Introduction

The word trigonometry is derived from the Greek words "**trigon**" and "**metron**" means the measurement of triangles. It is the branch of mathematics that deals with the measurement of the sides and the angles of a triangle and the problems allied with angles.

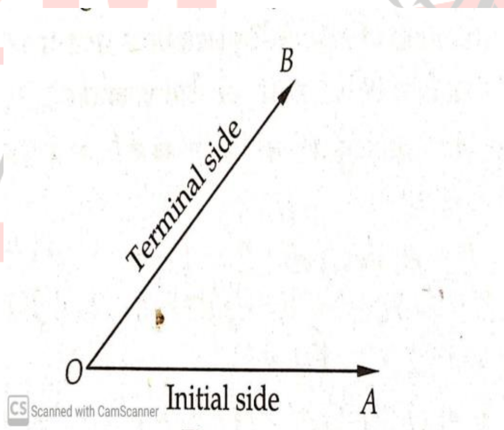
The Hindu mathematicians Aryabhatta, Varahamihira, Brahmagupta, and Bhaskara have a lot of contributions to Trigonometry.

Currently, Trigonometry is used in many areas such as the science of seismology, designing electrical circuits, describing the shape of an atom, predicting the height of the tide in the ocean, analyzing musical tones, and studying the occurrence of sunspots, forecasting fluctuations in the stock market.

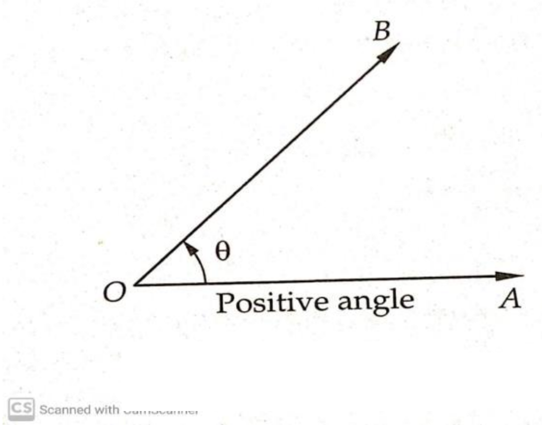
Angles

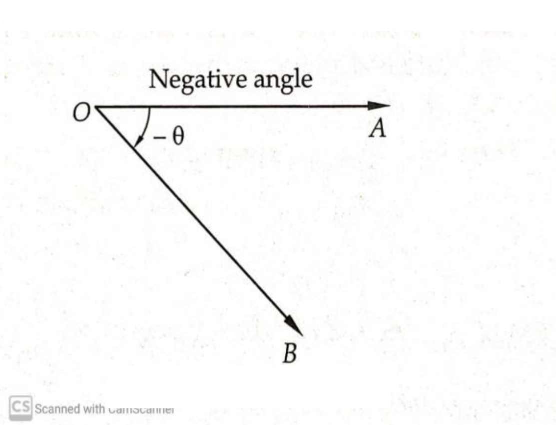
An angle is considered as the figure obtained by rotating a given ray about its endpoint.

The revolving ray is called the generating line of the angle. The initial position OA is called the initial side and the final position OB is called the terminal side. The endpoint O is called the vertex of the angle.



Sense of an Angle: The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according to the initial side rotates in the anticlockwise or clockwise direction to get to the terminal side.





Right Angle: If the revolving ray starting from its initial position to final position describes one-quarter of a circle, then we say that the measure of the angle formed is a right angle.

Systems of Measurement of Angles

There are three systems for measuring angles

(i) **Sexagesimal or English System or Degree measure:**

In this system, the unit of measurement is the degree.

1 right angle $= 90$ degrees$(90°)$

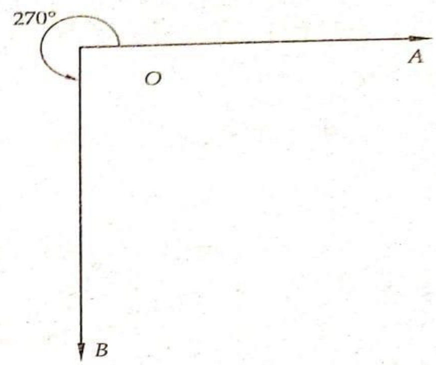
$1° = 60$ minutes$(60')$

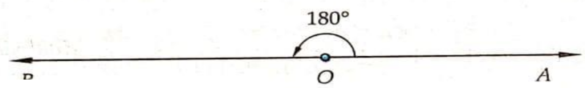
$1' = 60$ seconds$(60'')$

If a rotation from the initial side to the terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one degree.

So 1 complete rotation $= 360°$

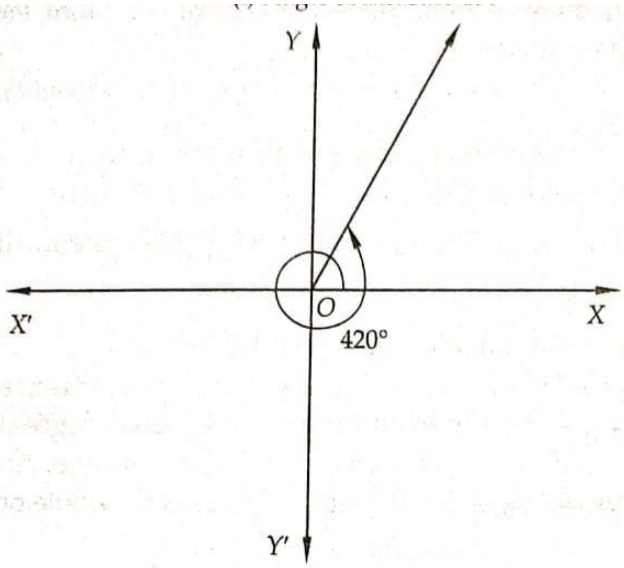
The angles of measures $180°,\ 270°,\ 420°,\ -420°,\ -30°$ are shown in the following figures.



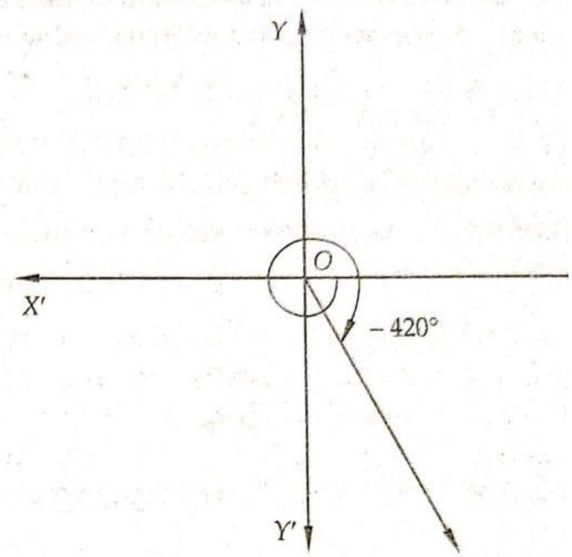


(i) Angle of measure 180°

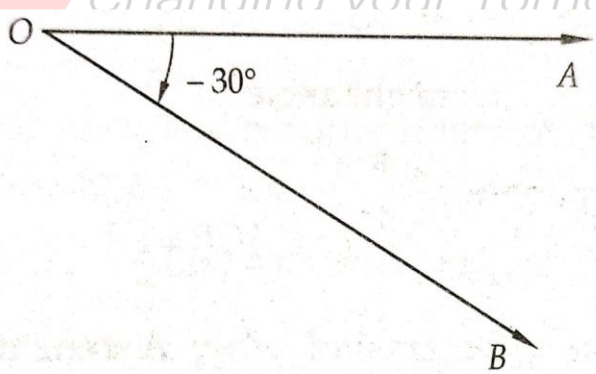
(ii) Angle of measure 270°



(iii) Angle of measure 420°



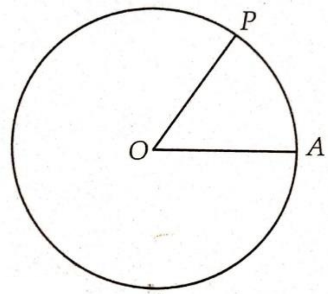
(iv) Angle of measure – 420°



(v) Angle of measure – 30°

(ii) **Circular System or Radian Measure:**

One radian, written as 1^c, is the measure of an angle subtended at the center of a circle by an arc of length equal to the radius of the circle.



Points to Remember:

- Radian is a constant angle.
- The number of radians in angle subtended by an arc of a circle at the center is equal to $\frac{arc}{radius}$.

Relations between Degrees and Radians

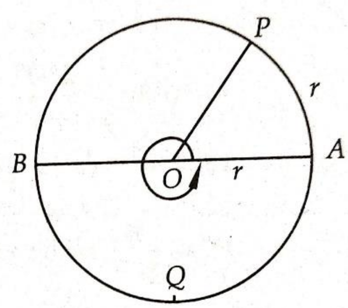
Consider a circle with center O and radius r. Let A be a point on the circle. Join OA and cut off an arc OP of length equal to the radius of the circle. Then $\angle AOP = 1^c$. Produce AO to meet the circle at B.

$\therefore\ \angle AOB =$ a straight angle $=$ 2 right a

We know that the angles at the center of a circle are proportional to the arcs subtending them.

$$\therefore\ \frac{\angle AOP}{\angle AOB} = \frac{arc\ AP}{arc\ APB} \Rightarrow \frac{\angle AOP}{2\ right\ angles} = \frac{r}{\pi r} \Rightarrow \angle AOP = \frac{2\ right\ angles}{\pi} \Rightarrow 1^c = \frac{180°}{\pi}$$

Hence $one\ radian = \frac{180°}{\pi} \Rightarrow \boldsymbol{\pi\ radians = 180°}$.



Points to Remember:

- 2π radians $= 360°$
- All integral multiples of $\frac{\pi}{2}$ are called quadrant angles.
- We have 1 $radian = 57°\,16'22''(approx)$ and $1° = 0.01746\,radians$
- The angle between two consecutive digits of a clock is $30°(=\frac{\pi}{6}radians)$
- The hour hand rotates through an angle of 30° in one hour$i.e.$ $\left(\frac{1}{2}\right)°$ in one minute.
- The minute hand rotates through an angle of 6° in one minute.

Conversion of Degree into Radians and vice – versa

Radian measure $= \frac{\pi}{180} \times$ Degree measure

Degree measure $= \frac{180}{\pi} \times$ Radian measure

The relation between degree measure and radian measure of some common angles are tabulated below.

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Example: Find the radian measure corresponding to the following degree measures:

(i) 340°

Sol: $340° = (340 \times \frac{\pi}{180})^{c} = \left(\frac{17\pi}{9}\right)^{c}$

(ii) $40°\,20'$

Sol: $40°20' = \left(40 + \frac{1}{3}\right)° = \left(\frac{121}{3}\right)° = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^{c} = \left(\frac{121\pi}{540}\right)^{c}$

(iii) $5°37'30''$

Sol: We have $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

So, $37'30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$

$\therefore\ 5°37'30'' = \left(5\frac{5}{8}\right)^\circ = \left(\frac{45}{8}\right)^\circ = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{32}\right)^c$

Example: Find the degree measure corresponding to the following radian measures:

$(i)\ \left(\frac{2\pi}{15}\right)^c$

Sol: $\left(\frac{2\pi}{15}\right)^c = \left(\frac{2\pi}{15} \times \frac{180}{\pi}\right)^\circ = 24°$

$(ii)\ \left(\frac{\pi}{8}\right)^c$

Sol: $\left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = \left(\frac{45}{2}\right)^\circ = \left(22\frac{1}{2}\right)^\circ = 22° \left(\frac{1}{2} \times 60\right)' = 22°\,30'$

$(iii)\ 6^c$

Sol: $6^c = \left(6 \times \frac{180}{\pi}\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90\times7\times6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\ \frac{7}{11}\right)^\circ$

$= 343° \left(\frac{7}{11} \times 60\right)' = 343° \left(\frac{420}{11}\right)' = 343° \left(38\frac{2}{11}\right)' = 343°\,38' \left(\frac{2}{11} \times 60\right)'' = 343°38'11''$

Example: Find the length of an of a circle of radius 5 cm subtending a central angle measuring $15°$.

Sol: Let l be the length of the arc subtending an angle θ at the center of a circle of radius r.

Here $r = 5\ cm$ and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$\therefore\ \theta = \frac{l}{r} \Rightarrow \frac{\pi}{12} = \frac{l}{5} \Rightarrow l = \frac{5\pi}{12}\ cm.$

Example: The angles of a triangle are in the ratio 3: 4: 5, find the smallest angle in degrees, and the greatest angle in radians.

Sol: Let the three angles be $3x$, $4x$, and $5x$ degrees. Then

$3x + 4x + 5x = 180° \Rightarrow x = 15°$

So, the smallest angle $= 3x = 45°$ and

The greatest angle $= 5x = 75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}\ radians$

Example: The minute hand of the watch is 35 cm long. How far does its tip move in 18 minutes? (Use $\pi = \frac{22}{7}$)

Sol: The minute hand of a watch completes one revolution in 60 minutes.

Therefore, the angle traced by a minute hand in 60 minutes $= 360° = 2\pi\, rad$

So, angle traced by the minute hand in 18 minutes $= 2\pi \times \frac{18}{60}\ rad = \frac{3\pi}{5}\ rad$

Let the distance moved by the tip in 18 minutes be l, then

$l = r\,\theta = 35 \times \frac{3\pi}{5} = 21\pi = 21 \times \frac{22}{7} = 66\ cm.$

Example: Find the angle between the minute hand of a clock and the hour hand when the time is $7{:}20\ AM$.

Sol: We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

So, angle traced by the hour hand in 12 hours $= 360°$

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min.

$i.e. \frac{22}{3}\ hrs = \left(\frac{360}{12} \times \frac{22}{3}\right)^{\circ} = 220°$

Also, the angle is traced by the minute hand in 60 min= 360°.

$\Rightarrow$ Angle traced by the minute hand in 20 min= $\left(\frac{360}{60} \times 20\right)^{\circ} = 120°$

Hence, the required angle between two hands= $220° - 120° = 100°$

Example: The wheel of a railway carriage is 40 cm in diameter and makes 6 revolutions in a second; how fast is the train going?

Sol: Diameter of the wheel $= 40\ cm \Rightarrow$ Radius of the wheel $= 20\ cm$

Circumference of the wheel $= 2\pi r = 2\pi \times 20 = 40\pi\ cm$

Number of revolutions made in 1 second $= 6$

So, distance covered in 1 second $= 40\pi \times 6 = 240\ \pi\ cm$

∴ Speed of the train $= 240\pi\ cm/sec$.

Example: A circular wire of radius 3 cm is cut and bent to lie along the circumference of a hoop whose radius is 48 cm. Find the angle in degrees which is subtended at the center of the hoop.

Sol: Given that circular wire is of radius 3 cm, so when it is cut then its length $= 2\pi \times 3 = 6\pi\ cm$.

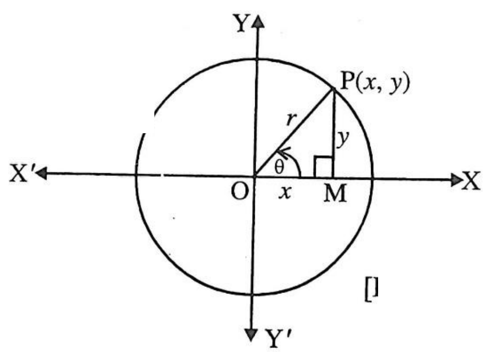
Again, it is being placed along a circular hoop of radius 48 cm.

Here $l = 6\pi\ cm$ is the length of the arc and $r = 48\ cm$ is the radius of the circle.

Therefore, the angle θ, in radian, subtended by the arc at the center of the circle is given by

$$\theta = \frac{arc}{Radius} = \frac{6\pi}{48} = \frac{\pi}{8} = 22.5°$$

TRIGONOMETRIC FUNCTIONS



Let $X'OX$ and $Y'OY$ be the coordinate axes and let a revolving line starts from OX in the anticlockwise direction and trace out an angle $\angle XOP = \theta$.

From P, draw $PM \perp OX$.

Let in the right-angled triangle POM, $OM = x$, $PM = y$ and $OP = r$.

There are six possible ratios among three sides of a triangle.

These six ratios are called trigonometrical ratios and defined as follows:

$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$ $\quad cos\theta = \frac{OM}{OP} = \frac{x}{r}$

$tan\theta = \frac{PM}{OM} = \frac{y}{x}$ $\quad cot\theta = \frac{OM}{PM} = \frac{x}{y}$

$sec\theta = \frac{OP}{PM} = \frac{r}{x}$ $\quad\quad cosec\,\theta = \frac{OP}{PM} = \frac{r}{y}$

Hence, the trigonometrical ratios of an angle are the numerical quantities. Each one of them represents the ratio of the length of one side to another side of a right-angled triangle. The functions $sin\theta$, $cos\theta$, $tan\theta$, $cot\theta$, $sec\theta$, and $cosec\theta$ are called trigonometric functions.

We define $cosec\theta = \frac{1}{sin\theta}$, $sec\theta = \frac{1}{cos\theta}$, $cot\theta = \frac{1}{tan\theta}$

Also, $tan\theta = \frac{sin\theta}{cos\theta}$ and $cot\theta = \frac{cos\theta}{sin\theta}$

Values of Trigonometric Functions

Angle / Trigonometric Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

Trigonometric Identities

An equation involving trigonometric functions which is true for all these values of the variable for which the function is defined is called a trigonometric identity.

We have the following three identities among trigonometrical ratios:

$(i)\ sin^2\theta + cos^2\theta = 1$

$(ii)\ sec^2\theta - tan^2\theta = 1$

$(iii)\ cosec^2\theta - cot^2\theta = 1$

These are called "Pythagorean Identities".

Signs of the Trigonometric Ratios / Functions

The signs of six trigonometric ratios depend on the quadrant in which the terminal side of the angle lies. The length $OP = r$ always positive.

Thus $sin\theta = \frac{y}{r}$ has the sign of y, $cos\theta = \frac{x}{r}$ has a sign of x. The sign of $tan\theta$ depends on the signs of x and y and similarly, the signs of other trigonometric ratios are decided by the signs of x and/or y.

In the first quadrant: We have $x > 0$ and $y > 0$.

So, $sin\theta = \frac{y}{r} > 0$ and $cos\theta > 0$.

Thus in the first quadrant, all trigonometric ratios are positive.

In the second quadrant: We have $x < 0$ and $y > 0$.

So, $sin\theta = \frac{y}{r} > 0$ and $cos\theta = \frac{x}{r} < 0$

Thus in the second quadrant sine and cosecant functions are positive and all others are negative.

In the third quadrant: We have $x < 0$ and $y < 0$.

So, $sin\theta = \frac{y}{r} < 0$ and $cos\theta = \frac{x}{r} < 0$

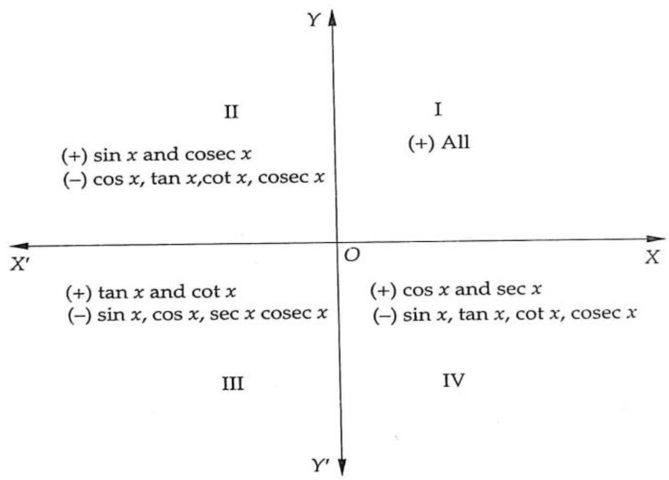
Thus, in the third quadrant, all trigonometric functions are negative except tangent and cotangent.

In the fourth quadrant: We have $x > 0$ and $y < 0$.

So, $sin\theta = \frac{y}{r} < 0$ and $cos\theta = \frac{x}{r} > 0$

Thus, in the fourth quadrant, all trigonometric functions are negative except cosine and secant.

The above rule is known as $ASTC$ Rule.



Points to Remember:

Since $sin^2\theta + cos^2\theta = 1$, so, $|sin\theta| \leq 1$ and $|cos\theta| \leq 1$

$\Rightarrow\ -1 \leq sin\theta \leq 1$ and $-1 \leq cos\theta \leq 1$

Also, $0 \leq sin^2\theta \leq 1,\ \ 0 \leq cos^2\theta \leq 1$

Since, $cosec\theta = \frac{1}{sin\theta}$, therefore $cosec\theta \geq 1$ or $cosec\theta \leq -1$

Since, $sec\ \theta = \frac{1}{cos\theta}$, therefore $sec\theta \geq 1$ or $sec\theta \leq -1$

Domain and Range of Trigonometric Functions

$sin: R \rightarrow [-1,\ 1]$

$cos: R \rightarrow [-1,\ 1]$

$tan: R - \left\{(2n+1)\frac{\pi}{2},\ n \in Z\right\} \rightarrow R$

$cot: R - \{n\pi,\ n \in Z\} \rightarrow R$

$sec:\ R - \left\{(2n+1)\frac{\pi}{2},\ n \in Z\right\} \rightarrow (-\infty,\ -1] \cup [1,\ \infty)$

$cosec:\ R - \{n\pi,\ n \in Z\} \rightarrow (-\infty,\ -1] \cup [1,\ \infty)$

Example: Find $sinx$ and $tanx$, if $cosx = -\frac{12}{13}$ and x lies in the third quadrant.

Sol: We know that $sin^2x + cos^2x = 1 \Rightarrow sinx = \pm\sqrt{1 - cos^2x}$

In the third quadrant $sinx$ is negative.

So, $sinx = -\sqrt{1 - cos^2x} \Rightarrow sinx = -\sqrt{1 - \left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$

and $tanx = \frac{sinx}{cosx} \Rightarrow tanx = \frac{-5}{13}\Big/\frac{-12}{13} = \frac{5}{12}$

Example: Find all other trigonometric ratios, if $sinx = -\frac{2\sqrt{6}}{5}$ and x lies in quadrant III.

Sol: We know that $sin^2x + cos^2x = 1 \Rightarrow cosx = \pm\sqrt{1 - sin^2x}$

In the third quadrant $cosx$ is negative.

$\therefore cosx = -\sqrt{1 - sin^2x} \Rightarrow cosx = -\sqrt{1 - \frac{24}{25}} = -\frac{1}{5}$

In the third quadrant $tanx$ is positive.

So, $tanx = \frac{sinx}{cosx} \Rightarrow tanx = \frac{-2\sqrt{6}}{5}\Big/\frac{-1}{5} = 2\sqrt{6}$

Now, $cosecx = \frac{1}{sinx} = -\frac{5}{2\sqrt{6}}$ $\quad secx = \frac{1}{cosx} = -5$ $\quad cotx = \frac{1}{tanx} = \frac{1}{2\sqrt{6}}$

Example: If $secx = \sqrt{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of $\frac{1+tanx+cosecx}{1+cotx-cosecx}$.

Sol: We have $secx = \sqrt{2} \Rightarrow cosx = \frac{1}{secx} = \frac{1}{\sqrt{2}}$

It is given that x lies in the fourth quadrant in which $sinx$ is negative.

So, $sinx = -\sqrt{1 - cos^2x} = -\sqrt{1 - \frac{1}{2}} = -\frac{1}{\sqrt{2}}$

$\Rightarrow cosecx = \frac{1}{sinx} = -\sqrt{2}$

and $tanx = \frac{sinx}{cosx} = -\frac{1}{\sqrt{2}} \times \sqrt{2} = -1 \Rightarrow cotx = -1$

$\therefore \frac{1+tanx+cosecx}{1+cotx-cosecx} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} = -1$

Example: Prove the following. $sec^2\theta + cosec^2\theta \geq 4.$

Sol: We have $sec^2\theta + cosec^2\theta = 1 + tan^2\theta + 1 + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta - 2\,tan\theta\,cot\theta + 2\,tan\theta\,cot\theta$

$= 2 + (tan\theta - cot\theta)^2 + 2$

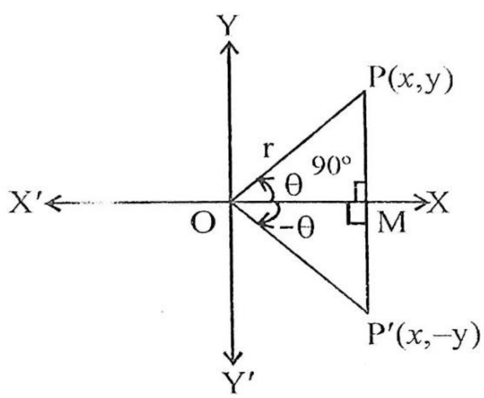
$= 4 + (tan\theta - cot\theta)^2 \geq 4$ [since $(tan\theta - cot\theta)^2 \geq 0$]

Values of Trigonometric Functions at Allied Angles

Two angles are said to be allied when their sum or difference is either zero or multiples of $\frac{\pi}{2}$.

The angles allied to θ are $-\theta, \frac{\pi}{2} \pm \theta,\ \pi \pm \theta, \frac{3\pi}{2} \pm \theta,\ 2\pi \pm \theta$, etc.

Trigonometrical Ratios of $(-\theta)$



Suppose a revolving line OP makes an angle $+\theta$ with OX by revolving in the anticlockwise direction and angle $(-\theta)$ by revolving in clockwise directions (position OP').

Let $\angle POM = \theta$. In $\Delta\,POM,\ OP = r,\ OM = x,\ PM = y.$

Now produce PM to P' such that $PM = MP'$. Join OP'.

Hence $\Delta OMP \equiv \Delta\,OMP'$

$\Rightarrow MP = MP'$ and $OP = OP'$

If we consider the sign of these distances, then $P'M = -PM$

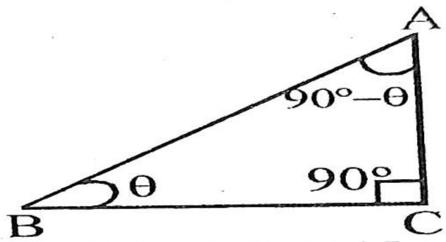
Also, we have $\angle P'OM = -\theta$, $OM = x,\ OP' = r$ and $P'M = -y$

In $\Delta\ P'OM, \sin(-\theta) = \frac{P'M}{OP'} = -\frac{y}{r} = -\frac{PM}{OP} = -sin\theta$

$\cos(-\theta) = \frac{OM}{OP'} = \frac{x}{r} = \frac{OM}{OP} = cos\theta$, $\tan(-\theta) = \frac{P'M}{OM} = -\frac{y}{x} = -\frac{PM}{OM} = -tan\theta$

Similarly, $cosec(-\theta) = -cosec\theta$, $\sec(-\theta) = sec\theta$ and $\cot(-\theta) = -cot\theta$

Trigonometrical Ratios of $\frac{\pi}{2} - \theta$



Let $\angle ACB = \frac{\pi}{2}$, $\angle ABC = \theta$

$\Rightarrow \angle BAC = \frac{\pi}{2} - \theta$, which is the complementary angle of θ.

In $\Delta\ ABC, \sin\left(\frac{\pi}{2} - \theta\right) = \frac{BC}{AB} = cos\theta$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{AC}{AB} = sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{BC}{AC} = cot\theta$$

Trigonometrical Ratios of $\frac{\pi}{2} + \theta$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2} - (-\theta)\right) = \cos(-\theta) = cos\theta$$

$$cos\left(\frac{\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2} - (-\theta)\right) = \sin(-\theta) = -sin\theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = \tan\left(\frac{\pi}{2} - (-\theta)\right) = \cot(-\theta) = -cot\theta$$

Trigonometrical Ratios of $\pi \pm \theta$

$$\sin(\pi - \theta) = \sin\left(\frac{\pi}{2} + \overline{\frac{\pi}{2} - \theta}\right) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Similarly, $\cos(\pi - \theta) = -\cos\theta$ and $\tan(\pi - \theta) = -\tan\theta$

Also, $\sin(\pi + \theta) = \sin\left(\frac{\pi}{2} + \overline{\frac{\pi}{2} + \theta}\right) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

Similarly, $\cos(\pi + \theta) = -\cos\theta$ and $\tan(\pi + \theta) = \tan\theta$

Trigonometrical Ratios of $\frac{3\pi}{2} \pm \theta$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = \sin\left(\pi + \overline{\frac{\pi}{2} - \theta}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos\theta$$

Similarly, $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$ and $\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$

Also, $\sin\left(\frac{3\pi}{2} + \theta\right) = \sin\left(\pi + \overline{\frac{\pi}{2} + \theta}\right) = -\sin\left(\frac{\pi}{2} + \theta\right) = -\cos\theta$

Similarly, $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$ and $\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

Trigonometrical Ratios of $2\pi \pm \theta$

$$\sin(2\pi - \theta) = \sin(\pi + \overline{\pi - \theta}) = -\sin(\pi - \theta) = -\sin\theta$$

Similarly, $\cos(2\pi - \theta) = \cos\theta$ and $\tan(2\pi - \theta) = \tan\theta$

Again, $\sin(2\pi + \theta) = \sin(2\pi - (-\theta)) = -\sin(-\theta) = \sin\theta$

Similarly, $\cos(2\pi + \theta) = \cos\theta$ and $\tan(2\pi + \theta) = \tan\theta$

To remember the above results we adopt the following technique.

$$(i)\ \sin\left(n\frac{\pi}{2} + \theta\right) = \begin{cases} (-1)^{\frac{n-1}{2}} \cos\theta, \ if\ n\ be\ an\ odd\ integer \\ (-1)^{\frac{n}{2}} \sin\theta\ ,\ if\ n\ be\ an\ even\ integer \end{cases}$$

$$(ii)\ \cos\left(n\frac{\pi}{2} + \theta\right) = \begin{cases} (-1)^{\frac{n+1}{2}} \sin\theta, if\ n\ be\ an\ odd\ integer \\ (-1)^{\frac{n}{2}}\ \cos\theta\ ,\ if\ n\ be\ an\ even\ integer \end{cases}$$

$$(iii)\ \tan\left(n\frac{\pi}{2} + \theta\right) = \begin{cases} -\cot\theta, if\ n\ be\ an\ odd\ integer \\ \tan\theta, if\ n\ be\ an\ even\ integer \end{cases}$$

Periodic Functions

A function $f(x)$ is said to be a periodic function if there exists a positive real number T such that $f(x+T)=f(x)$ for all x. Here T is called the period of f.

Periodicity of Trigonometric Functions

(i) $sinx,\ cosx,\ secx,\ cosecx$ are all periodic with period 2π.

(ii) $tanx$ and $cotx$ are periodic with period π.

(iii) $sin^n x,\ cos^n x,\ sec^n x,\ cosec^n x$ are periodic with period 2π and π, according to n is odd or even.

(iv) $tan^n x,\ cot^n x$ are periodic with period π for n being odd or even.

$(v)|sinx|,\ |cosx|,\ |secx|,\ |cosecx|,\ |tanx|,\ |cotx|$ are periodic with period π.

Even and Odd Functions

A function $f(x)$ is said to be an even function if $f(-x)=f(x)$ for all x and is said to be odd if $f(-x)=-f(x)$ for all x.

$sine$ and $tangent$ are odd functions, where $cosine$ is an even function.

Example: Evaluate the following:

$(i)\ \sin\dfrac{31\pi}{3}$

Sol: $\sin\frac{31\pi}{3}=\sin\left(10\pi+\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$(ii)\ \cos\frac{7\pi}{6}$

Sol: $\cos\frac{7\pi}{6}=\cos\left(2\times\frac{\pi}{2}+\frac{\pi}{6}\right)=-\cos\frac{\pi}{6}\ =-\frac{\sqrt{3}}{2}$

$(iii)\ \sin\left(-\dfrac{25\pi}{4}\right)$

Sol: $\sin\left(-\frac{25\pi}{4}\right)=-\sin\left(\frac{25\pi}{4}\right)=-\sin\left(12\ \times\frac{\pi}{2}+\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$

$(iv)\ \tan\left(\dfrac{19\pi}{3}\right)$

Sol: $\tan\left(\frac{19\pi}{3}\right) = \tan\left(12 \times \frac{\pi}{2} + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

$(v)\ \cot\left(-\frac{15\pi}{4}\right)$

Sol:$\cot\left(-\frac{15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot\left(7 \times \frac{\pi}{2} + \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

Example: Evaluate the following:

$(i)\ \tan 480°$

Sol: $\tan 480° = \tan\frac{8\pi}{3} = \tan\left(5 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$

$(ii)\ \cos(-1710°)$

Sol: $\cos 1710° = \cos\frac{19\pi}{2} = 0$

Example: Prove that $cos510°\ cos330° + sin390°\ cos120° = -1$

Sol: $LHS = cos510°\ cos330° + sin390°\ cos120°$

$= \cos(5 \times 90° + 60°)\cos(3 \times 90° + 60°) + \sin(4 \times 90° + 30°)\cos(1 \times 90° + 30°)$

$= (-1)^3\ sin60°.\ (-1)^2\ sin60° + (-1)^2\ sin30°\ (-1)^1\ sin30°$

$= -sin^2\ 60° - sin^2\ 30° = -\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = -\frac{3}{4} - \frac{1}{4} = -1 = RHS$

Example: Prove that $\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = cot^2x$

Sol: $LHS = \frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \frac{-cosx\ cosx}{sinx\ (-sinx)} = \frac{cos^2x}{sin^2x} = cot^2x = RHS.$

Example: Prove that $\frac{\cos(2\pi+x)cosec(2\pi+x)\tan\left(\frac{\pi}{2}+x\right)}{\sec\left(\frac{\pi}{2}+x\right)cosx\cot(\pi+x)} = 1.$

Sol: $LHS = \frac{\cos(2\pi+x)cosec(2\pi+x)\tan\left(\frac{\pi}{2}+x\right)}{\sec\left(\frac{\pi}{2}+x\right)cosx\cot(\pi+x)} = \frac{cosx\ cosecx\ (-cotx)}{-\ cosecx\ cosx\ cotx} = 1 = RHS$

Example: Find the value of $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°}$

Sol: $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°} = \frac{\cot(90° - 36°)}{\tan 36°} + \frac{tan20°}{\cot(90° - 20°)} = \frac{\tan 36°}{tan36°} + \frac{tan20°}{tan20°} = 1 + 1 = 2$

Example: State the sign of $sin201° + cos201°$

Sol: Since 201° lies in 3rd quadrant, so $sin201° < 0$ and $cos201° < 0$

Thus $sin201° + cos201° < 0$

Example: Find the value of $sin1° . sin2°. sin3° \ .... \sin 200°$.

Sol: $sin1° . sin2°. sin3° \ .... \sin 200° = 0$, since $sin180° = 0$

Trigonometric Functions of Compound Angles

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example, if A, B, C are three angles then $A \pm B$, $A + B + C$, $A - B + C$, etc. are compound angles.

Trigonometric Ratios of Sum and Difference of two angles

Let the rotating line start from initial line OX and trace out $\angle XOS = \angle A$ in the anticlockwise direction and let the rotating line further rotates to trace out $\angle SOP = \angle B$.

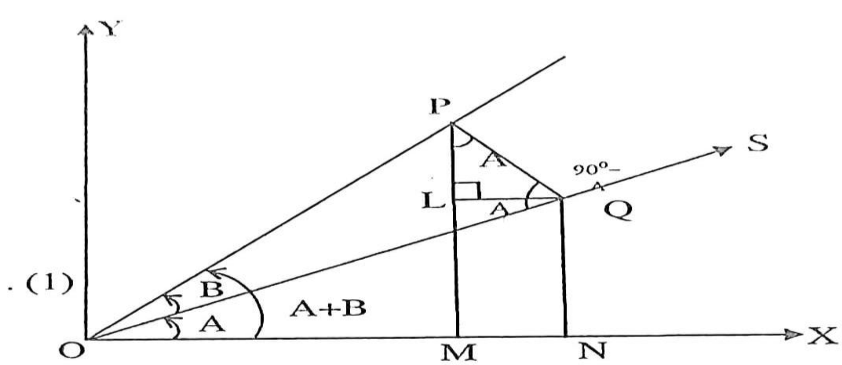
So that $\angle XOP = \angle (A + B)$

Draw $PM \perp OX$ and $PQ \perp OS$. Again draw $QL \perp PM$ and $QN \perp OX$.

Then $\angle QPL = 180° - 90° - \angle PQL = 90° - (90° - A) = A$

$\therefore \sin(A + B) = sin \angle POM = \frac{PM}{OP} = \frac{PL+LM}{OP} = \frac{PL+QN}{OP} = \frac{PL}{OP} + \frac{QN}{OP}$

$= \frac{PL}{PQ} . \frac{PQ}{OP} + \frac{QN}{OQ} . \frac{OQ}{OP} = cosA \ sinB + sinA \ . \ cosB = sinA \ cosB + cosA \ sinB \ .... (1)$



$\cos(A+B) = \frac{OM}{OP} = \frac{ON-MN}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OP} - \frac{LQ}{OP}$

$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{LQ}{PQ} \cdot \frac{PQ}{OP} = cosA\ cosB\ - sinA\ sinB$ .... (2)

$\tan(A+B) = \frac{PM}{OM} = \frac{PL+LM}{OM} = \frac{QN+PL}{ON-MN} = \frac{QN+PL}{ON-LQ}$

$= \frac{\frac{QN}{ON}+\frac{PL}{ON}}{1-\frac{LQ}{ON}} = \frac{\frac{QN}{ON}+\frac{PL}{ON}}{1-\frac{LQ}{PL}\cdot\frac{PL}{ON}} = \frac{\tan A + tanB}{1-tanA.\ tanB}$ .... (3)

Replacing B by $(-B)$ in (1), (2) and (3) we have

$\sin(A-B) = sinA\cos(-B) + cosA\sin(-B) = sinA\ cosB - cosA\sin B$

$\cos(A-B) = cosA\cos(-B) - sinA\sin(-B) = cosA\ cosB + sinAsinB$

$\tan(A-B) = \frac{tanA+\tan(-B)}{1-tanA\tan(-B)} = \frac{tanA-tanB}{1+tanA\ tanB}$

Similarly, it can be proved that

$\cot(A+B) = \frac{cotA\ cotB-1}{cotB+cotA}$ and $\cot(A-B) = \frac{cotA\ cotB+1}{cotB-cotA}$

More Useful Result:

$(i)\ \sin(A+B)\sin(A-B) = sin^2A - sin^2B = cos^2B - cos^2A$

$(ii)\ \cos(A+B)\cos(A-B) = cos^2A - sin^2B = cos^2B - sin^2A$

$(iii)\ \sin(A+B+C)$
$= sinA\ cosB\ cosC + cosA\ sinB\ cosC + cosA\ cosB\ sinC - sinAsinBsinC$

$(iv)\ \cos(A+B+C)$
$= cosA\ cosBcosC - cosA\ sinB\ sinC - sinA\ cosBsinC - sinAsinBcosC$

$(v)\ \tan(A+B+C) = \frac{tanA+tanB+tanC-tanA\ tanB\ tanC}{1-tanA\ tanB-tanB\ tanC-tanC\ tanA}$

Example: Find the value of $\sin 15°$.

Sol: We have $\sin 15° = \sin(45° - 30°)$

$= \sin 45°\ cos30° - cos45°\ sin30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Example: Prove that $\frac{\sin(x+y)}{\sin(x-y)} = \frac{tanx+tany}{tanx-tany}$

Sol: $LHS = \frac{\sin(x+y)}{\sin(x-y)} = \frac{sinx\ cosy+cosx\ siny}{sinx\ cosy-cosx\sin y}$

$= \frac{\frac{sinx\ cosy}{cosx\ cosy}+\frac{cosx\ siny}{cosx\ cosy}}{\frac{sinx\ cosy}{cosx\ cosy}-\frac{cosx\ siny}{cosx\ cosy}} = \frac{tanx+tany}{tanx-tany} = RHS$

Example: Prove that $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\,cosx$

Sol: $LHS = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$= (\cos\frac{\pi}{4}\ cosx - \sin\frac{\pi}{4}\ sinx) + (\cos\frac{\pi}{4}\ cosx + \sin\frac{\pi}{4}\ sinx)$

$= 2\cos\frac{\pi}{4}\ cosx = 2 \times \frac{1}{\sqrt{2}} \times cosx = \sqrt{2}\,cosx = RHS$

Example: Show that $tan3x\ tan2x\ tanx = tan3x - tan2x - tanx$

Sol: We know that $3x = 2x + x$

So, $\tan 3x = \tan(2x+x)$

$\Rightarrow \tan 3x = \frac{tan2x+tanx}{1-\tan 2x\ tanx}$

$\Rightarrow tan3x - tan3x\ tan2x\ tanx = tan2x + tanx$

$\Rightarrow \tan 3x - tan2x - tanx = tan3x\ tan2x\ tanx$

Example: Prove that $\tan 75° + \cot 75° = 4$.

Sol: $LHS = \tan 75° + \cot 75° = (2+\sqrt{3}) + \frac{1}{2+\sqrt{3}} = 2+\sqrt{3}+2-\sqrt{3} = 4 = RHS$

Example: If $tanA = \frac{1}{2}$ and $tanB = \frac{1}{3}$, show that $cos2A = \sin 2B$

Sol: We have $\tan(A+B) = \frac{tanA+tanB}{1-tanA\ tanB} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \frac{5}{6}\Big/\frac{5}{6} = 1$

$\Rightarrow A+B = \frac{\pi}{4} \Rightarrow A = \frac{\pi}{4} - B$

Now $\cos 2A = \cos 2\left(\frac{\pi}{4}-B\right) = \cos\left(\frac{\pi}{2}-2B\right) = \sin 2B$

Example: If $tanA - tanB = x$ and $cotB - \cot A = y$, prove that $\cot(A-B) = \frac{1}{x} + \frac{1}{y}$

Sol: We have $\cot B - \cot A = y$

$\Rightarrow \frac{1}{tanB} - \frac{1}{tanA} = y \Rightarrow \frac{tanA - tanB}{tanA\, tanB} = y$

$\Rightarrow \frac{x}{tanA\, tanB} = y \Rightarrow tanA\, tanB = \frac{x}{y}$

Now $\cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+tanA\, tanB}{tanA - tanB} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$

Example: If $A + B = \frac{\pi}{4}$, prove that $(1+tanA)(1+tanB) = 2$

Sol: We have $A + B = \frac{\pi}{4}$

$\Rightarrow \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \frac{\tan A + tanB}{1 - tanA\, tanB} = 1$

$\Rightarrow tanA + tanB = 1 - tanA\, tanB$

$\Rightarrow tanA + tanB + tanA\, tanB = 1$

$\Rightarrow 1 + tanA + tanB + tanA\, tanB = 2$

$\Rightarrow (1+tanA) + tanB(1+tanA) = 2$

$\Rightarrow (1+tanA)(1+tanB) = 2$

Example: If angle θ is divided into two parts such that the tangent of one part is k times the tangent of other and φ is their difference, then show that $sin\theta = \frac{k+1}{k-1}\, sin\varphi$

Sol: Let $= \alpha + \beta$. Then $tan\alpha = k\, tan\beta$

$\Rightarrow \frac{tan\alpha}{tan\beta} = \frac{k}{1}$

$\Rightarrow \frac{tan\alpha + tan\beta}{tan\alpha - tan\beta} = \frac{k+1}{k-1}$ (Applying componendo and dividendo)

$\Rightarrow \frac{sin\alpha\, cos\beta + cos\alpha\, sin\beta}{sin\alpha\, cos\beta - cos\alpha\, sin\beta} = \frac{k+1}{k-1}$

$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{k+1}{k-1}$

$\Rightarrow \frac{\sin\theta}{\sin\varphi} = \frac{k+1}{k-1}$

$\Rightarrow sin\theta = \frac{k+1}{k-1} sin\varphi$

Maximum and Minimum Values of Trigonometrical Expressions

The maximum and minimum values of $a\ sin\theta + b\ cos\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

Example: Find the maximum and minimum values of $7\ cos\theta + 24\ sin\theta$.

Sol: Here $a = 24$ and $b = 7$

So,$\sqrt{a^2+b^2} = \sqrt{24^2+7^2} = 25$

Thus the maximum and minimum values of $7\ cos\theta + 24\ sin\theta$ are 25 and -25 respectively.

Transformation Formulae

We have $\sin(A+B) = sinA\ cosB + cosA\ sinB \ldots (i)$

$\sin(A-B) = sinA\ cosB\ - cosA\ sinB \ldots (ii)$

$\cos(A+B) = cosA\ cosB\ - sinA\ sinB \ldots (iii)$

$\cos(A-B) = cosA\ cosB + sinA\ sinB \ldots (iv)$

Adding (i) and (ii), we obtain

$\sin(A+B) + \sin(A-B) = 2\ sinA\ cosB$

Subtracting (ii) from (i), we obtain

$\sin(A+B) - \sin(A-B) = 2\ cosA\ sinB$

Adding (iii) and (iv), we obtain

$\cos(A+B) + \cos(A-B) = 2\ cosA\ .\ cosB$

Subtracting (iii) from (iv), we obtain

$\cos(A-B) - \cos(A+B) = 2\ sinA\ sinB$

In above formulae $A > B$

Formulae to transform the sum or difference into products

Let $A + B = C$ and $A - B = D$. Then, $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

Substituting the values of A, B, C and D in the above formulae, we get

$\sin C + sinD = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$sinC - sinD = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$cosC + cosD = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$cosD - cosC = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Or, $\cos C - cosD = \ -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Or, $cosC - cosD = \ 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Example: Convert each of the following products into the sum or difference of sines and cosines.

$(i) 2\sin 5\theta\ cos\theta$

Sol: $2\sin 5\theta\ cos\theta = \sin(5\theta + \theta) + \sin(5\theta - \theta) = \sin 6\theta + \sin 4\theta$

$(ii)\ cos75° \cos 15°$

Sol: $cos75° \cos 15° = \frac{1}{2}(\ 2cos75° \cos 15°) = \frac{1}{2}\{\ \cos(75° + 15°) + \cos(75°\ - 15°)\}$

$= \frac{1}{2}(\cos 90° + \cos 60°)$

Example: Express each of the following as a product:

$(i) \sin 6\theta\ - \sin 2\theta$

Sol: $\sin 6\theta\ - \sin 2\theta = 2\sin\left(\frac{6\theta-2\theta}{2}\right)\ cos\left(\frac{6\theta+2\theta}{2}\right) = 2\sin 2\theta \cos 4\theta$

$(ii) \cos 6\theta\ - \cos 8\theta$

Sol: $\cos 6\theta\ - \cos 8\theta = 2\sin\left(\frac{8\theta+6\theta}{2}\right)\sin\left(\ \frac{8\theta-6\theta}{2}\right) = 2\ sin7\theta \sin\theta$

Example: Prove that $\frac{\cos 7x+\cos 5x}{sin7x\ -\sin 5x} = cotx$

Sol: $LHS = \frac{\cos 7x+\cos 5x}{sin7x-\sin 5x} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{cosx}{sinx} = cotx = RHS$

Example: Prove that $\frac{\sin 5x-2\sin 3x+sinx}{\cos 5x-\cos x} = tanx$

Sol: $LHS = \frac{\sin 5x-2\sin 3x+sinx}{\cos 5x-\cos x} = \frac{sin5x+sinx-2\,sin3x}{\cos 5x-cosx}$

$= \frac{2\,sin3x\cos 2x-2\,sin3x}{-2\sin 3x\,sin2x} = -\frac{sin3x(\,cos2x-1)}{sin3x\,sin2x}$

$= \frac{1-\cos 2x}{sin2x} = \frac{2\,sin^2x}{2\,sinx\,cosx} = tanx = RHS$

Example: Prove that $cos18° - sin18° = \sqrt{2}\,sin27°$

Sol: $LHS = cos18° - sin18°$

$= \cos(90° - 72°) - sin18°$

$= sin72° - sin18°$

$= 2\sin\left(\frac{72°-18°}{2}\right)\cos\left(\frac{72°+18°}{2}\right) = 2\sin 27°\cos 45°$

$= 2\,sin27°\,\frac{1}{\sqrt{2}} = \sqrt{2}\,sin27° = RHS$

Example: Prove that $sin10°\,sin30°\,sin50°\,sin70° = \frac{1}{16}$

Sol: $LHS = sin10°\,sin30°\,sin50°\,sin70°$

$= sin30°\,(sin10°\,sin50°)sin70°$

$= \frac{1}{2}\,(sin50°\,sin10°)\sin 70°$

$= \frac{1}{2} \times \frac{1}{2}(\,2\,sin50°\,sin10°)sin70°$

$= \frac{1}{4}\,[\{\cos(50° - 10°) - \cos(50° + 10°)\}sin70°]$

$= \frac{1}{4}\{(cos40° - \cos 60°)\sin 70°\}$

$= \frac{1}{4}\,(\,sin70°\cos 40° - sin70°\cos 60°)$

$= \frac{1}{4}\left(\,sin70°\,cos40° \;-\frac{1}{2}\,sin70°\right)$

$= \frac{1}{8}\ (2sin70° \ cos40° - sin70°)$

$= \frac{1}{8}\ \{\sin(70° + 40°) + \sin(70° \ - 40°) - sin70°\}$

$= \frac{1}{8}\ (sin110° + \sin 30° - sin70)$

$= \ \frac{1}{8}\ \left\{\sin(180° - 70°) + \frac{1}{2}\ - sin70°\right\}$

$= \frac{1}{8}\ \left(sin70° + \frac{1}{2}\ - sin70°\right) = \frac{1}{8}\ \times \frac{1}{2} = \frac{1}{16} = RHS$

Example: Prove that $\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right) = 0$

Sol: $LHS = \ \sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right)$

$= sinx + 2\sin\left(\frac{x + \frac{2\pi}{3} + x + \frac{4\pi}{3}}{2}\right)\cos\left(\frac{x + \frac{2\pi}{3} - x - \frac{4\pi}{3}}{2}\right)$

$= sinx + 2\sin(\pi + x)\cos\left(-\frac{\pi}{3}\right) = sinx + 2\ (-sinx)\cos\frac{\pi}{3}$

$= sinx \ - 2\ sinx\, .\frac{1}{2} = sinx \ - sinx \ = 0 = RHS$

Example: Prove that $\frac{\sin 5A - sin7A + sin8A \ - sin4A}{cos4A + cos7A \ - cos5A - cos8A} = \cot 6A$

Sol: $LHS = \frac{\sin 5A - sin7A + sin8A \ - sin4A}{cos4A + cos7A \ - cos5A - cos8A}$

$= \ \frac{sin8A - sin4A \ - (sin7A - sin5A)}{cos4A - cos8A \ - (cos5A \ - cos7A)}$

$= \frac{2\cos 6A \ sin2A - 2cos6A \ sinA}{2\ sin6A \ sin2A - 2\ sin6A \ sinA}$

$= \frac{cos6A\ (sin2A - sinA)}{sin6A(sin2A - sinA)}$

$= \frac{\cos 6A}{\sin 6A} = \cot 6A = RHS$

Example: Prove that $cos3A + \cos 5A + cos7A + cos15A = 4\ cos4A\ cos5A\cos 6A$

Sol: $LHS = \cos 3A + cos5A + cos7A + cos15A$

$= \cos 15A + cos5A + cos7A + cos3A$

$= 2\cos\left(\frac{15A + 5A}{2}\right)\cos\left(\frac{15A - 5A}{2}\right) + 2\cos\left(\frac{7A + 3A}{2}\right)\cos\left(\frac{7A - 3A}{2}\right)$

$= 2\cos 10A \cos 5A + 2\cos 5A \cos 2A$

$= 2\cos 5A(\cos 10A + \cos 2A)$

$= 2\cos 5A\left\{2\cos\left(\frac{10A+2A}{2}\right)\cos\left(\frac{10A-2A}{2}\right)\right\}$

$= 2\cos 5A \,.\, 2\cos 6A \cos 4A$

$= 4\cos 4A\, cos5A \cos 6A = RHS$

Example: If $a\, sinx = b\, \sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right)$, prove that $ab+bc+ca=0$.

Sol: We have $a\, sinx = b\, \sin\left(x+\frac{2\pi}{3}\right) = c\sin\left(x+\frac{4\pi}{3}\right) = k\ (say)$

$\Rightarrow \frac{k}{a} = sinx,\ \frac{k}{b} = \sin\left(x+\frac{2\pi}{3}\right),\ \frac{k}{c} = \sin\left(x+\frac{4\pi}{3}\right)$

$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c} = \sin x + \sin\left(x+\frac{\pi}{3}\right) + \sin\left(x+\frac{4\pi}{3}\right)$

$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c} = \left\{\sin\left(x+\frac{4\pi}{3}\right) + sinx\right\} + \sin\left(x+\frac{2\pi}{3}\right)$

$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c} = 2\sin\left(x+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\left(x+\frac{2\pi}{3}\right)$

$\Rightarrow k\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = -\sin\left(x+\frac{2\pi}{3}\right) + \sin\left(x+\frac{2\pi}{3}\right)$

$\Rightarrow k\left(\frac{ab+bc+ca}{abc}\right) = 0$

$\Rightarrow ab+bc+ca = 0$

Example: Prove that $\frac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°} = \tan 54°$

Sol: RHS $= \tan 54° = \tan(45°+9°)$

$$= \frac{\tan 45° + \tan 9°}{1 - tan45° \tan 9°}$$

$$= \frac{1+\dfrac{\sin 9°}{\cos 9°}}{1-\dfrac{\sin 9°}{\cos 9°}} = \frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = LHS$$

Example: Prove that $\frac{sinx}{cos3x}+\frac{sin3x}{cos9x}+\frac{sin9x}{cos27x} = \frac{1}{2}(tan27x - tanx)$

Sol: $LHS = \frac{sinx}{cos3x}+\frac{sin3x}{cos9x}+\frac{sin9x}{cos27x}$

$= \frac{1}{2}\left(\frac{2\,sinx\,cosx}{cos3x\,cosx} + \frac{2\,sin3x\,cos3x}{cos9x\,cos3x} + \frac{2\,sin9x\,cos9x}{cos27x\,cos9x}\right)$

$= \frac{1}{2}\left(\frac{\sin(x+x)}{cos3x\,cosx} + \frac{\sin(3x+3x)}{cos9x\,cos3x} + \frac{\sin(9x+9x)}{co27x\,cos9x}\right)$

$= \frac{1}{2}\left(\frac{sin2x}{cos3x\,cosx} + \frac{sin6x}{cos9x\,cos3x} + \frac{sin18x}{cos27x\,cos9x}\right)$

$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{cos3x\,cosx} + \frac{\sin(9x-3x)}{cos9x\,cos3x} + \frac{\sin(27x-9x)}{cos27x\,cos9x}\right\}$

$= \frac{1}{2}\left(\frac{sin3xcosx - cos3x\,sinx}{cos3xcosx} + \frac{sin9x\,cos3x - cos9x\,sin3x}{cos9x\,cos3x} + \frac{sin27xcos9x - cos27xsin9x}{cos27x\,cos9x}\right)$

$= \frac{1}{2}\{(tan3x - tanx) + (tan9x - tan3x) + (tan27x - tan9x)\}$

$= \frac{1}{2}\,(tan27x - tanx) = RHS$

Example: Prove that $cos2\alpha \cos 2\beta + sin^2(\alpha - \beta) - sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$.

Sol: $LHS = cos2\alpha \cos 2\beta + sin^2(\alpha - \beta) - sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$

$= \cos 2\alpha cos2\beta + sin\,(\alpha - \beta + \alpha + \beta) \sin(\alpha - \beta - \alpha - \beta)$

$= \cos 2\alpha \cos 2\beta - sin2\alpha \sin 2\beta$

$= \cos(2\alpha + 2\beta)$

$= \cos 2(\alpha + \beta) = RHS$

Example: If α and β are the solutions of the equation $a\,tanx + b\,secx = c$, show that $\tan(\alpha + \beta) = \frac{2ac}{a^2 - c^2}$

Sol: We have, $a\,tanx + b\,secx = c \ldots (i)$

$\Rightarrow c - a\,tanx = b \sec x$

$\Rightarrow (c - a\,tanx)^2 = b^2 sec^2 x$

$\Rightarrow c^2 + a^2 tan^2 x - 2ac\,tanx = b^2(1 + tan^2 x)$

$\Rightarrow tan^2 x\,(a^2 - b^2) - 2ac\,tanx + (c^2 - b^2) = 0 \ldots (ii)$

It is given that α and β are the solutions of the given equation(i). Therefore $tan\alpha$ and $tan\beta$ are roots of the equation(ii).

So, $tan\alpha + tan\beta = \frac{2ac}{a^2-b^2}$ and $tan\alpha\, tan\beta = \frac{c^2-b^2}{a^2-b^2}$

Hence $\tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1-tan\alpha\, tan\beta} = \frac{2ac}{a^2-c^2}$

Example: Prove that: $\cos A \cos\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right) = \frac{1}{4}\cos 3A$

Sol: $LHS = \cos A \cos\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right)$

$= \frac{1}{2}\cos A\left\{2\cos\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right)\right\}$

$= \frac{1}{2}\cos A\left[\cos\left(\frac{\pi}{3} + A + \frac{\pi}{3} - A\right) + \cos\left(\frac{\pi}{3} + A - \frac{\pi}{3} + A\right)\right]$

$= \frac{1}{2}\, cosA\left[\cos\frac{2\pi}{3} + \cos 2A\right]$

$= \frac{1}{2}\cos A\left\{\cos 2A - \frac{1}{2}\right\} = \frac{1}{2}\cos A\cos 2A - \frac{1}{4}\, cosA$

$= \frac{1}{4}\, 2\cos 2A\, cosA - \frac{1}{4}\, cosA = \frac{1}{4}\,(\cos 3A + cosA) - \frac{1}{4}cosA$

$= \frac{1}{4}\cos 3A + \frac{1}{4}\cos A - \frac{1}{4}cosA = \frac{1}{4}\cos 3A = RHS$

Note: We have $\cos A \cos\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right) = \frac{1}{4}\cos 3A$

Similarly $\text{sinA} \sin\left(\frac{\pi}{3} - A\right)\sin\left(\frac{\pi}{3} + A\right) = \frac{1}{4}\sin 3A$

and $\tan A \tan\left(\frac{\pi}{3} - A\right)\tan\left(\frac{\pi}{3} + A\right) = \tan 3A$

Example: If three angles $A,\ B$ and C are in $A.P.$, prove that $\cot B = \frac{sinA - sinC}{\cos C - \cos A}$

Sol: Since $A,\ B$ and C are in $A.P.$, so $2B = A + C$

Now, $RHS = \frac{sinA - sinC}{\cos C - \cos A} = \frac{2\sin\frac{A-C}{2}\cos\frac{A+C}{2}}{2\sin\frac{A+C}{2}\sin\frac{A-C}{2}}$

$= \cot\left(\frac{A+C}{2}\right) = \cot\left(\frac{2B}{2}\right) = \cot B = LHS$

Example: If $A,\ B$, and C are the angles of a triangle, then prove that

$sin^2A + sin^2B + sin^2C - 2\, cosA\cos B\cos C = 2$

Sol: $LHS = sin^2A + sin^2B + sin^2C - 2\, cosA\cos B\cos C$

$= \ 1 - cos^2A + 1 - \ cos^2B + sin^2C \ - 2\cos A \cos B \cos C$

$= 2 \ - \ cos^2A - (cos^2B \ - \ sin^2C) - 2\ cosA \cos B \cos C$

$= 2 \ - \ cos^2A \ - \cos(B+C)\cos(B-C) - 2\cos A \cos B \cos C$

$= 2 \ - \ cos^2A + \cos A \cos(B-C) - 2\cos A\ cosB \cos C \ \ [\ since\ A+B+C = \pi\ ]$

$= 2 \ - cosA\ [\cos A \ - \cos(B-C)] \ - 2\cos\ A\cos\ B\cos\ C$

$= 2 \ - \cos A\ [\ - \cos(B+C) - \cos(B-C)] - 2\cos A \cos B \cos C$

$= 2 + \cos A\ \ [\cos(B+C) + \cos\ (B-C)] \ - \ \ 2\cos\ A\cos\ B\cos\ C$

$= 2 + \cos A\ [\ 2\cos B\cos C] \ - \ \ 2\cos A\cos B\cos C$

$= 2 = RHS$

Trigonometric Functions of Multiple and Submultiple Angles

If A is any angle, then angles $2A$, $3A$, $4A$, ... etc are called multiple angles and the angles $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$, ... etc are called sub - multiple angles of A.

Trigonometric Ratios of angle $2A$ in terms of that of angle A

(i) $\sin 2A = \sin(A+A) = sinA\ cosA + cosA\ sinA = 2\ sinA\ cosA$

(ii) We have $sin2A = 2\ sinA\ cosA = \frac{2\ sinA\ cos^2A}{cosA} = \frac{2\ tanA}{sec^2A} = \frac{2\ tanA}{1+\ tan^2A}$

(iii) $\cos 2A = \cos(A+A) = cosA\ cosA \ - sinA\ sinA = \ cos^2A \ - \ sin^2A$

(iv) $\cos 2A = cos^2A \ - \ sin^2A = \ cos^2A \ - (1 - cos^2A) = 2cos^2A - 1$

(v) $\cos 2A = cos^2A \ - \ sin^2A = 1 - sin^2A - \ sin^2A = 1 - 2sin^2A$

(vi) $cos2A = cos^2A \ - \ sin^2A = \frac{cos^2A - sin^2A}{cos^2A + sin^2A} = \frac{1 - tan^2A}{1 + tan^2A}$

(vii) We have $\cos 2A = 1 - 2sin^2A \Rightarrow \ \ sin^2A = \frac{1 - cos2A}{2}$

$(viii)$ We have $cos2A = 2cos^2A - 1 \Rightarrow cos^2A = \frac{1 + \cos 2A}{2}$

(ix) $\tan 2A = \tan(A+A) = \frac{\tan A + tanA}{1 - tanA\ tanA} = \frac{2\ tanA}{1 - \ tan^2A}$

(x) $tan^2\ A = \frac{1\ - \cos 2A}{1 + \cos 2A}$

Trigonometric Ratios of the Angle A in terms of Angle $\frac{A}{2}$

Replacing A by $\frac{A}{2}$ in the relations of multiple angles of $2A$, we obtain

$(i)\ sinA = 2\sin\frac{A}{2}\cos\frac{A}{2}$

$(ii)\ sinA = \frac{2\tan\frac{A}{2}}{1+tan^2\frac{A}{2}}$

$(iii)\ cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2}$

$(iv)\ cosA = 2\ cos^2\frac{A}{2} - 1$

$(v)\ cosA = 1 - 2\ sin^2\frac{A}{2}$

$(vi)\ cosA = \frac{1-tan^2\frac{A}{2}}{1+tan^2\frac{A}{2}}$

$(vii)\ sin^2\frac{A}{2} = \frac{1-cosA}{2}$

$(viii)\ cos^2\frac{A}{2} = \frac{1+cosA}{2}$

$(ix)\ tanA = \frac{2\tan\frac{A}{2}}{1-tan^2\frac{A}{2}}$

$(x)\ tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$

Trigonometric Ratios of Angle $3A$ in terms of Angle A

(i) $\sin 3A = 3\ sinA - 4sin^3A$

Sol: $\sin 3A = \sin(A + 2A) = sinA\ cos2A + sin2A\ cosA$

$= sinA\ (1 - 2sin^2A) + (2\ sinA\ cosA)cosA$

$= sinA - 2sin^3A + 2\ sinA\ (1 - sin^2A)$

$= sinA\ - 2sin^3A + 2\ sinA - 2\ sin^3A = 3\ sinA\ - 4sin^3A$

$(ii)\ cos3A = 4cos^3A\ - 3\ cosA$

$(iii)\ tan3A = \frac{3\ tanA - tan^3A}{1-3tan^2A}$

Points to Remember:

1. $\sin\frac{A}{2} = \begin{cases} \sqrt{\frac{1-\cos A}{2}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ I\ or\ II \\ -\sqrt{\frac{1-\cos A}{2}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ III\ or\ IV \end{cases}$

2. $\cos\frac{A}{2} = \begin{cases} \sqrt{\frac{1+cosA}{2}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ \text{I or } IV \\ -\sqrt{\frac{1+cosA}{2}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ \text{II or III} \end{cases}$

3. $\tan\frac{A}{2} = \begin{cases} \sqrt{\frac{1-cosA}{1+cosA}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ \text{I or III} \\ -\sqrt{\frac{1-cosA}{1+cosA}} & if\ \frac{A}{2}\ lies\ in\ quadrant\ \text{II or } IV \end{cases}$

Example: If $sinA = \frac{3}{5}$, where $0° < A < 90°$, find the values of $sin2A,\ cos2A,\ tan2A$.

Sol: We have $sinA = \frac{3}{5}$, where $0° < A < 90°$

Since $cos^2A = 1 - \ sin^2A$

So, $cosA = \sqrt{1 - sin^2A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore\ tanA = \frac{sinA}{cosA} = \frac{3}{4}$

Now, $sin2A = 2\ sinA\ cosA = 2.\frac{3}{5}.\frac{4}{5} = \frac{24}{25}$

$cos2A = 1 - 2\ sin^2A = 1 - 2\ \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

$\tan 2A = \frac{2\ tanA}{1-tan^2A} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{24}{7}$

Example: Prove that $\frac{1+sin2\theta+cos2\theta}{1+sin2\theta-cos2\theta} = cot\theta$

Sol: $LHS = \frac{1+sin2\theta+cos2\theta}{1+sin2\theta-cos2\theta} = \frac{(1+cos2\theta)+sin2\theta}{(1-cos2\theta)+sin2\theta} = \frac{2cos^2\theta+2\ sin\theta\ cos\theta}{2\ sin^2\theta+2\ sin\theta\ cos\theta}$

$= \frac{2\ cos\theta\ (\ cos\theta+sin\theta)}{2\ sin\theta\ (\ cos\theta+sin\theta)} = cot\theta = RHS$

Example: Prove that $\frac{\sin 5x - 2\, sin3x + sinx}{\cos 5x - cosx} = tanx$

Sol: $LHS = \frac{\sin 5x - 2\, sin3x + sinx}{\cos 5x - cosx} = \frac{\sin 5x + sinx - 2\sin 3x}{cos5x - cosx}$

$= \frac{2\, sin3x\, cos2x - 2\, sin3x}{-\, 2\, sin3x\, sin2x} = \frac{sin3x(\, 2\, cos2x - 1)}{-\, 2sin3x\, sin2x}$

$= \frac{1 - cos2x}{sin2x} = \frac{2\, sin^2 x}{2\, sinx\, cosx} = tanx = RHS$

Example: Prove that $\left(1 + \cos\frac{\pi}{8}\right)\left(\, 1 + \cos\frac{3\pi}{8}\right)(1 + \cos\frac{5\pi}{8})\; (\, 1 + \cos\frac{7\pi}{8}\,) = \frac{1}{8}$

Sol: We have $\cos\frac{7\pi}{8} = \cos\left(\, \pi\, - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

Also $\cos\frac{5\pi}{8} = \cos\left(\, \pi\, - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$LHS = \left(1 + \cos\frac{\pi}{8}\right)\left(\, 1 + \cos\frac{3\pi}{8}\right)(1 + \cos\frac{5\pi}{8})\; (\, 1 + \cos\frac{7\pi}{8}\,)$

$= \left(1 + \cos\frac{\pi}{8}\right)\left(\, 1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$

$= \left(1 - \, cos^2\frac{\pi}{8}\right)\left(1 - \, cos^2\frac{3\pi}{8}\right) = \, sin^2\frac{\pi}{8}\; sin^2\frac{3\pi}{8}$

$= \frac{1 - \cos\frac{\pi}{4}}{2} \times \frac{1 - \cos\frac{3\pi}{4}}{2} = \frac{1}{4}\; \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1}{4}\; \left(\, 1 - \frac{1}{2}\right) = \frac{1}{8} = RHS$

Example: Prove that $cos^2 x + \, cos^2\left(\, x + \frac{\pi}{3}\right) + \, cos^2\, \left(\, x - \frac{\pi}{3}\right) = \frac{3}{2}$

Sol: $LHS = cos^2 x + \, cos^2\left(\, x + \frac{\pi}{3}\right) + \, cos^2\, \left(\, x - \frac{\pi}{3}\right)$

$= \, \frac{1 + cos2x}{2} + \, \frac{1 + \cos 2\left(\, x + \frac{\pi}{3}\right)}{2} + \, \frac{1 + \cos 2\left(\, x - \frac{\pi}{3}\right)}{2}$

$= \frac{1}{2}\left[3 + \cos 2x + \cos\left(\, 2x + \frac{2\pi}{3}\right) + \cos\left(\, 2x - \frac{2\pi}{3}\right)\right]$

$= \frac{1}{2}\left[3 + \cos 2x + 2\cos 2x \cos\frac{2\pi}{3}\right]$

$= \frac{1}{2}\; \left[3 + \cos 2x + 2\cos 2x\; \left(\, -\frac{1}{2}\right)\right]$

$= \frac{1}{2}\; [3 + \cos 2x\; - \cos 2x] = \frac{3}{2} = RHS$

Example: Find the value of $\tan\frac{\pi}{8}$

Sol: Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$

Now, $\tan 2x = \frac{2\, tanx}{1-tan^2 x} \Rightarrow \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-tan^2\frac{\pi}{8}} \Rightarrow 1 = \frac{2\tan\frac{\pi}{8}}{1-tan^2\frac{\pi}{8}}$

$\Rightarrow\ 1 - tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$

$\Rightarrow tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

$\Rightarrow \tan\frac{\pi}{8} = \frac{-2 \pm\sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $\frac{\pi}{8}$ lies in the first quadrant so $\tan\frac{\pi}{8}$ is positive.

Hence $\tan\frac{\pi}{8} = -1 + \sqrt{2} = \sqrt{2} - 1$

Example: If $tanx = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, find the values of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Sol: Since $< x < \frac{3\pi}{2}$, so $\cos x$ is negative.

Also, $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$, so $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $sec^2 x = 1 + tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$

$\Rightarrow cos^2 x = \frac{16}{25} \Rightarrow cosx = -\frac{4}{5}$

Now $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$

$$\cos\frac{x}{2} = -\sqrt{\frac{1+cosx}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = -3$$

Example: Show that $2\, sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$

Sol: $LHS = 2\, sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

$= 2\, sin^2\beta + 2\cos(\alpha+\beta)\,(2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$

$= 2sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$

$= 2sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\,cos^2(\alpha+\beta) + 2\,cos^2\,(\alpha+\beta) - 1$

$= 2sin^2\beta + 2(\,cos^2\alpha - \,sin^2\beta) - 1$

$= 2sin^2\beta + 2\,cos^2\alpha\, - 2sin^2\beta\, - 1$

$= 2cos^2\alpha - 1 = \cos 2\alpha = RHS$

Example: Prove that $\sqrt{3}\,cosec\,20° - \sec 20° = 4$

Sol: $LHS = \sqrt{3}\,cosec\,20° - \sec 20°$

$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cos 20°}$

$= \frac{2(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°)}{\frac{1}{2}2\sin 20°\cos 20°}$

$= \frac{4\,(\sin 60°\cos 20° - \cos 60°\sin 20°)}{\sin 40°} = \frac{4\sin 40°}{\sin 40°} = 4 = RHS$

Example: Prove that $\cos 20°\cos 40°\cos 60°\cos 80° = \frac{1}{16}$

Sol: $LHS = \cos 20°\cos 40°\cos 60°\cos 80°$

$= \frac{1}{2}\cos 20°\cos 40°\cos 80°$

$= \frac{1}{2}\cos 20°\cos(\,60° - 20°)\cos(60° + 20°)$

$= \frac{1}{2}\cos 20°\,(cos^2 20° - sin^2\,60°)$

$= \frac{1}{2}\cos 20°\,\left(\,cos^2 20° - \frac{3}{4}\right) = \frac{1}{8}\cos 20°\,(4\,cos^2\,20° - 3)$

$= \frac{1}{2}\,(\,4\,cos^3\,20° - 3\cos 20°)$

$= \frac{1}{2}\cos 60° = \frac{1}{2}\,.\frac{1}{2} = \frac{1}{4} = RHS$

Example: If $cos\theta = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$, prove that $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$

Sol: We have $cos\theta = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$

$\Rightarrow \quad \frac{1 - tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$

$\Rightarrow \quad \frac{\left(1 - tan^2\frac{\theta}{2}\right) + \left(1 + tan^2\frac{\theta}{2}\right)}{\left(1 - tan^2\frac{\theta}{2}\right) - \left(1 + tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha - \cos\beta) + (1 - cos\alpha\cos\beta)}{(\cos\alpha - \cos\beta) - (1 - cos\alpha\cos\beta)}$

$\Rightarrow \quad \frac{2}{-2\,tan^2\frac{\theta}{2}} = \frac{1 + cos\alpha - cos\beta - cos\alpha\,cos\beta}{-(1 + cos\alpha - cos\beta - cos\alpha\,cos\beta)}$

$\Rightarrow \quad \frac{1}{tan^2\frac{\theta}{2}} = \frac{(1 + cos\alpha)(1 - cos\beta)}{(1 - cos\alpha)(1 + cos\beta)}$

$\Rightarrow \quad tan^2\frac{\theta}{2} = tan^2\frac{\alpha}{2}\,cot^2\frac{\beta}{2}$

$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$

Example: Prove that $sin3x\,sin^3x + \cos 3x\,cos^3x = cos^3 2x$

Sol: $LHS = sin3x\, sin^3x + \cos 3x\, cos^3x$

$= \frac{1}{4}\,(sin3x\, 4sin^3x + \cos 3x\, 4cos^3x)$

$= \left\{\sin 3x \left(\frac{3\, sinx - sin3x}{4}\right) + \cos 3x \left(\frac{\cos 3x + 3\, cosx}{4}\right)\right\}$

$= \frac{1}{4}\,\{3\, sin3x\, sinx - sin^2 3x + cos^2 3x + 3\cos 3x\, cosx\}$

$= \frac{1}{4}\{3(\cos 3x\, cosx + sin3x\, sinx) + (cos^2 3x - sin^2 3x)\}$

$= \frac{1}{4}\,\{3\cos(3x - x) + \cos 2(3x)\}$

$= \frac{1}{4}\,(3\cos 2x + \cos 3(2x))$

$= \frac{1}{4}\, 4\, cos^3 2x = cos^3 2x = RHS$

Example: Prove that $\frac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x} = 4\cos 2x \cos 4x$

Sol: $LHS = \frac{\tan 5x + \tan 3x}{\tan 5x - \tan 3x}$

$= \frac{\frac{sin5x}{\cos 5x} + \frac{sin3x}{cos3x}}{\frac{sin5x}{\cos 5x} + \frac{sin3x}{cos3x}} = \frac{sin5x \cos 3x + \cos 5x\, sin3x}{\sin 5x \cos 3x - \cos 5x\, sin3x}$

$= \frac{\sin(5x+3x)}{\sin(5x-3x)} = \frac{sin8x}{\sin 2x}$

$= \frac{2\sin 4x \cos 4x}{\sin 2x}$

$= \frac{2\,(2\sin 2x \cos 2x)\cos 4x}{\sin 2x}$

$= 2\cos 2x \cos 4x$

$= RHS$

Example: Prove that $\frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = -(\cos 2x + cosx)$

Sol: $LHS = \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = \frac{sin3x\,(\cos 5x + \cos 4x)}{sin3x\,(1 - 2\cos 3x)}$

$= \frac{(2\sin\frac{3x}{2}\cos\frac{3x}{2})\,(2\cos\frac{9x}{2}\cos\frac{x}{2})}{\sin 3x - 2\sin 3x \cos 3x}$

$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x}$

$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin(\frac{3x-6x}{2})\cos(\frac{3x+6x}{2})} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin(-\frac{3x}{2})\cos\frac{9x}{2}}$

$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\sin(\frac{3x}{2})\cos\frac{9x}{2}} = -2\cos\frac{3x}{2}\cos\frac{x}{2} = -(cos2x + cosx) = RHS$

Example: If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, prove that $\cos\alpha = \frac{\cos\theta - e}{1 - e\cos\theta}$

Sol: We have $\cos\alpha = \frac{1 - tan^2\frac{\alpha}{2}}{1 + tan^2\frac{\alpha}{2}}$

$\Rightarrow \cos\alpha = \frac{1 - \frac{1+e}{1-e}tan^2\frac{\theta}{2}}{1 + \frac{1+e}{1-e}tan^2\frac{\theta}{2}}$

$\Rightarrow \cos\alpha = \frac{(1-e)-(1+e)tan^2\frac{\theta}{2}}{(1-e)+(1+e)\frac{tan^2\theta}{2}} \Rightarrow \cos\alpha = \frac{\left(1-tan^2\frac{\theta}{2}\right)-e\left(1+tan^2\frac{\theta}{2}\right)}{\left(1+tan^2\frac{\theta}{2}\right)-e\left(1-tan^2\frac{\theta}{2}\right)}$

$\Rightarrow \cos\alpha = \frac{\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}-e}{1-e\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}}$ [Dividing numerator and denominator by $1+\ tan^2\frac{\theta}{2}$]

$\Rightarrow \cos\alpha = \frac{\cos\theta - e}{1-e\cos\theta}$ [since $\cos\theta = \frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$]

Trigonometric Equations

Definition

The equations involving trigonometric functions of unknown angles are called trigonometric equations.

$cosx = \frac{1}{2}$, $sinx = 0$, $tanx + secx = -\sqrt{3}$ etc. are trigonometric equations.

The solution of a Trigonometric Equations

A solution of a trigonometric equation is the value of the unknown angle ($i.e.$ variable) that satisfies the equation.

Consider the equation $sinx = \frac{1}{2}$. This equation is satisfied by $x = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. So, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Solutions of trigonometric equations are of two types.

Principal Solution

The solution of a trigonometric equation for which the value of unknown angle says x lies between 0 and 2π $i.e.,\ 0 \le x < 2\pi$ is called its principal solution.

Example: Find the principal solutions of the following equations.

$(i) sinx = \frac{\sqrt{3}}{2}$

Sol: We have known that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Therefore, principal solutions are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$(ii)\ tanx = -\frac{1}{\sqrt{3}}$

Sol: We know that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Thus $\tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ and $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

Thus $\tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

Therefore, principal solutions are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

General Solution

Since the trigonometric functions are periodic, so if a trigonometric equation has a solution, it will have infinitely many solutions. We know that the values of $sinx$ and $cosx$ repeat after an interval of 2π and the values of $tanx$ repeat after an interval of π.

A solution of a trigonometric equation, generalized through periodicity, is known as the general solution.

In other words, the expression involving integer n which gives all solutions of a trigonometric equation is called the general solution.

The general solution of a trigonometric equation can find out with the help of the following rules:

1) For any real number x, $sinx = 0 \Rightarrow x = n\pi$, $n \in Z$

2) For any real number x, $cosx = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$, $n \in Z$

3) For any real number x, $tanx = 0 \Rightarrow x = n\pi$, $n \in Z$

4) For any real number x, $cotx = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$, $n \in Z$

5) For any real numbers x and y, $sinx = siny \Rightarrow x = n\pi + (-1)^n y$, $n \in Z$

6) For any real numbers x and y, $cosx = \cos y \Rightarrow x = 2n\pi \pm y$, $n \in Z$

7) If x and y are not odd multiples of $\frac{\pi}{2}$, then $tanx = tany \Rightarrow x = n\pi + y$, $n \in Z$

Points to Remember:

1) Since $secx \geq 1$ or $secx \leq -1$, therefore $secx = 0$ does not have any solution. Similarly, $cosecx = 0$ has no solution.
2) The equation $cosecx = cosec\, y$ is equivalent to $sinx = siny$. Thus $cosec\, x = cosec\, y$ and $\sin x = \sin y$ have the same general solution.
3) Since $tanx = tany \Leftrightarrow cotx = coty$. So, general solutions of $cotx = \cot y$ and $\tan x = \tan y$ are same.
4) Since $ecx = \sec y \Leftrightarrow \cos x = \cos y$. So, the general solutions of $\cos x = cosy$ and $secx = tany$ are the same.

Example: Find the general solutions of the following equations:

(i) $2\, sinx + 1 = 0$

Sol: We have $sinx = -\frac{1}{2}$

Again consider $sinx = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

So, the general solution is $x = n\pi + \ (-1)^n$, $n \in Z$

(ii) $\tan 2x = \sqrt{3}$

Sol: We have $\tan 2x = \sqrt{3}$

$\Rightarrow \tan 2x = \tan\frac{\pi}{3} \Rightarrow 2x = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \ x = \frac{n\pi}{2} + \frac{\pi}{6}, n \in Z$

(iii) $\cos\left(\frac{3x}{2}\right) = \frac{1}{2}$

Sol: We have $\cos\left(\frac{3x}{2}\right) = \frac{1}{2} \Rightarrow \cos\left(\frac{3x}{2}\right) = \cos\frac{\pi}{3}$

$\Rightarrow \frac{3x}{2} = 2n\pi \ \pm \frac{\pi}{3}, n \in Z \ \Rightarrow x = \frac{4n\pi}{3} \ \pm \frac{2\pi}{9}$, $n \in Z$

Example: Solve $sin2x \ - sin4x + sin6x = 0$

Sol: The given equation can be written as

$sin6x + sin2x \ - sin4x = 0$

$\Rightarrow\ 2\sin 4x\cos 2x\ - \sin 4x = 0$

$\Rightarrow\ \sin 4x\ (\ 2\cos 2x\ - 1) = 0$

So, $sin4x = 0$ or $2\cos 2x\ - 1 = 0$

$\Rightarrow\ \ \sin 4x = 0$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow\ \ \sin 4x = 0$ or $\cos 2x = \cos\frac{\pi}{3}$

$\Rightarrow\ \ \ 4x = n\pi$ or $2x = 2n\pi\ \pm\frac{\pi}{3},\ \ n \in Z$

$\Rightarrow x = \frac{n\pi}{4}$ or $x = n\pi\ \pm\frac{\pi}{6}\ ,\ n \in Z$

Example: Solve $2\ cos^2x + 3\ sinx = 0$

Sol: The given equation can be written as $2\ (1 - sin^2x) + 3\ sinx = 0$

$\Rightarrow\ 2\ sin^2x\ - 3\ sinx\ - 2 = 0$

$\Rightarrow\ \ (2\ sinx + 1)(sinx - 2) = 0$

$\Rightarrow\ \ sinx = -\frac{1}{2}$ or $sinx = 2$

But $sinx = 2$ is not possible.

So $sinx = -\frac{1}{2} \Rightarrow sinx = \sin\frac{7\pi}{6} \Rightarrow x = n\pi + \ (-1)^n\frac{7\pi}{6},\ n \in Z$

Example: Solve $sin3x + \cos 2x = 0$

Sol: The given equation can be written as $\cos 2x = -\sin 3x$

$\Rightarrow\ \ \ \cos 2x = \cos\left(\frac{\pi}{2} + 3x\right) \Rightarrow 2x = 2n\pi \pm \frac{\pi}{2} + 3x,\ \ n \in Z$

$\Rightarrow\ \ \ 2x - 3x = 2n\pi\ \pm\frac{\pi}{2}$

$\Rightarrow\ \ \ -x = 2n\pi\ \pm\frac{\pi}{2}$

$\Rightarrow\ \ \ x = -2n\pi\ \mp\frac{\pi}{2}\ ,\ n \in Z$

Example: Solve $2tanx\ - cotx = -1$

Sol: The given equation becomes $2\ tanx\ - \frac{1}{tanx} = -1$

$\Rightarrow \quad 2\,tan^2x - 1 = -tanx \Rightarrow 2\,tan^2x + tanx - 1 = 0$

$\Rightarrow \quad (tanx + 1)(\,2\,tanx - 1) = 0$

$\Rightarrow \quad tanx + 1 = 0$ or $2\,tanx - 1 = 0$

$\Rightarrow \quad tanx = -1$ or $tanx = \frac{1}{2}$

Now, $tanx = -1 \Rightarrow tanx = \tan\left(-\frac{\pi}{4}\right) \Rightarrow x = n\pi + \left(-\frac{\pi}{4}\right) = n\pi - \frac{\pi}{4}$, $n \in Z$

and $\tan x = \frac{1}{2} \Rightarrow tanx = \tan\alpha$, where $tan\alpha = \frac{1}{2}$

$\Rightarrow x = n\pi + \alpha$, where $tan\alpha = \frac{1}{2}$ and $n \in Z$

Problems on Trigonometric Equations

Example: Solve: $tanx + \tan 2x + \tan 3x = tanx\tan 2x\,tan3x$

Sol: We have $tanx + \tan 2x + \tan 3x = tanx\tan 2x\,tan3x$

$\Rightarrow \quad tanx + \tan 2x = -\tan 3x + tanx\tan 2x\,tan3x$

$\Rightarrow \quad tanx + \tan 2x = -\tan 3x\,(\,1 - tanx\tan 2x)$

$\Rightarrow \quad \frac{\tan x + \tan 2x}{1 - tanx\tan 2x} = -\tan 3x$

$\Rightarrow \quad \tan(x + 2x) = -\tan 3x$

$\Rightarrow \quad \tan 3x + \tan 3x = 0$

$\Rightarrow \quad 2\tan 3x = 0 \Rightarrow \tan 3x = 0 \Rightarrow 3x = n\pi,\ n \in Z \Rightarrow x = \frac{n\pi}{3},\ n \in Z$

Example: Solve: $2\,tan^2x + sec^2x = 2$ for $0 \le x \le 2\pi$

Sol: We have $2\,tan^2x + sec^2x = 2 \Rightarrow 2\,tan^2x + 1 + tan^2x = 2$

$\Rightarrow 3\,tan^2x = 1 \Rightarrow tanx = \pm\frac{1}{\sqrt{3}}$

If we take $tanx = \frac{1}{\sqrt{3}}$, then $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

Again, if we take $nx = -\frac{1}{\sqrt{3}}$, then $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

Therefore, the possible solutions of the given equation are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

General Solutions of Trigonometrical Equations of some particular form

$(i)\ sin^2x = sin^2y$

$\Rightarrow\ 2\ sin^2x = 2\ sin^2y \Rightarrow 1 - \cos 2x = 1 - \cos 2y$

$\Rightarrow\ \cos 2x = \cos 2y$

$\Rightarrow\ 2x = 2n\pi\ \pm 2y\ ,\ n \in Z$

$\Rightarrow\ x = n\pi \pm y\ ,\ n \in Z$

Similarly

$(ii)\ cos^2x = cos^2y \Rightarrow x = n\pi \pm y\ ,\ n \in Z$

and $(iii)\ tan^2x = \ tan^2\ y \Rightarrow\ x = n\pi \pm y\ ,\ n \in Z$

Example: Solve: $7\ cos^2x + 3sin^2x = 4$

Sol: We have $7\ cos^2x + 3sin^2x = 4$

$\Rightarrow\ 3(\ 1 -\ cos^2x) + 7\ cos^2x = 4$

$\Rightarrow\ 4\ cos^2x = 1$

$\Rightarrow\ cos^2x = \frac{1}{4} = \ cos^2\frac{\pi}{3}$

$\Rightarrow\ x = n\pi\ \pm\frac{\pi}{3}\ ,\ n \in Z$

Example: Solve: $81^{sin^2x} +\ 81^{cos^2x} = 30,\ 0 \leq x \leq \pi.$

Sol: We have $81^{sin^2x} +\ 81^{cos^2x} = 30$

$\Rightarrow\ 81^{sin^2x} +\ 81^{1-sin^2x} = 30$

$\Rightarrow\ 81^{sin^2x} +\ \frac{81}{81^{sin^2x}} = 30$

$\Rightarrow\ y + \frac{81}{y} = 30$, where $y =\ 81^{sin^2x}$

$\Rightarrow\ y^2 - 30y + 81 = 0$

$\Rightarrow\ (y - 27)(y - 3) = 0 \Rightarrow y = 27$ or $y = 3$

Now $y = 27 \Rightarrow 81^{sin^2x} = 27$

$\Rightarrow\ (3^4)^{sin^2x} = 3^3 \Rightarrow 4\,sin^2x = 3 \Rightarrow sin^2x = \frac{3}{4} = sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3}\,,\ n \in Z$

and $y = 3 \Rightarrow 81^{sin^2x} = 3$

$\Rightarrow (3^4)^{sin^2x} = 3^1 \Rightarrow 4\,sin^2x = 1 \Rightarrow sin^2x = \frac{1}{4} = sin^2\frac{\pi}{6} \Rightarrow x = n\pi \pm \frac{\pi}{6}\,,\ n \in Z$

Hence, $x = n\pi \pm \frac{\pi}{3}$ or $x = n\pi \pm \frac{\pi}{6}\,,\ n \in Z$

Trigonometric Equations of the form

$a\,cosx + b\,sinx = c$, where $a,\ b,\ c \in R$ such that $|c| \le \sqrt{a^2 + b^2}$.

Example: Solve: $\sqrt{3}\,cosx + sinx = \sqrt{2}$

Sol: Dividing both sides by $\sqrt{(\sqrt{3})^2 + 1^2} = 2,$ we obtain

$$\frac{\sqrt{3}}{2}\,cosx + \frac{1}{2}\,sinx = \frac{\sqrt{2}}{2}$$

$\Rightarrow cosx\ \cos\frac{\pi}{6} + sinx \sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}\ \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$\Rightarrow\ x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}\,,\ n \in Z \Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}\,,\ n \in Z$

$\Rightarrow\ x = 2n\pi + \frac{5\pi}{12}$ or, $x = 2n\pi - \frac{\pi}{12}\,,\ n \in Z$

Example: Solve: $\sqrt{2}\,secx + tanx = 1$

Sol: We have $\sqrt{2}\,secx + tanx = 1$

$\Rightarrow\ \frac{\sqrt{2}}{cosx} + \frac{sinx}{cosx} = 1 \Rightarrow \sqrt{2} + sinx = cosx$

$\Rightarrow\ cosx - sinx = \sqrt{2}$

Dividing throughout by $\sqrt{1^2 + (-1)^2} = \sqrt{2},$ we obtain

$\frac{1}{\sqrt{2}}\,cosx - \frac{1}{\sqrt{2}}\,sinx = 1 \Rightarrow cosx\cos\frac{\pi}{4} - sinx\sin\frac{\pi}{4} = 1$

$\Rightarrow\ \cos\left(x + \frac{\pi}{4}\right) = \cos 0 \Rightarrow x + \frac{\pi}{4} = 2n\pi \pm 0,\ n \in Z$

$\Rightarrow x = 2n\pi - \frac{\pi}{4},\ n \in Z$

Example: Solve: $4\ sinx \sin 2x\ sin4x = sin3x$

Sol: We have $4\ sinx \sin 2x\ sin4x = sin3x$

$\Rightarrow\ 4\ sinx \sin(3x - x)\sin(3x + x) = \sin 3x$

$\Rightarrow\ 4\ [\ sinx\ (\ sin^2 3x\ -\ sin^2 x)] = \sin 3x$

$\Rightarrow\ 4\ sinx\ sin^2 3x\ - 4\ sin^3 x = 3\ sinx\ - 4sin^3 x$

$\Rightarrow\ 4 \sin x\ sin^2 3x\ - 3\ sinx = 0 \Rightarrow \sin x\ (\ 4\ sin^2 3x\ - 3) = 0$

$\Rightarrow\ \sin x = 0\ or\ 4\ sin^2 3x\ - 3 = 0$

$\Rightarrow\ sinx = 0\ or\ sin^2 3x = \frac{3}{4}$

Now, $sinx = 0 \Rightarrow x = n\pi\ ,\ n \in Z$

and $sin^2 3x = \frac{3}{4} \Rightarrow sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow sin^2 3x = sin^2 \frac{\pi}{3}$

$\Rightarrow 3x = m\pi\ \pm \frac{\pi}{3} \Rightarrow x = \frac{m\pi}{3}\ \pm \frac{\pi}{9}\ ,\ m \in Z$

Example: Solve: $cot^2 x + \frac{3}{sinx} + 3 = 0.$

Sol: We have, $cot^2 x + \frac{3}{sinx} + 3 = 0$

$\Rightarrow\ cosec^2 x\ - 1 + 3\ cosecx + 3 = 0$

$\Rightarrow\ (cosecx + 2)(cosecx + 1) = 0$

$\Rightarrow\ cosecx + 2 = 0$ or $cosecx + 1 = 0$

Now, $cosecx + 2 = 0 \Rightarrow \frac{1}{sinx} + 2 = 0$

$\Rightarrow\ sinx = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) \Rightarrow x = n\pi + (-1)^n \left(-\frac{\pi}{6}\right) = n\pi + (-1)^{n+1}\frac{\pi}{6}\ ,\ n \in Z$

and $cosecx + 1 = 0\ \Rightarrow \frac{1}{sinx} + 1 = 0$

$\Rightarrow\ sinx = -1 = \sin\left(-\frac{\pi}{2}\right) \Rightarrow x = m\pi + (-1)^m \left(-\frac{\pi}{2}\right) = m\pi + (-1)^{m+1}\frac{\pi}{2}\ ,\ m \in Z$