Introduction:

Logic is a field of study which deals with methods of reasoning. One of the main aims of mathematical logic is to provide rules utilizing which we can determine whether a given argument or reasoning is valid or not. The Greek Philosopher and Scientist Aristotle (381 – 322 BC) is said to be the first person to have studied logical reasoning.

Logical reasoning is the essence of mathematics.

The word Logic comes from the Greek word “Logos” which means “word” or sometimes “reason” in science, Philosophy, Law, or the ordinary happening of everyday life.

Logic is the science of reasoning by which we conclude, known statements or assertions.

The study of reasoning through the use of mathematical symbols is called mathematical reasoning.

Logical reasoning has two components. The first component deals with statements with values true or false and is concerned with the analysis of propositions. The second component deals with the predicates which are propositions containing variables.

Definition: A sentence from which we can infer whether it is true or false is called a **statement** or a **proposition**.

Or, A proposition (mathematically acceptable statement) is a declarative sentence that is either true or false, but not both simultaneously.

Consider the following sentences:

(i) $2 + 3 = 5$.

(ii) Sunrises in the west.

The above two sentences are statements because we can immediately decide that they are true or false.

Consider the following examples:

Where are you going?

How beautiful is that flower!

Sit down.

Bring a glass of water.

Wish you a happy birthday.

It may rain today.

The above sentences are not acceptable as statements, because we cannot say whether they are always true or false.

Sentences like

x is less than 5,

u is the father of v

are also not statements since they involve variables, though they become statements when the variables x, u, v are specified.

Consider the following sentences.

"Socrates was a wise man."

"Rose is beautiful."

"Rama is a good boy."

These are also not statements since they contain words like " wise", "beautiful", "good" whose truth cannot be asserted. Such sentences are called fuzzy propositions.

Remember:

No sentence can be called a statement if

- It involves variable time such as 'today', 'tomorrow', yesterday' etc.
- It involves variable places such as ' here', 'there', 'everywhere' etc.
- It involves pronouns such as 'she', 'he', 'they' etc.
- It is an exclamation.
- It is an order or request.
- It is a question.
- It involves adjectives/undefined terms or words such as 'good', beautiful', 'wise' etc.

Example: Check whether the following sentences are statements. Give reason.

$i)$ 7 is less than 5.

Sol: It is false, so it is a statement.

$ii)$ Please open the door.

Sol: This is not a statement as it is meaningless to talk about its truth value. It is a request.

$iii)$ Ram is an intelligent student.

Sol: This is not a statement since it contains the undefined term intelligent.

$iv)$ $\sqrt{3}$ is an irrational number.

Sol: It is true, so it is a statement.

$v)$ If $x = 1,\ x^2 + 2x + 5 = 7.$

Sol: This is a statement since x is defined and the truth value of this statement can be defined.

$vi)$ x is less than 5.

Sol: This is not a statement since x is unknown.

- Statements are represented by small Roman letters like $p,\ q,\ r,\ s,$ etc. These letters are known as **statement letters**.
- The truth or falsity of a statement is called its **truth value**.
- The truth value of a true statement is **True** and is represented by 'T' or '1'. The truth value of a false statement is **False** and is represented by 'F' or '0'. True statements are called valid statements and false statements are called invalid statements.

For example

p : 2 is a prime number. (T)

q: There are 32 days in a month. (F)

Example: Check whether the following sentences are statements. Give reasons for your answer.

(i) 8 is less than 6.

Sol: This sentence is false because 8 is greater than 6. Hence it is a statement.

(ii) Every set is a finite set.

Sol: This sentence is also false since there are sets that are not finite. Hence it is a statement.

(iii) The sun is a star.

Sol: It is a scientific fact that the sun is a star and, therefore, this sentence is always true. Hence it is a statement.

(iv) Mathematics is fun.

Sol: This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others, it may not be. This means that this sentence is not always true. Hence it is not a statement.

(v) There is no rain without clouds.

Sol: it is a scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence it is a statement.

Formation of New Statements from Old

Here we look into a method for producing new statements from those that we already have.

We use two techniques:

(i) The negation of a statement

(ii) Compound statement.

The negation of a Statement

If a proposition p is modified by the word "not", a new sentence results. We call it the negation or denial of p and denote it by the symbol $\sim p$.

Let p : 2 is a composite number.

$\sim p$: 2 is not a composite number.

q: Six is not a prime number.

$\sim q$: It is not true that six is not a prime number.

(It is also correct to write $\sim q$ as "Six is a prime number.")

r: Nitin went to school today and yesterday.

$\sim r$: It is not true that Nitin went to school today and yesterday.

(It is not correct to write $\sim r$ as Nitin did not go to school today and yesterday.)

Axiom of negation:

For any proposition p, if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

The axiom of negation can be represented by the following truth table.

The negation table

p	$\sim p$
T	F
F	T

Remark:

(1) A statement and its negation have opposite truth values.

(2) Negation is the only operator that acts on a single proposition.

Example: Write the negation of the following statements.

(i) All students are intelligent.

Sol: The negation of the statement is 'Some students are intelligent'.

This can be written as 'There exists a student who is intelligent'.

Again this can be written as 'At least one student is not intelligent'.

(ii) No student is intelligent.

Sol: The negation of the statement is 'Some students are intelligent'.

(iii) Tajmahal is in America.

Sol: The negation of the statement is 'It is false that Tajmahal is in America'.

(iv) $\sqrt{5}$ is rational.

Sol: The negation of the statement is '$\sqrt{5}$ is not rational'.

This can also be written as 'It is not the case that $\sqrt{5}$ is rational'.

(v) Both the diagonals of a rectangle have the same length.

Sol: The negation of the statement is 'It is false that both in a rectangle have the same length'.

Example: Write the negation of the following statements and check whether the resulting statements are true.

(i) There does not exist a quadrilateral that has all its sides equal.

Sol: The negation of the statement is 'It is not the case that there does not exist a quadrilateral which has all its sides equal'.

This also means the following:

'There exists a quadrilateral which has all its sides equal'.

This statement is true because we know that square is a quadrilateral such that its four sides are equal.

(ii) The sum of 3 and 4 is 9.

Sol: The negation of the statement is ‘It is false that the sum of 3 and 4 is 9’.

This can be written as ‘The sum of 3 and 4 is not equal to 9’.

This statement is true.

Compound Statements

Simple Statements: Any statement whose truth value does not explicitly depend on another statement is said to be a simple statement.

- Two or more propositions can be combined to form new propositions. There are four such keywords and phrases, called connectives, playing a major role in combining propositions, which are:

$$or,\ and,\ only\ if,\ if\ and\ only\ if$$

- A proposition in which one or more of these connectives appear is called a **composite** or **compound proposition** while its constituents are called its **prime components or component statements**.

Or, a **compound statement** is the one that is made up of two or more simple statements.

- The fundamental property of a compound statement is that its truth value is completely determined by the truth values of the component statements together with how they are connected to form the compound statement.

Example: Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

Sol: The component statements are

p: The sky is blue.

q: The grass is green.

The connecting word is ‘ and ‘.

(ii) It is raining and it is cold.

Sol: The component statements are

p: It is raining.

q: It is cold.

The connecting word is ‘and ‘.

(iii) 0 is a positive number or a negative number.

Sol: The component statements are

p: 0 is a positive number.

q : 0 is a negative number.

The connecting word is 'or '.

(iv) The constituents of water are oxygen and nitrogen.

Sol: The component statements are

p: The constituent of water is oxygen

q: The constituent of water is nitrogen

The connecting word is 'and'.

(v) Time and Tide wait for none.

Sol: The component statements are

p: Time waits for none.

q: Tide waits for none.

The connecting word is 'and '.

Example: Find the component statements of the following and check whether they are true or not.

(i) A square is a quadrilateral and its four sides equal.

Sol: The component statements are

p: A square is a quadrilateral.

q: A square has all its sides equal.

We know that both these statements are true. Here the connecting word is 'and'.

(ii) A person who has taken Mathematics or Computer Science can go for MCA.

Sol: The component statements are

p: A person who has taken Mathematics can go for MCA.

q: A person who has taken Computer Science can go for MCA.

Both of these statements are true. Here the connecting word is 'or'.

(iii) 24 is a multiple of 2, 4, and 8.

Sol: The component statements are p: 24 is a multiple of 2.

q: 24 is a multiple of 4. r: 24 is a multiple of 8.

All these statements are true. Here the connecting words are 'and'.

The Connecting word "AND"

Conjunction

If two propositions are connected by the word 'and' then the new proposition so formed is called a conjunction.

If $p,\ q$ be two propositions, then the conjunction is 'p and q'. It is denoted by $p \wedge q$. The conjunction $p \wedge q$ becomes a proposition when a definite truth value is assigned to it.

Consider the following two propositions.

p: Raju went to the market.

q: Raju bought a notebook.

Then the conjunction $p \wedge q$ is 'Raju went to the market and bought a notebook'.

Axiom of Conjunction:

A conjunction $p \wedge q$ is true if both p and q are true and false if at least one of $p,\ q$ is false. The axiom of conjunction can be represented by the following table:

The conjunction table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) $\sqrt{3}$ is irrational and 3 is a complete square.

Sol: The component statements are

p: $\sqrt{3}$ is irrational and q: 3 is a complete square.

Since p is true and q is false so conjunction $p \wedge q$ is false.

(ii) Bhubaneswar is in India and $2 + 3 = 5$.

Sol: The component statements are

p: Bhubaneswar is in India.

q: $2 + 3 = 5$.

Both these statements are true, therefore the compound statement is true.

(iii) Mumbai is the capital of Maharashtra and $10 < 5$.

Sol: The component statements are

p: Mumbai is the capital of Maharashtra.

q: $10 < 5$.

This compound statement is false since q is false.

(iv) A-line straight and extends indefinitely in both directions.

Sol: The component statements are

p: A line is straight.

q: A-line extends indefinitely in both directions.

Both the statements are true, therefore, the compound statement is true.

(v) 0 is less than every positive integer and every negative integer.

Sol: The component statements are

p: 0 is less than every positive integer.

q: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

(vi) All living things have two legs and two eyes.

Sol: The component statements are

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statement is false.

Remember:

- It should be noted that the word "and" is used as a connective as we use the English language. But 'and' is also used with other meanings. For example, in the statement "Ravish and Ravi are good friends" the word 'and' is not a connective. Similarly, in the statement "Mohan opened the door and ran away", the word 'and' is used in the sense of 'and then' because the action described in "Mohan ran away" occurs after the action described in "Mohan opened the door".

- In our day – to – day life, the word 'and' is used between two statements that have some kind of relation. But, in reasoning, it can be used even for the statements which are not related to each other. For example, "It is raining and 5 is a prime number" is a compound statement.

The Connecting word "OR"

Disjunction

When two statements p, q are combined by the word "or" to form a new sentence, the latter is called their disjunction and is written as $p \vee q$.

Consider the following propositions:

p: Sunday is a holiday.

q: Thursday is a holiday.

Then the disjunction $p \vee q$ is "Either Sunday or Thursday is a holiday".

A few more examples of disjunction are:

Three is greater than five or it is not.

The number $\sqrt{2}$ is either rational or irrational.

The disjunction of two propositions is also a proposition.

Axiom of disjunction:

A disjunction $p \vee q$ is true if at least one of p, q is true and false if both p and q are false.

The axiom of disjunction can be represented by the following table.

The disjunction table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: Find the disjunction of the following propositions.

(i) p: An employee goes on leave.

q: An employee attends his duty.

Sol: The disjunction of the given propositions is

$p \vee q$: An employee either goes on leave or attends to his duty.

(ii) p: The rivers are rising.

q: There is a flood this year.

Sol: The disjunction $(p \vee q)$ is the proposition

"The rivers are rising or there is flood this year'.

Exclusive "Or", Inclusive "Or"

Let us consider the following examples:

(i) Pineapple juice or Pepsi is available with a meal in a restaurant. It means that a person who does not want pineapple juice can have a Pepsi along with a meal or one does not want Pepsi can have pineapple juice along with the meal. But a person cannot have both pineapple juice and Pepsi. This is called an **exclusive "Or".**

(ii) A student who has taken physics or electronics can apply for M.Tech. Here we mean that the students who have taken both physics and electronics can apply for M.Tech. As well as, a student who has taken one of these subjects can apply for M.Tech. In this case, we are using **inclusive "Or".**

Example: For each of the following statements, determine whether an inclusive "Or" or exclusive "Or" is used. Give reasons for your answer.

(i) To enter a country, you need a passport or a voter registration card.

Sol: Here "Or" is inclusive, since a person can have both a passport and a voter registration card to enter a country.

(ii) Two lines intersect at a point or are parallel.

Sol: Here "Or" is exclusive because two lines can't intersect and parallel together.

(iii) The school is closed if it is a holiday or a Sunday.

Sol: Here "Or" is inclusive since school is closed on holiday as well as Sunday.

(iv) On 4^{th} December 2020 at 3 O'clock, Aswini will play cricket or tennis.

Sol: Here "Or" is exclusive, because at 3 O'clock on 4^{th} December, Aswini can play cricket or he can play tennis but not both.

Example: Identify the type of "Or" used in the following statements and check whether the statements are true or false:

(i) $\sqrt{2}$ is a rational number or an irrational number.

Sol: The component statements are

p: $\sqrt{2}$ is a rational number.

q: $\sqrt{2}$ is an irrational number.

Here, we know that p is false and q is true. So, the compound statement is true. Also "Or" is exclusive.

(ii) A rectangle is a quadrilateral or a 5-sided polygon.

Sol: The component statements are

p: A rectangle is a quadrilateral.

q: A rectangle is a 5 – sided polygon.

The compound statement is true. Here "Or" is exclusive.

Quantifiers (Universal and Existential)

In Mathematics we come across many mathematical statements containing phrases "There exists" and "For every".

"Girls are beautiful" is an open sentence. It can be converted into a statement of the variable by a specific word. Thus, one way of converting open sentences into a statement by using the phrases

(i) All (For every)

(ii) Some (There exists)

(iii) None

Thus the phrases all, some, and none tell us the quantity of the variable in an open sentence as such they are known as quantifiers.

(1) **For every:**

The phrase "For every (or for all)" is called the **universal quantifier**.

Let us consider the statement "All human beings are mortal".

Let $p(x)$ denote 'is mortal'. Above statement can be written as $\forall\, x \in A,\ p(x)$ or $\forall\ (x)\ p(x)$, where A is the set of all human beings. The symbol ∀ stands for 'for all'.

Here $\forall\ (x)\ p(x)$ represents each of the following phrases.

For all x, x is mortal

or For every x, x is mortal

or For each x, x is mortal

Thus the universal quantification of $p(x)$ is the statement ' (x) for all values of x in the domain'.

$i.e.$, $\forall\ (x)\ p(x)$ is the universal quantification of $p(x)$

Remember:

- An element x for which $p(x)$ is false is called a counterexample of $\forall\ (x)\ p(x)$.
- Let all the elements in the domain be $x_1, x_2,\ \ldots, x_n$. The universal quantification $\forall(x)p(x)$ is same as $p(x_1) \wedge p(x_2) \wedge \ldots. \wedge p(x_n)$.
- $\forall\ (x)p(x)$ is true if each of $p(x_1),\ p(x_2),\ \ldots,\ p(x_n)$ is true.

Example: Consider the following statements.

$p: x + 4 > 3$ for all $x \in N$

This statement p means that for every natural number $x,\ x + 4 > 3$.

q: For every prime number x, $\sqrt{x}$ is an irrational number.

This statement means that if S denotes the set of all prime numbers, then for all the members x of the set S, $\sqrt{x}$ is an irrational number.

(2) **There exists:**

There exists can be replaced by the words "there is at least one" or "there is "or "it is possible to find" or "some".

The phrase "There exists" is known as the **existential quantifier**.

The existential quantification of $p(x)$ is the proposition "There exists an element x in the domain such that $p(x)$".

It is denoted by $\exists\ (x)p(x)$.

Here $\exists\ (x)$ represents each of the following phrases.

There exists an x.

There is an x.

For some x.

There is at least one x.

Example: Let $p(x)$ be the statement '$x^2 = 2$'.

If the domain be the set of all integers, then $\exists\ (x)p(x)$ is false. If the domain be the set of all real numbers, then $\exists\ (x)p(x)$ is true.

Example: Consider the following statements.

p: There exists a rectangle whose all sides are equal.

This statement p means that there is at least one rectangle whose all sides are equal.

q: There exists $x \in N$ such that $x + 4 < 7$.

This statement q means that there exists at least one natural number x such that $x + 4 < 7$.

The negation of Quantified Statements

(i) Let us consider the statement 'Every student in the class has taken a course on computer'.

This statement is a universal quantification and is written as $\forall\ (x)\ p(x)$, where $p(x)$ is the statement 'x has taken a course in computer'. Here domain is the set of all students in the class.

The negation of the above statement is "It is not the case that every student in the class has taken a course in computer'.

The above is equivalent to 'There is a student in the class who has not taken a course in computer'.

This is simply the existential quantification of the negation of the original propositional function and is written as $\exists\ (x) \sim p(x)$.

Thus we have the following equivalence $\sim \forall\ (x)p(x) \equiv\ \exists\ (x) \sim p(x)$ …. (1)

Here we see that the negation of a universal quantification $\sim \forall\ (x)p(x)$ is logically equivalent to an existential quantification $\exists\ (x)\ p(x)$

(ii) Let us consider a statement 'There is a student in the class who has taken a course in computer'. This statement is an existential quantification and is written as $\exists\ (x)\ p(x)$, where $p(x)$ is the statement 'has taken a course in computer'. The negation of this statement is 'It is not the case that there is a student who has taken a course in computer'.

Above statement is equivalent to

'All student in the class has not taken a course in computer' which is the universal quantification of the negation of the original propositional function and is written as $\forall\ (x) \sim p(x)$.

Thus we have the following equivalence

$\sim \exists\ (x)\ p(x) \equiv\ \forall\ (x) \sim p(x)$ …. (2)

Thus $\sim \exists\ (x)p(x)$ and $\forall\ (x) \sim p(x)$ are logically equivalent.

The logical expressions in (1) and (2) are known as $De\ Morgan's\ Laws\ of\ quantifiers$.

Example: Identify the quantifier in the following statement and write the negation of the statement.

(i) There exists a number that is equal to its square.

Sol: Here quantifier of the given statement is "there exists". It is true.

$\sim p$: There does not exist a number which is equal to its square.

(ii) For every real number x, x is less than $x + 1$.

Sol: Here quantifier of the given statement is "For every". It is true.

The negation of the given statement is

$\sim p$: For every real number x, x is not less than $x + 1$.

(iii) There exists a capital for every state in India.

Sol: Here quantifier of the given statement is "there exists". It is true.

The negation of the given statement is

$\sim p$: There exists a state in India that does not have a capital.

Implications, Contrapositive, Inverse, and Converse

Conditional

A proposition of the type " if p then q " is called a conditional. It can also be written as " is sufficient for q ", " q is necessary for p ", " p only if q ", " q provided that p" and so on. It is denoted by $p \rightarrow q$. Here p is called the antecedent or hypothesis and q is called the consequent or conclusion.

"If in $\Delta\, ABC$, $\angle$ C is right-angled, then $AB^2 = BC^2 + AC^2$ " is an example of a conditional in which the antecedent is ' in $\Delta\, ABC$, $\angle$ C is right-angled' and the consequent is '$AB^2 = BC^2 + AC^2$'.

Axiom of Conditional:

A conditional $p \rightarrow q$ is false only when p is true and q is false. In all other cases, it is true.

The tabular representation of the axiom of conditional is as follows.

The conditional Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: If a conditional $p \to q$ is always true, then we say that p implies q and we write $p \Rightarrow q$ to indicate the implication.

The symbol $\Rightarrow$ is not a connective.

The implication $p \Rightarrow q$ is the same as each of the following.

(i) p is a sufficient condition for q.

(ii) p only if q.

(iii) q is a necessary condition for p.

(iv) $\sim q \Rightarrow \sim p$

For example, consider the statement.

r : If you are born in some country, then you are a citizen of that country.

This statement corresponds to two statements p and q given by

p: you are born in some country.

q: you are a citizen of that country.

Then the sentence "if p then q" says that in the event if p is true then q must be true.

Example: Write each of the following statements in the form "if p then q".

(i) To be a Professor, it is sufficient to be $Ph.\ D.$ Holder.

Sol: If you are a $Ph.D.$ Holder, then you will be a Professor.

(ii) It snows whenever the wind blows from the south-west.

Sol: If the wind blows from the south-west, then it snows.

Example: For each of the following compound statements, first identify the corresponding component statements. Then check whether the statements are true or not.

(i) If a triangle ABC is equilateral, then it is isosceles.

Sol: The component statements are given by

p: Triangle ABC is equilateral.

q: Triangle ABC is isosceles.

Since an equilateral triangle is isosceles, we infer that the given statement is true.

(ii) If a and b are integers, then ab is a rational number.

Sol: The component statements are given by

p: a and b are integers.

q: ab is a rational number.

Since the product of two integers is an integer and therefore a rational number, the compound statement is true.

Converse, Inverse, and Contrapositive

Given a conditional $p \to q$.

The converse of $p \to q$ is given by $q \to p$.

The inverse of $p \to q$ is given by $\sim p \to \sim q$.

The contrapositive of $p \to q$ is given by $\sim q \to \sim p$.

Remember: The contrapositive statement is the converse of the inverse statement.

Example: Write the converse, inverse, and contrapositive of the statement "If x^2 is a multiple of 3 then x is a multiple of 3".

Sol: Converse: If x is a multiple of 3 then x^2 is a multiple of 3.

Inverse: If x^2 is not a multiple of 3 then x is not a multiple of 3.

Contrapositive: If x is not a multiple of 3 then x^2 is not a multiple of 3.

Example: Write the contrapositive of the following statements:

(i) If you are born in India, then you are a citizen of India.

Sol: The contrapositive of the given statement is "If you are not a citizen of India, then you were not born in India".

(ii) If a triangle is equilateral, it is isosceles.

Sol: The contrapositive of the given statement is "If a triangle is not isosceles, then it is not equilateral".

Example: Write the converse of the following statements.

(i) If you do all the exercises in the book, you get an A grade in the class.

Sol: The converse of the given statement is "If you get an A grade in the class, then you have done all the exercises of the book".

(ii) If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.

Sol: The converse of the given statement is "If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.

Validating Statements

Checking the validity of a statement means checking when the statement is true and when it is not true. This depends upon which of the connectives, quantifiers, and implication is being used in the given statement.

Let us now discuss some techniques or rules to find when a statement is valid or true.

The validity of the statements with " AND "

If p and q are mathematical statements, then to show that the statement "p and q" is true, we follow the following steps:

Step I: Show that the statement p is true.

Step II: Show that the statement q is true.

Example: Consider two statements:

p: 80 is a multiple of 5.

q: 80 is a multiple of 4.

Write the compound statement connecting these two statements with "and" and checks its validity.

Sol: The compound statement is "80 is a multiple of 5 and 4".

We know that 80 is a multiple of 5 as well as 4. So, p and q are true statements.

Hence the compound statement is also true.

Thus, the compound statement " p and q" is a valid statement.

Example: If p and q are two statements given by:

p: 25 is a multiple of 5.

q: 25 is a multiple of 8.

Write the compound statement connecting these two statements with "and" and check its validity.

Sol: The compound statement is "25 is a multiple of 5 and 8".

Since 25 is a multiple of 5 but it is not a multiple of 8, so, p is true but q is not true.

Hence, the compound statement is not true $i.e.$, the statement "p and q" is not a valid statement.

The validity of Statements with "OR"

If p and q are mathematical statements, then to show that the compound statement " p or q "is true, we proceed as follows:

Assuming that p is false, show that q must be true.

or Assuming that q is false, show that p must be true.

Example: Given below are two statements:

p: 25 is a multiple of 5.

q: 25 is a multiple of 8.

Write the compound statement connecting these two statements with "OR" and check its validity.

Sol: The compound statement is " 25 is a multiple of 5 or 8".

Let us assume that the statement q is false $i.e.$ 25 is not a multiple of 8. Here, p is true.

Thus, if we assume that q is false, then p is true. Hence the compound statement is true $i.e.$ valid.

Example: Check the validity of the following statement: " Square of an integer is positive or negative".

Sol: The given statement is a compound statement with "OR" whose component statements are

p: Square of an integer is positive.

q: Square of an integer is negative.

Let us assume that p is false $i.e.$, a square of an integer is not positive. Then, for any integer x, we have x^2 is not greater than or equal to zero $\Rightarrow x^2 < 0 \Rightarrow q$ is true.

Thus, if we assume that p is false, then q is true.

Hence, " or q" is a valid statement. In other words, the given statement is true.

The validity of Statements with " IF-THEN "

If p and q are two mathematical statements, to prove the validity of the statement " if p, then q", we may use any one of the following methods.

(i) **Direct Method**

Step I: Assume that p is true.

Step II: Prove that q is true.

(ii) **Contrapositive Method:**

Step I: Assume that q is false.

Step II: Prove that p is false.

(iii) **Contradiction Method**

Step I: Assume that p is true and q is false.

Step II: Obtain a contradiction from step I.

Example: Check whether the following statement is true or not:

" If x and y are odd integers, then xy is an odd integer".

Sol: Let p and q be the statements given by

p: x and y are odd integers.

q: xy is an odd integer.

Then, the given statement is: If p, then q.

Direct Method: Let p is true.

Then, p is true

$\Rightarrow$ x and y are odd integers

$\Rightarrow$ $x = 2m + 1,\ y = 2n + 1$ for some integers $m,\ n$

$\Rightarrow$ $xy = (2m + 1)(2n + 1)$

$\Rightarrow$ $xy = 2(2mn + m + n) + 1$

$\Rightarrow$ xy is an odd integer

$\Rightarrow$ q is true.

Thus p is true $\Rightarrow q$ is true.

Hence, " If p, then q" is a true statement.

Contrapositive method: Let q is not true. Then,

q is not true

$\Rightarrow$ xy is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow$ p is not true.

Thus, q is false $\Rightarrow p$ is false

Hence, "If p, then q" is a true statement.

Example: Show that the statement p: If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by

(i) direct method (ii) method of contradiction (iii) method of contrapositive.

Sol: Let q and r be the statements given by

q: x is a real number such that $x^3 + 4x = 0$.

r : x is 0.

Then p: If q, then r.

(i) Direct Method: Let q is true. Then,

q is true

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$

$\Rightarrow$ x is a real number such that $x(x^2 + 4) = 0$

$\Rightarrow$ $x = 0$

$\Rightarrow$ r is true.

Thus, q is true $\Rightarrow r$ is true.

Hence, p is true.

(ii) Method of Contradiction: If possible, let p is not true. Then,

p is not true

$\Rightarrow \sim p$ is true

$\Rightarrow \sim (q \Rightarrow r)$ is true

$\Rightarrow q$ and $\sim r$ is true

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$ and $x \neq 0$.

$\Rightarrow$ $x = 0$ and $x \neq 0$

This is a contradiction.

Hence, p is true.

(iii) Method of Contrapositive: Let r is not true. Then,

r is not true

$\Rightarrow$ $x \neq 0,\ x \in R$

$\Rightarrow$ $x(x^2 + 4) \neq 0,\ x \in R$

$\Rightarrow$ q is not true.

Thus, $\sim r \Rightarrow \sim q$

Hence, $p: q \Rightarrow r$ is true.

The validity of Statements by Contradiction

Sometimes to check whether a statement p is true or not, we assume that p is not true $i.e.$ $\sim p$ is true.

Then, we arrive at some results which contradict our supposition. Therefore, we conclude that p is true. This method is known as a contradiction method.

Example: Verify by the method of contradiction that $\sqrt{7}$ is irrational.

Sol: Let p be the statement given by p : $\sqrt{7}$ is irrational.

If possible, let p be not true $i.e.$, let p be false. Then,

p is false

$\Rightarrow$ $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7} = \frac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow$ $7 = \frac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 7\, b^2$

$\Rightarrow$ 7 divides a^2

$\Rightarrow$ 7 divides a.

$\Rightarrow$ $a = 7c$ for some integer c

$\Rightarrow$ $a^2 = 49\, c^2$

$\Rightarrow$ $7\, b^2 = 49\, c^2$

$\Rightarrow$ $b^2 = 7\, c^2$

$\Rightarrow$ 7 divides b^2

$\Rightarrow$ 7 divides b

Thus, 7 is a common factor of both a and b. This contradicts that a and b have no common factor. So, the supposition $\sqrt{7}$ is rational is wrong. Hence, the statement " $\sqrt{7}$ is irrational" is true.

The invalidity of Statements by Counter Examples

To show that a statement is false, we may give an example of a situation where the statement is not valid. Such an example is called a counterexample. The name itself suggests that this is an example to counter the statement.

Example: Giving an example, show that the following statement is false.

" If n is an odd integer, then n is prime".

Sol: We observe that 9 is an odd integer which is not prime. Similarly, 21, 25, etc are odd integers that are not primes.

Hence, the given statement is false.

Example: By giving a counterexample, show that the following statement is not true:

p: "The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2"

Sol: We observe that $x = 1$ is a root of $x^2 - 1 = 0$ and $x = 1$ lies between 0 and 2.

So, the given statement is not true.

Statements with if and only if

Biconditional

The conjunction of a conditional $p \rightarrow q$ and its converse $q \rightarrow p$ is called a biconditional and is written as $p \leftrightarrow q$.

Equivalent ways of expressing a biconditional $p \leftrightarrow q$ are:

p if and only if q

p iff q

q if and only if p

p is necessary and sufficient for q

q is necessary and sufficient for p

The statement " 10 is prime if and only if it has no proper divisor" is an example of a biconditional.

Axiom of Biconditional

The biconditional $p \leftrightarrow q$ is true when p and q are both true or both false, otherwise it is false.

The axiom of biconditional can be represented by the following table.

The table for Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: Given below are two pairs of statements. Combine these two statements using " if and only if ".

(i) p: If a rectangle is a square, then all its four sides are equal.

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

Sol: A rectangle is a square if and only if all its four sides are equal.

(ii) p: If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

Sol: A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Example: Translate the following biconditional into a symbolic form:

" C is an equilateral triangle if and only if it is equiangular".

Sol: Let p: ABC is an equilateral triangle.

and q: ABC is an equiangular triangle.

Then the given statement in symbolic form is given by $p \leftrightarrow q$.

The validity of Statements with "IF AND ONLY IF"

To prove the validity of the statement " if and only if q", we proceed as follows.

Step I: Show that: If p is true, then q is true.

Step II: Show that: If q is true, then p is true.

Example: Using the words "necessary and sufficient" rewrite the statement

"The integer n is odd if and only if n^2 is odd "

Also, check whether the statement is true.

Sol: The given statement can be written as

"The necessary and sufficient condition that the integer n is odd is n^2 must be odd".

Let p and q be the statements given by

p: the integer n is odd.

q: n^2 is odd.

The given statement is "if and only if q".

To check its validity, we have to check the validity of the following statements.

(i) "If p, then q" (ii) "If q, then p"

Checking the validity of " If p, then q" :

The statement "if p, then q " is given by :

" If the integer n is odd, then n^2 is odd"

Let us assume that n is odd. Then,

$n = 2m + 1$, where m is an integer

$\Rightarrow \quad n^2 = (2m + 1)^2$

$\Rightarrow \quad n^2 = 4m(m + 1) + 1$

$\Rightarrow \quad n^2$ is an odd integer

$\Rightarrow \quad n^2$ is odd.

Thus n is odd $\Rightarrow n^2$ is odd

$\therefore$ " If p, then q is true.

Checking the validity of " If q, then p" :

The statement "If q, then p " is given by

"If n is an integer and n^2 is odd, then n is odd"

To check the validity of this statement, we will use the contrapositive method. So, let n be an even integer. Then,

n is even

$\Rightarrow \quad n = 2k$ for some integer k

$\Rightarrow \quad n^2 = 4k^2$

$\Rightarrow \quad n^2$ is an even integer

$\Rightarrow \quad n^2$ is not an odd integer.

Thus, n is not odd $\Rightarrow n^2$ is not odd

$\therefore$ "If q, then p" is true.

Hence, " if and only if q" is true.

Miscellaneous Examples

Example: Check whether "Or" used in the following compound statement is exclusive or inclusive? Write the component statements of the compound statements and use them to check whether the compound statement is true or not. Justify your answer.

t: you are wet when it rains or you are in a river.

Sol: "Or" used in the given statement is inclusive because it is possible that it rains and you are in the river.

The component statements of the given statement are

p: You are wet when it rains. $\quad$ q: You are wet when you are in a river.

Here both the component statements are true and therefore, the compound statement is true.

Example: Write the negation of the following statements:

(i) p: For every real number x, $x^2 > x$.

Sol: The negation of p is “It is false that p is” which means that the condition $x^2 > x$ does not hold for all real numbers. This can be expressed as

$\sim p$: There exist a real number x such that $x^2 < x$.

(ii) q: There exist a rational number x such that $x^2 = 2$.

Sol: Negation of q is “ It is false that q". Thus $\sim q$ is a statement.

$\sim q$: There does not exist a rational number x such that $x^2 = 2$.

This statement can be rewritten as

$\sim q$: For all real numbers x, $x^2 \neq 2$.

(iii) r: All birds have wings.

Sol: The negation of the statement is

$\sim r$: There exists a bird that has no wings.

Example: For the given statements identify the necessary and sufficient conditions.

t: If you drive over 80 km per hour, then you will get a fine.

Sol: Let p and q denote the statements:

p: you drive over 80 km per hour.

q: you will get a fine.

The implication if p, then q indicates that p is sufficient for q.

That is driving over 80 km per hour is sufficient to get a fine.

Here the sufficient condition is “ driving over 80 km per hour”.

Similarly, if p, then q also indicates that q is necessary for p.

That is when you drive over 80 km per hour, you will necessarily get a fine.

Here the necessary condition is “getting a fine”.