -6)$ respectively, find the coordinates of the point C.

Sol: Let the coordinates of C be $(x,\ y,\ z)$ and the coordinates of the centroid Gbe $(1,\ 1,\ 1)$. Then

$$\frac{x+3-1}{3}=1\ i.e.,\ x=1$$

$$\frac{y-5+7}{3}=1\ i.e.,\ y=1$$

$$\frac{z+7-6}{3}=1\ i.e.,\ z=2$$

Hence, coordinates of C are $(1,\ 1,\ 2)$

Example: Find the coordinates of the points which trisect the line segment AB, given that $A(2,\ 1,\ -3)$and $B(5,\ -8,\ 3)$.

Sol: Let P and Q be the points which trisect AB. Then $AP=PQ=QB$.

Therefore P divides AB in the ratio of $1:2$ and Q divides it in the ratio of $2:1$.

As P divides AB in the ratio of $1:2$, so coordinates of P are

$$\left(\frac{1\times5+2\times2}{1+2},\ \frac{1\times(-8)+2\times1}{1+2},\ \frac{1\times3+2\times(-3)}{1+2}\right)=(3,\ -2,\ -1)$$

Since Q divides AB in the ratio of $2:1$, so coordinates of Q are

$$\left(\frac{2\times5+1\times2}{2+1},\ \frac{2\times(-8)+1\times1}{2+1},\ \frac{2\times3+1\times(-3)}{2+1}\right)=(4,\ -5,\ 1)$$

Example: The midpoints of the sides of a triangle are $(1,\ 5,\ -1)$, $(0,\ 4,\ -2)$and $(2,\ 3,\ 4)$. Find its vertices.

Sol: Let $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ be the vertices of the given triangle and let $D(1,\ 5,\ -1)$, $E(0,\ 4,\ -2)$ and $F(2,\ 3,\ 4)$ be the midpoints of the sides BC, CA, and AB respectively.

$D$is the midpoint of $BC\Rightarrow\frac{x_2+x_3}{2}=1,\ \frac{y_2+y_3}{2}=5,\ \frac{z_2+z_3}{2}=-1$

$$\Rightarrow\ x_2+x_3=2,\ y_2+y_3=10,\ z_2+z_3=-2\ \dots(i)$$

$E$is the midpoint of $CA\Rightarrow\frac{x_1+x_3}{2}=0,\frac{y_1+y_3}{2}=4,\ \frac{z_1+z_3}{2}=-2$

$$\Rightarrow\ x_1+x_3=0,\ y_1+y_3=8,\ z_1+z_3=-4\ \dots(ii)$$

$F$is the midpoint of $AB\Rightarrow\frac{x_1+x_2}{2}=2,\ \frac{y_1+y_2}{2}=3,\ \frac{z_1+z_2}{2}=4$

$\Rightarrow x_1 + x_2 = 4,\ y_1 + y_2 = 6,\ z_1 + z_2 = 8\ \ldots(iii)$

Adding the first three equations in (i), $(ii)$and (iii), we obtain

$2(x_1 + x_2 + x_3) = 2 + 0 + 4 \Rightarrow x_1 + x_2 + x_3 = 3$

Solving the first equations in (i), (ii) and $(iii)$with $x_1 + x_2 + x_3 = 3$, we obtain

$x_1 = 1,\ x_2 = 3,\ x_3 = -1.$

Adding second equations in (i), $(ii)$and (iii), we obtain

$2(y_1 + y_2 + y_3) = 10 + 8 + 6 \Rightarrow y_1 + y_2 + y_3 = 12$

Solving second equations in (i), (ii) and $(iii)$with $y_1 + y_2 + y_3 = 12$, we obtain

$y_1 = 2,\ y_2 = 4,\ y_3 = 6.$

Adding the last three equations in (i), $(ii)$and (iii), we obtain

$2(z_1 + z_2 + z_3) = -2 - 4 + 8 \Rightarrow z_1 + z_2 + z_3 = 1$

Solving the last equations in (i), (ii) and $(iii)$with $z_1 + z_2 + z_3 = 1$, we obtain

$z_1 = 3,\ z_2 = 5, z_3 = -7.$

Thus, the vertices of the triangle are $A(1,\ 2,\ 3)$, $B(3,\ 4,\ 5)$and $C(-1,\ 6,\ -7)$.