Introduction:

A sentence or description which can be judged either true or false is called a statement or proposition.

For example: (i) 3 divides 9.

(ii) Jaipur is the capital of India.

A proposition involving mathematical relations is known as the mathematical proposition.

A theorem is a proposition that can be proved to be true. An argument which establishes the truth of a theorem is called a proof.

Proof of a mathematical statement involves a sequence of statements and each statement is established with a definition or an axiom, which is previously established by the method of deduction using logical rules.

Motivation:

Let us understand the basic principles of mathematical induction.

Suppose we have a ladder and the number of steps in the ladder is not known. We aim to reach every step on this ladder.

Can we reach it? We know two things:

(i) We can reach the first step of the ladder.

(ii) If we can reach a particular step, then we can reach the next step.

Can we conclude that we reach every step of the ladder?

By (i) we know that we can reach the first step of the ladder. Again by (ii), we can reach the second step, because we can reach the first step.

Again by (ii) we can reach the third step because we can reach the second step. Continuing in this way we can reach the 4^{th} step, 5^{th} step, 6^{th} step, and so on. If we reach the 29^{th} step, then we can reach the 30^{th} step. But can we conclude that we can reach every step of this ladder? This is possible if we can verify an important proof technique called **mathematical induction.**

We know that $N \subset R$. Also, N is the smallest subset of R with the following property:

" A set S is called inductive set if $1 \in S$ and $x + 1 \in S$ whenever $x \in S$." Hence, we take the natural numbers as the least inductive subset of a set of real numbers.

Principle of Mathematical Induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true $i.e.$ $P(n)$ is true for $n = 1$.

and (ii) $P(k+1)$ is true, whenever $P(k)$ is true.

$i.e.$ $P(k)$ is true $\Rightarrow P(k+1)$ is true.

Then $P(n)$ is true for all natural numbers n.

Example: If $P(n)$: "$49^n + 16^n + k$ is divisible by 64 for $\in N$ "is true, then find the least negative value of k.

Sol: Here the given statement is true for all $n \in N$, therefore it is true for $n = 1$ also.

So, we have $P(1)$: $49 + 16 + k$ is divisible by 64.

$i.e.$ $65 + k$ is divisible by 64.

Here k should be -1.

Example: If $P(n)$: " $3.\ 5^{2n+1} + 2^{3n+1}$ is divisible by λ for all $n \in N$" is true, then find the value of λ.

Sol: Here, the given statement is true for all $n \in N$.

So it is true for $n = 1$ and $n = 2$.

For $n = 1$, $P(1)$: $3.5^3 + 2^4 = 375 + 16 = 391$

and for $n = 2$, $P(2)$: $3.\ 5^5 + 2^7 = 9375 + 128 = 9503$

Since the HCF of 391 and 9503 is 17, so, $3.\ 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

Hence, the value of λ is 17.

Algorithm for Principle of Mathematical Induction:

The steps used to prove a statement is true for all natural numbers by using the principle of mathematical induction are given below.

Step I: Consider the given statement as $P(n)$.

Step II: Put $n = 1$ and then show that it is a true statement.

Step III: Suppose that the statement is true for $n = k$ $i.e.$ $P(k)$ is true.

Step IV: Show that the statement is true for $n = k + 1$.

Combining the results of steps II and IV, we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \in N$.

This algorithm is used to solve the various problems, which can be categorized into the following types

Identity Type Problems

In this type of problem, we use the principle of mathematical induction to show LHS is equal to RHS.

Example: Prove by the principle of mathematical induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$i.e.,\ 1 + 2 + 3 + \ \dots + n = \frac{n(n+1)}{2}$

Sol: Step I: Let $P(n)$ be the given statement, $i.e.$

$P(n): 1 + 2 + 3 + \ \dots + n = \frac{n(n+1)}{2}$

Step II: For $n = 1$, we have $LHS = 1$ and $RHS = \frac{1(1+1)}{2} = 1$

Since $LHS = RHS$, so $P(1)$ is true.

Step III: Let us assume that $P(n)$ is true for $n = k$. Then, we have

$P(k) : 1 + 2 + 3 + \ \dots + k = \frac{k(k+1)}{2}$

Step IV: Now, we shall prove the statement for $n = k + 1$. For this, we have to show that

$1 + 2 + 3 + \ \dots . + (k + 1) = \frac{(k+1)(k+1+1)}{2}$

Consider $LHS = 1 + 2 + 3 + \ \dots + (k + 1)$

$= 1 + 2 + 3 + \ \dots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$

$= (k + 1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+1+1)}{2} = RHS$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all natural numbers n.

Example: Prove by mathematical induction that

$$1^2 + 2^2 + 3^2 + \ ... + \ n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sol: Step I: Let $P(n)$ be the given statement $i.e.$

$$P(n):\ 1^2 + 2^2 + 3^2 + \ ... + \ n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step II: For $n = 1$, we have $LHS = \ 1^2 = 1$ and $RHS = \frac{1(1+1)(2\times1+1)}{6} = 1$

So, $LHS = RHS$. Thus $P(1)$ is true.

Step III: Let us assume that $P(n)$ is true for $n = k$. Then, we have

$$P(k):\ \ 1^2 + 2^2 + 3^2 + \ ... + \ k^2 = \frac{k(k+1)(2k+1)}{6}$$

Step IV: Now, we shall prove that $P(k+1)$ is true.

For this, we have to show that $1^2 + 2^2 + 3^2 + \ ... + \ (k+1)^2 = \frac{(k+1)(\ k+1+1)\{2(k+1)+1\}}{6}$

Consider $LHS = 1^2 + 2^2 + 3^2 + \ ... + \ (k+1)^2$

$$= 1^2 + 2^2 + 3^2 + \ ... + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \ (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(\ k+1+1)\{2(k+1)+1\}}{6}$$

Thus, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

Example: Using the principle of mathematical induction, prove that

$$1.3 + 2.3^2 + 3.\ 3^3 + \ \ .... \ + n\,.\,3^n = \frac{(2n-1)3^{n+1}+3}{4} \text{ for all } n \in N$$

Sol: Step I: Let $P(n)$ be the given statement. Then

$$P(n):\ 1.3 + 2.3^2 + 3.\ 3^3 + \ \ .... \ + n\,.\,3^n = \frac{(2n-1)3^{n+1}+3}{4}$$

Step II: For $n = 1$, we have $LHS = 1.3 = 3$ and $RHS = \frac{(2-1)3^{1+1}+3}{4} = \frac{9+3}{4} = 3$

Since $LHS = RHS$ so $P(1)$ is true.

Step III: Let us assume that $P(n)$ is true for $n = k$. Then we have

$P(k) : 1.3 + 2.3^2 + 3.\ 3^3 + \ \dots. \ + k\,.3^k = \frac{(2k-1)3^{k+1}+3}{4}$

Step III: Now we shall prove the statement for $n = k + 1$.For this, we have to show that

$P(k+1): 1.3 + 2.3^2 + 3.\ 3^3 + \ \dots. \ + (k+1).3^{k+1} = \frac{(2(k+1)-1)3^{(k+1)+1}+3}{4}$

Then $LHS = 1.3 + 2.3^2 + 3.\ 3^3 + \ \dots. \ + (k+1).3^{k+1}$

$= 1.3 + 2.3^2 + 3.\ 3^3 + \ \dots. \ + k.\ 3^k + (k+1).3^{k+1}$

$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1).3^{k+1} = \frac{(2k-1)3^{k+1}+3+4(k+1).3^{k+1}}{4}$

$= \frac{(2k-1+4k+4)3^{k+1}+3}{4} = \frac{(6k+3)3^{k+1}+3}{4}$

$= \frac{3(2k+1)3^{k+1}+3}{4} = \frac{(2k+1+1-1)3^{k+2}+3}{4} = \frac{(2(k+1)-1)3^{(k+1)+1}+3}{4} = RHS$

Thus $P(k+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

Example: Prove by mathematical induction that

$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in N$.

Sol: Step I: Let $P(n)$ be the given statement. Then

$P(n)$: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step II: For $n = 1$, we have $LHS = \frac{1}{1.2} = \frac{1}{2}$ and $RHS = \frac{1}{1+1} = \frac{1}{2}$

Since $LHS = RHS$ so $P(1)$ is true.

Step III: Let us assume that $P(n)$ is true for $n = k$

Then we have $P(k)$: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Step IV: Now we shall prove the statement for $n = k + 1$. For this, we have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$$

Now $LHS = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{(k+1)(k+1+1)}$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+1)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1} = RHS$$

Thus $P(k+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Using the principle of mathematical induction, prove that

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$ for all $n \in N$

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n){:}\ \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

Step II: For $n = 1$, we have $LHS = \left(1+\frac{3}{1}\right) = 4$ and $RHS = (1+1)^2 = 4$

Since $HS = RHS$, so $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(n)$ is true for $n = k.$ Then we have

$P(k){:}\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2k+1}{k^2}\right) = (k+1)^2$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show that $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) = (k+1+1)^2$

Then $LHS = \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$

$= \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$

$= (k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) = (k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$

$= (k+1)^2 + 2(k+1) + 1$

$= k^2 + 2k + 1 + 2k + 2 + 1 = k^2 + 4k + 4 = (k+2)^2 = (k+1+1)^2 = RHS$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Divisibility Type Problems:

In this type of problem, we use the principle of mathematical induction to show that the given statement $P(n)$ is divisible by a given number or $P(n)$ is a multiple of a given number.

Example: Using the principle of mathematical induction prove that $4^n + 15n - 1$ is divisible by 9 for all natural numbers n.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n)$: $4^n + 15n - 1$ is divisible by 9

Step II: For $n = 1$, we have $P(1)$: $4^1 + 15 \times 1 - 1 = 18 = 9 \times 2$, which is divisible by 9.

Thus $P(1)$ is true.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k)$: $4^k + 15k - 1$ is divisible by 9

Then $4^k + 15k - 1 = 9m$, for some $m \in N \Rightarrow 4^k = 9m - 15k + 1$

Step IV: Now, we shall prove that the statement for $n = k + 1$

For this, we have to show that $4^{k+1} + 15(k+1) - 1$ is divisible by 9.

Then $4^{k+1} + 15(k+1) - 1 = 4.\ 4^k + 15(k+1) - 1$

$= 4\,(9m - 15k + 1) + 15(k+1) - 1$

$= 36m - 60k + 4 + 15k + 15 - 1 = 36m - 45k + 18$

$= 9(4m - 5k + 2)$, which is divisible by 9.

$\therefore P(k + 1)$ is true.

Thus $P(k)$ is true $\Rightarrow P(k + 1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example: Using the principle of mathematical induction, prove that $7^n - 3^n$ is divisible by 4 for all $n \in N$

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n)$: $7^n - 3^n$ is divisible by 4.

Step II: For $n = 1$, we have $P(1)$: $7^1 - 3^1 = 4$, which is divisible by 4.

Thus $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(k)$ be true for some natural number k.

$i.e.\ P(k)$: $7^k - 3^k$ is divisible by 4.

We can write $7^k - 3^k = 4m$, where $m \in N \Rightarrow 7^k = 3^k + 4m$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show that $7^{k+1} - 3^{k+1}$ is divisible by 4.

Then, $7^{k+1} - 3^{k+1} = 7.7^k - 3.3^k = 7(3^k + 4m) - 3.\ 3^k$

$= 7.\ 3^k + 28m - 3.\ 3^k = 4.\ 3^k + 28m = 4(3^k + 7m)$, which is divisible by 4

Thus $P(k + 1)$ is true when $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement is true for every positive integer n.

Note: If x and y are any two distinct integers, then $x^n - y^n$ is divisible by $x - y$ for all $n \in N$.

Example: Using the principle of mathematical induction, prove that $2.\ 7^n + 3.\ 5^n - 5$ is divisible by 24, for all $n \in N$.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n)$: $2.\ 7^n + 3.\ 5^n - 5$ is divisible by 24.

Step II: For $n = 1$, we have

$P(1)$: $2.\ 7^1 + 3.\ 5^1 - 5 = 24$, which is divisible by 24.

Thus $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k)$: $2.\ 7^k + 3.\ 5^k - 5$ is divisible by 24.

Then $2.\ 7^k + 3.\ 5^k - 5 = 24\,m$, where $m \in N$

$\Rightarrow\ 2.\ 7^k = -3.\ 5^k + 5 + 24\,m$

Step IV: Now, we wish to prove that the statement is true for $n = k + 1$

For this, we have to show that $2.\ 7^{k+1} + 3.\ 5^{k+1} - 5$ is divisible by 24.

Then, $2.\ 7^{k+1} + 3.\ 5^{k+1} - 5 = 2.\ 7.\ 7^k + 3.\ 5.\ 5^k - 5$

$= 7(-3.\ 5^k + 5 + 24m) + 15.\ 5^k - 5$

$= -21.\ 5^k + 35 + 168m + 15\ .\ 5^k - 5$

$= \ -6.\ 5^k + 168\,m + 30$

$= 168m - 6\,(5^k - 5)$

$= 168m - 6\,(4p)$ [$5^k - 5$ is a multiple of 4. So, let $5^k - 5 = 4p$]

$= 24\,(7m - p)$, which is divisible by 24.

Thus $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Using the principle of mathematical induction, prove that $2^{3n} - 1$ is divisible by 7 for all $n \in N$.

Sol: Step I: Let $P(n)$ be the statement given by

$P(n)$: $2^{3n} - 1$ is divisible by 7.

Step II: For $n = 1$, we have $P(1)$: $2^{3\times1} - 1 = 8 - 1 = 7$, which is divisible by 7.

So, $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k)$: $2^{3k} - 1$ is divisible by 7.

$\Rightarrow\ 2^{3k} - 1 = 7m$, for some $m \in N$.

$\Rightarrow\ 2^{3k} = 7m + 1$

Step IV: Now, we shall prove the statement for $n = k + 1$. For this, we have to show that $2^{3(k+1)} - 1$ is divisible by 7.

Now, $2^{3(k+1)} - 1 = 2^{3k} \times 2^3 - 1 = 8(7m + 1) - 1 = 56m + 8 - 1$

$= 56m + 7 = 7(8m + 1)$, which is divisible by 7.

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Prove by the principle of mathematical induction that for all $n \geq 2$, $6^n - 5n$ when divided by 25 the remainder is always 1.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n)$: $6^n - 5n$ when divided by 25 leaves the remainder 1.

$i.e.\ P(n)$: $6^n - 5n = 25\,p + 1$ for some $p \in N$.

Step II: For $n = 2$, we have

$LHS = 6^2 - 5 \times 2 = 36 - 10 = 26 = 25 \times 1 + 1 = RHS$ for $p = 1$

Thus $P(2)$ is true.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k)$: $6^k - 5k = 25p + 1$

$\Rightarrow 6^k = 5k + 25\,p + 1$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show that

$6^{k+1} - 5(k + 1) = 25\,\text{q} + 1$ for some $q \in N$

Then, $6^{k+1} - 5(k + 1) = 6.\ 6^k - 5k - 5$

$= \ 6\,(\,5k + 25\,p + 1) - 5k \ - 1 = 30k + 150\,p + 6 \ - 5k - 5$

$= 25k + 150p + 1 = 25\,(k + 6p) + 1$

$= 25\,q + 1$, where $q = k + 6p \in N$.

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \ \geq 2$.

Inequality Type Problems:

In this type of problem, we have inequality symbols like $>$, $<$, $\geq$ or $\leq$. For such problems, we use the principle of mathematical induction to show that LHS and RHS satisfy the given inequality.

Example: Prove by the principle of mathematical induction that $2^n \ > n$ for all positive integers n.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e. P(n)\colon\ 2^n > n$

Step II: For $n = 1$, we have $2^1 > 1 \Rightarrow 2 > 1$, which is true.

Thus, $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k)\colon\ 2^k > k$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show $2^{k+1} > (k+1)$.

Since $P(k)$ is true, so $2^k \ > k$

Multiplying both sides by 2, we get $2.\ 2^k \ > 2k$

$i.e.\ 2^{k+1} > 2k = k + k > k + 1$

Therefore $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Prove that $(1+x)^n \ \geq 1 + nx$, for all-natural number n, where $x \succ 1$.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n):\ (1+x)^n \geq 1+nx$

Step II: For $n = 1$, we have $(1+x) \geq (1+x)$, which is true.

Thus $P(1)$ is true.

Step III: Let us assume that $P(k)$ is true.

$i.e. (1+x)^k \geq 1+kx \ldots (1)$

Step IV: Now, we shall prove the statement for $n = k+1$.

For this, we have to show that $(1+x)^{k+1} \geq 1+(k+1)x$

From eq. (1) we have $(1+x)^k \geq 1+kx \ \ldots (2)$

Since $1 > 0$, so $x+1 > 0$

Multiplying both sides of eq. (2) by $x+1$, we get $(1+x)(1+x)^k \geq (1+x)(1+kx)$

$\Rightarrow (1+x)^{k+1} \geq 1+x+kx+kx^2 \ldots (3)$

Here, k is a natural number and $x^2 \geq 0$, therefore $kx^2 \geq 0$ and so

$1+x+kx+kx^2 \geq 1+x+kx$

Then from Eq. (3) , we have $(1+x)^k \geq 1+x+kx$

$\Rightarrow \ (1+x)^k \geq 1+(k+1)x$

Therefore $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Prove that $1^2+2^2+\ \ldots+n^2 > \frac{n^3}{3}$, $n \in N$

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n):\ 1^2+2^2+\ \ldots+n^2 > \dfrac{n^3}{3}$

Step II: Since $1^2 = 1 > \frac{1}{3} = \frac{1^3}{3}$, so $P(n)$ is true for $n = 1$.

Step III: Let us assume that $P(k)$ is true.

$i.e.\ P(k): 1^2 + 2^2 + \ \dots + k^2 > \frac{k^3}{3} \dots (1)$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show that $1^2 + 2^2 + \ \dots + (k+1)^2 > \frac{(k+1)^3}{3}$

From eq. (1) , we get $1^2 + 2^2 + \ \dots + k^2 > \frac{k^3}{3}$

Adding $(k+1)^2$ on both sides,

$$1^2 + 2^2 + \ \dots + k^2 + (k+1)^2 > \frac{k^3}{3} + (k+1)^2$$

$$\Rightarrow \quad 1^2 + 2^2 + \ \dots + (k+1)^2 > \frac{1}{3}[k^3 + 3(k+1)^2] = \frac{1}{3}[k^3 + 3k^2 + 6k + 3]$$

$$\Rightarrow \quad 1^2 + 2^2 + \ \dots + (k+1)^2 > \frac{1}{3}[(k+1)^3 + 3k + 2] > \frac{1}{3}(k+1)^3$$

Therefore $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: Prove by mathematical induction that

$\frac{1}{n+1} + \frac{1}{n+2} + \ \dots + \frac{1}{2n} > \frac{13}{24}$ for all $n \in N,\ n > 1$.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n): \frac{1}{n+1} + \frac{1}{n+2} + \ \dots + \frac{1}{2n} > \frac{13}{24}$

Step II: For $n = 2$, we have $\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{13}{24}$, which is true.

Thus $P(2)$ is true.

Step III: Let us assume that $P(k)$ is true, where $k > 1$

$i.e.\ P(k): \frac{1}{k+1} + \frac{1}{k+2} + \ \dots + \frac{1}{2k} > \frac{13}{24} \dots (i)$

Step IV: Now, we shall prove the statement for $n = k + 1$.

For this, we have to show that $\frac{1}{k+1+1} + \frac{1}{k+1+2} + \ \dots + \frac{1}{2(k+1)} > \frac{13}{24}$

$i.e.\ \frac{1}{k+2} + \frac{1}{k+3} + \ \dots + \frac{1}{2k+2} > \frac{13}{24}$

From eq. (i), we have $\frac{1}{k+1}+\frac{1}{k+2}+\ \ldots+\frac{1}{2k}>\frac{13}{24}$

$\Rightarrow \frac{1}{k+2}+\ \ldots+\frac{1}{2k}>\frac{13}{24}\ -\frac{1}{k+1}$

On adding $\frac{1}{2k+1}+\frac{1}{2k+2}$ both sides, we get

$\frac{1}{k+2}+\ \ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}>\frac{13}{24}\ -\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$

$\Rightarrow\ \frac{1}{k+2}+\ \ldots+\frac{1}{2k+2}>\frac{13}{24}+\frac{1}{2k+1}+\frac{1}{2k+2}\ -\frac{1}{k+1}$

$=\frac{13}{24}+\frac{1}{2k+1}+\ \frac{1-2}{2(k+1)}=\frac{13}{24}+\frac{1}{2k+1}\ -\ \frac{1}{2(k+1)}$

$=\frac{13}{24}+\ \frac{2(k+1)-2k-1}{2(k+1)(2k+1)}=\frac{13}{24}+\ \frac{1}{2(k+1)(2k+1)}>\frac{13}{24}$

Therefore $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$, $n>1$.

An Extra Problem

Example: Prove the rule of exponents $(ab)^n = a^n b^n$ by using the principle of mathematical induction for every natural number.

Sol: Step I: Let $P(n)$ be the given statement.

$i.e.\ P(n)\colon\ (ab)^n = a^n b^n$

Step II: For $n=1$, $P(1)\colon\ (ab)^1 = ab = a^1 b^1$

So, $P(n)$ is true for $n=1$.

Step III: Let us assume that $P(k)$ is true, where $k \in N$.

$i.e.\ P(k)\colon (ab)^k = a^k b^k$ ... (1)

Step IV: Now, we shall prove the statement for $n=k+1$.

For this, we have to show that$(ab)^{k+1} = a^{k+1}b^{k+1}$

Then, $(ab)^{k+1} = (ab)^k\,(ab) = (a^k b^k)(ab) = (a^k\,.\,a)(b^k.\,b) = a^{k+1}b^{k+1}$

Therefore $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.