Chapter- 11

Conic Sections

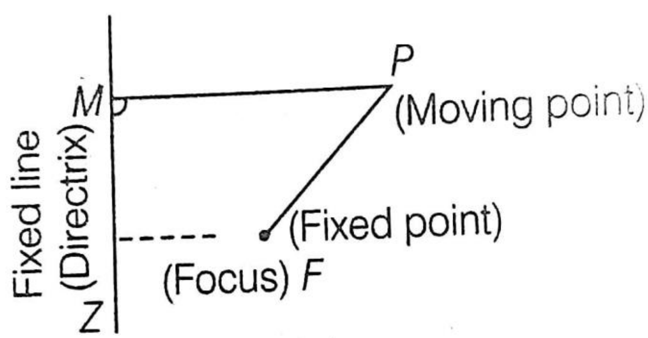
Introduction to Conic Sections

Circle, parabola, ellipse, and hyperbola are called conic sections. Conic Sections were studied extensively by ancient Greeks, who discovered properties that enable us to state their definitions in terms of a fixed point and a fixed line in the plane.

Definition

The locus of a second-degree equation in x and y is called a conic.

The conic section is the locus of a point that moves in a plane such that its distance from a fixed point always bears a constant ratio to its perpendicular distance from a fixed straight line in the plane.



Then

(i) the fixed point is called **focus** and is denoted by F.

(ii) the constant line ZM is called **the directrix.**

(iii) the constant ratio is called **eccentricity** and is denoted by e.

In the above figure, $\frac{PM}{PM} = e = constant$

If $e = 0$, then the conic is called a circle.

If $e = 1$, then the conic is called a parabola.

If $e < 1$, then conic is called an ellipse.

If $e > 1$, then conic is called a hyperbola.

(iv) The straight line passing through the focus and perpendicular to the directrix is called the **axis** of the conic.

(v) The point of intersection of the cone and its axis is called **the vertex** of the conic.

(vi) The point which bisects every chord of the conic passing through it is called the **cent**er of the conic.

The name conic section is derived from the fact that these are the curves from a cone by taking its cross-section in various ways.

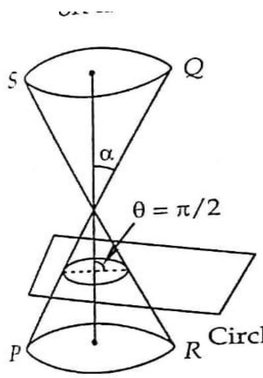


Fig. 25.2

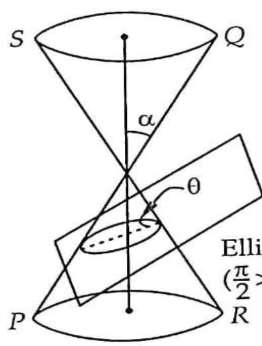


Fig. 25.3

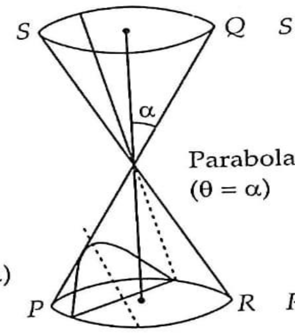


Fig. 25.4

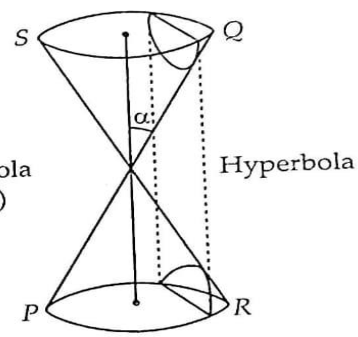


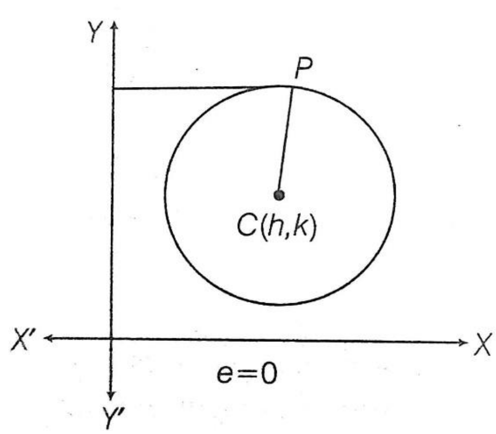
Fig. 25.5

Equation of a Circle with a given center and radius

A circle is the set of all points in a plane that are equidistant from a fixed point in that plane.

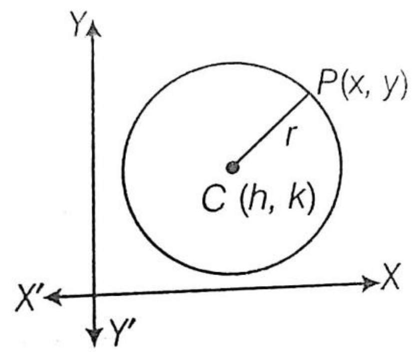
Or, The locus of a point that moves on the plane such that it remains at a constant distance from a fixed point of the plane, is called a circle.

The fixed point is called the center and the constant distance is called the radius of the circle.



Standard Equation of a Circle.

Let $C(h,\ k)$ be the center of the circle, $P(x,\ y)$ be any point on the circumference of the circle and r be the radius of the circle.



Then $|CP| = r$

$$\Rightarrow \sqrt{(x-h)^2 + (y-k)^2} = r$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2 \dots (1)$$

which is the required equation of the circle.

This equation is also known as a central form of the equation of a circle.

If the center is at origin $i.e.\ h = 0,\ k = 0$, then the equation of the circle is $x^2 + y^2 = r^2$.

Example: Find the equation of a circle with a center $(2,\ 3)$ and radius is 5.

Sol: The required equation of the circle is $(x-2)^2 + (y-3)^2 = 5^2 = 25$

Equation of Circle in Special Cases:

1) When the circle passes through the origin

If the circle passes through the origin $(0,\ 0)$, then radius $r = |OC| = \sqrt{h^2 + k^2}$

Hence the equation of the circle is $(x-h)^2 + (y-k)^2 = h^2 + k^2$

2) When the center lies on $x - axis$ or $y - axis$

If the center lies on $x - axis$, then $k = 0$

Then, the equation of the circle is $(x-h)^2 + y^2 = r^2$

If the center lies on the $y - axis$, then $h = o$

Then, the equation of the circle is $x^2 + (y-k)^2 = r^2$.

3) When the circle touches the $x - axis$

Since the circle touches $the\ x - axis$, so $r = |k|$. In this case, circles may lie on the upper of $the\ x - axis$ or the lower of the $x - axis$. Therefore, the equations of such circles are $(x - h)^2 + (y \mp r)^2 = r^2$

4) When the circle touches the $y - axis$.

Since, the circle touches $y - axis$, so $|h| = r$

In this case, circles may lie on the right of $the\ y - axis$ or o the left of the $y - axis$. Therefore, the equation of such circles are $(x \mp r)^2 + (y - k)^2 = r^2$

5) When circles touch both the coordinate axes

Since the circle touches both the axes, so $|h| = |k| = r$

In this case, circles may lie in any of the four quadrants.

Then, the equation of such circles are $(x \pm r)^2 + (y \pm r)^2 = r^2$

Example: Find the equation of a circle whose center is $(1, 2)$ and touches $the\ x - axis$.

Sol: Given, center $(h, k) = (1, 2)$ and circle touches $x - axis$, so radius $r = 2$

Thus, the equation of the circle is $(x - 1)^2 + (y - 2)^2 = 2^2$

$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 4$

$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0$

Example: Find the equation of the circle which touches both the axes and whose radius is 5.

Sol: Given, radius $= 5$ and circle touches both the axes.

So, the center of the circle is $(\pm 5, \pm 5)$

Hence, the required equation of the circle is $(x \pm 5)^2 + (y \pm 5)^2 = 5^2$

$\Rightarrow x^2 + 25 \pm 10x + y^2 + 25 \pm 10y = 25$

$\Rightarrow x^2 + y^2 \pm 10x \pm 10y + 25 = 0.$

Diameter Form of the Equation of the Circle

Let (x_1, y_1) and (x_2, y_2) be the endpoints of the diameter of a circle. Then, the equation of the circle drawn on the diameter is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

Example: Find the equation of the circle, whose endpoints diameter are $A(1,\ 5)$ and $B(-1,\ 3)$.

Sol: The required equation of the circle is

$(x-1)(x+1)+(y-5)(y-3)=0$

$\Rightarrow\ x^2-1+y^2-8y+15=0$

$\Rightarrow\ x^2+y^2-8y+14=0$

General Equation of a Circle

Equation (1) $(i.e. (x-h)^2+\ (y-k)^2=\ r^2\ )$ can be written as $x^2-2hx+h^2+y^2-2ky+k^2=r^2$

$\Rightarrow\ x^2+y^2-2hx-2ky+\ h^2+k^2-r^2=0$

which is of the form $x^2+y^2+2gx+2fy+c=0$

The above equation is called the general equation of a circle.

Conversely, consider an equation of form $x^2+y^2+2gx+2fy+c=0$ … (2)

$\Rightarrow\ x^2+y^2+2gx+2fy=-c$

$\Rightarrow\ x^2+y^2+2gx+2fy+c=0$

$\Rightarrow\ x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c$

$\Rightarrow\ \{x-(-g)\}^2+\{y-(-f)\}^2=\ \left(\sqrt{g^2+f^2-c}\right)^2$ … (3)

Comparing equations (1) and (3), we get eq. (2) always represents a circle with a cent at $(-g,\ -f)$ and radius $r=\ \sqrt{g^2+f^2-c}$

Notes:

- Eq. (2) represents a real circle if $g^2+f^2>c$
- If $g^2+f^2=c$, then $r=0$, then the circle is called a point circle.
- If $g^2+f^2<c$, then we get an imaginary circle.
- The general second-degree equation in x and y in which the coefficients of x^2 and y^2 are equal and there is no term containing the product XY always represents a circle.
- If $ax^2+ay^2+2gx+2fy+c=0$, represents a circle, then the center is at $\left(-\frac{g}{a},\ -\frac{f}{a}\right)$ and radius $=\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}\ -\frac{c}{a}}$

Example: Find the centre and radius of each of the following circles.

$(i)\ x^2 + (y+2)^2 = 9$

Sol: Given equation of the circle is $x^2 + (y+2)^2 = 9$

$\Rightarrow\ (x-0)^2 + \{y-(-2)\}^2 = 3^2$

Hence, the centre of the circle $= (0,\ -2)$ and radius $= r = 3$.

$(ii)\ x^2 + y^2 + 8x + 10y - 8 = 0$

Sol: The given equation can be written as $x^2 + 8x + y^2 + 10y = 8$

$\Rightarrow\ (x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$

$\Rightarrow\ (x+4)^2 + (y+5)^2 = 49$

$\Rightarrow\ \{x-(-4)\}^2 + \{y-(-5)\}^2 = 7^2$

Therefore, the given circle has centre at $(-4,\ -5)$ and radius 7.

$(iii)\ 3x^2 + 3y^2 + 5x - 6y - 2 = 0$

Sol: Dividing throughout 3, we get $x^2 + y^2 + \frac{5}{3}x - 2y - \frac{2}{3} = 0$.

Here $2g = \frac{5}{3},\ 2f = -2,\ c = -\frac{2}{3}$

$\Rightarrow\ g = \frac{5}{6},\ f = -1,\ c = -\frac{2}{3}$

Then, the c is at $\left(-\frac{5}{6},\ 1\right)$ and radius $r = \sqrt{\left(\frac{5}{6}\right)^2 + (-1)^2 - \left(-\frac{2}{3}\right)} = \frac{\sqrt{85}}{6}$

Example: Find the equation of the circle which passes through the points $(3,\ 7)$ and $(5,\ 5)$ and has its center lies on the line $x - 4y = 1$.

Sol: Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

Its centre is at $(-g,\ -f)$. As the centre lies on the line $x - 4y = 1$, so $-g + 4f = 1$ ... (2)

Again, since the circle (1) passes through the points $(3,\ 7)$ and $(5,\ 5)$, so

$9 + 49 + 6g + 14f + c = 0$

and $25 + 25 + 10g + 10f + c = 0$

$\Rightarrow\ 6g + 14f + c = --58$ ... (3) and $10g + 10f + c = -50$... (4)

Subtracting (3) from (4), we get $4g - 4f = 8$

$\Rightarrow\ g - f = 2$ ... (5)

Adding (2) and (5), we get $3f = 3 \Rightarrow f = 1$

Then from eq. (5), $g - 1 = 2 \Rightarrow g = 3$

From eq. (3), $18 + 14 + c = -58 \Rightarrow c = -90$

Therefore, the required equation of the circle is $x^2 + y^2 + 6x + 2y - 90 = 0$

Position of a point with respect to a Circle

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle and $P(\alpha,\ \beta)$ be a given point. Then the centre is at $(-g,\ -f)$ and radius $=$.

The point P will lie inside or on or outside the given circle according to as

$$|CP| < \ = \ > r$$

$$\Rightarrow \sqrt{(\alpha + g)^2 + (\beta + f)^2} < \ = \ > \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \alpha^2 + 2g\alpha + g^2 + \beta^2 + 2f\beta + f^2 < \ = \ > g^2 + f^2 - c$$

$$\Rightarrow \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c < \ = \ > 0.$$

Example: Does the point $(-3,\ 2)$ lies inside or outside or on the circle $x^2 + y^2 = 9$?

Sol: The equation of the circle is $x^2 + y^2 - 9 = 0$

Putting $x = -3$ and $y = 2$ in $x^2 + y^2 - 9$, we get $(-3)^2 + (2)^2 - 9 = 4 > 0$.

So, the given point lies outside the circle.

Example: Find the equation of the circle whose centre $(1,\ 2)$, which passes through the point $(4,\ 6)$.

Sol: Here radius $r = \sqrt{(4-1)^2 + (6-2)^2} = 5$

Therefore, the equation of the circle is $(x-1)^2 + (y-2)^2 = 5^2$

$$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 25$$

$$\Rightarrow x^2 + y^2 - 2x - 4y - 20 = 0$$

Example: Find the equation of the circle of radius 5 whose centre lies on the $x - axis$ and passes through the point $(2,\ 3)$.

Sol: Let the coordinates of the centre of the required circle be $C(h,\ 0)$.

So, the equation of circle with radius 5 is $(x-h)^2 + y^2 = 25$

Since, the circle passes through the point $(2,\ 3)$, so, $(2-h)^2 + 3^2 = 25$

$$\Rightarrow (2-h)^2 = 16$$

$$\Rightarrow (2-h) = \pm 4$$

$$\Rightarrow h = 2 \pm 4 = 6,\ -2$$

Therefore, the required equations of the circles are

$(x-6)^2 + y^2 = 25$ and $(x+2)^2 + y^2 = 25$

$\Rightarrow\ x^2 - 12x + 36 + y^2 = 25$ and $x^2 + 4x + 4 + y^2 = 25$

$\Rightarrow\ x^2 + y^2 - 12x + 11 = 0$ and $x^2 + y^2 + 4x - 21 = 0$

Example: Find the equation of the circle having centre at $(3,\ -4)$ and touching the line $5x + 12y - 12 = 0$.

Sol: Given that the centre is at $(3,\ -4)$. Since the line $5x + 12y - 12 = 0$ is a tangent to the circle, so the length of the perpendicular drawn from the centre to the line is the radius of the circle.

So, radius $r = \left|\frac{5\times3+12(-4)-12}{\sqrt{5^2+(12)^2}}\right| = \frac{45}{13}$

Therefore, the equation of the circle is

$(x-3)^2 + (y+4)^2 = \left(\frac{45}{13}\right)^2 = \frac{2025}{169}$

Equation of a Circle passing through three non–collinear points

Example: Find the equation of the circle which passes through points $(1,\ 2),\ (3,\ -4)$ and $(5,\ -6)$.

Sol: Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ … (1)

Since, it passes through points $(1,\ 2),\ (3,\ -4)$ and $(5,\ -6)$, so we have

$1 + 4 + 2g + 4f + c = 0$

$9 + 16 + 6g - 8f + c = 0$

$25 + 36 + 10g - 12f + c = 0$

$\Rightarrow\ 2g + 4f + c = -5$ … (2)

$6g - 8f + c = -25$ … (3)

$10g - 12f + c = -61$ … (4)

Subtracting (2) from (3), we get $4g - 12f = -20$

$\Rightarrow -g + 3f = 5$ … (5)

Subtracting (4) from (3) , we get $-4g + 4f = 36$

$\Rightarrow\ g - f = -9$ … (6)

Adding (5) and (6), we get $2f = -4 \Rightarrow f = -2$

From (6), $g + 2 = -9 \Rightarrow g = -11$

Again from (2), we get $-22 - 8 + c = -5$

$\Rightarrow \ c = 25$

Therefore, the required equation of the circle is

$x^2 + y^2 + 2(-11)x + 2(-2)y + 25 = 0$

$\Rightarrow x^2 + y^2 - 22x - 4y + 25 = 0$

Concentric Circles

Two circles having the same center $C(h, \ k)$ but different radii r_1 and r_2 are called concentric circles.

Thus the circles $(x-h)^2 + (y-k)^2 = \ r_1^2$ and $(x-h)^2 + (y-k)^2 = \ r_2^2, \ r_1 \neq r_2$ are concentric circles.

Example: Find the equation of the circle which passes through the center of the circle $x^2 + y^2 + 8x + 10y - 7 = 0$ and is concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$.

Sol: Given, circles are $x^2 + y^2 + 8x + 10y - 7 = 0 \ldots (i)$ and

$2x^2 + 2y^2 - 8x - 12y - 9 = 0 \ldots (ii)$

The Centre of circle (i) is $C_1(-4, \ -5)$

Equation of any circle concentric with circle (ii) is $2x^2 + 2y^2 - 8x - 12y + c = 0 \ldots (iii)$

If this circle passes through $(-4, \ -5)$, then $2(-4)^2 + 2(-5)^2 - 8(-4) - 12(-5) + c = 0$

$\Rightarrow \ c = -174$

Hence, the required equation of the circle is $2x^2 + 2y^2 - 8x - 12y - 174 = 0$

$\Rightarrow x^2 + y^2 - 4x - 6y - 87 = 0$

Example: Find the equation of the circle concentric with the circle $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ and having its area equal to 16π sq. units.

Sol: The given equation of the circle is $x^2 + y^2 + 4x + 5y - \frac{39}{2} = 0$.

The coordinates of the centre are $\left(-2, \ -\frac{5}{2}\right)$.

Since the required circle is concentric with the given circle so, the coordinates of its centre are $\left(-2, \ -\frac{5}{2}\right)$.

Let r be the radius of the required circle.

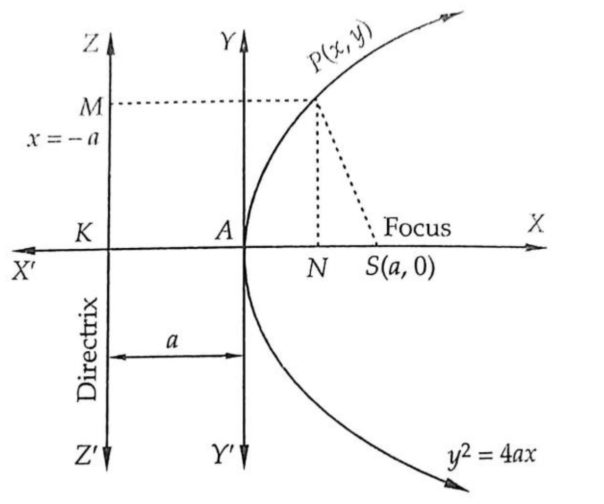
Since the area is $16\,\pi\ sq.\ units$, so $\pi r^2 = 16\pi \Rightarrow r = 4$

Therefore, the required equation of the circle is $(x+2)^2 + \left(y+\frac{5}{2}\right)^2 = 4^2$

$\Rightarrow 4x^2 + 4y^2 + 16x + 20y - 23 = 0$

Equation of a Parabola in different Forms

Consider a parabola whose vertex is at origin and focus along the x-axis at $F(a, 0)$. If $P(x, y)$ be any point on the parabola, then $|\text{PF}| = |\text{PM}|$



$\Rightarrow \sqrt{(x-a)^2 + y^2} = x + a$

$\Rightarrow \ (x-a)^2 + y^2 = (x+a)^2$

$\Rightarrow \ y^2 = (x+a)^2 - (x-a)^2$

$\Rightarrow \ y^2 = 4ax$, which is the equation of the parabola.

Properties:

1. The parabola $y^2 = 4ax$ opens to right.
2. The equation of the directrix is $x = -a \; i.e \; x + a = 0$.
3. The equation of the axis is $y = 0$.
4. The parabola $y^2 = 4ax$ is symmetrical about the x-axis. Since x cannot be negative, so no portion of the curve lies in the 2nd and 3rd quadrant.

5. The distance of any point on the parabola from the focus is called the focal distance. For the parabola $y^2 = 4ax$, the focal distance is $x + a$.

6. The line segment joining any two points of the parabola is called a chord. If a chord is passing through the focus, it is called a focal chord.

   If a focal chord is perpendicular to the axis, it is called the latus rectum.

   The endpoints of the latus rectum are at $L(a, 2a), R(a. -2a)$. The length of the latus rectum is 4a.

Corollary:

1. Let the vertex of the parabola is at $(0,0)$. If the focus of is at$(-a, 0), i.e.$

   the parabola opens to the left, then the equation of the parabola is $y^2 = 4ax$.

   Then the equation of the directrix is $x = a\, i.e\; x - a = 0$.

   The equation of the axis is $y = 0$.

   The endpoints of the latus rectum are $L(-a, 2a), R(-a, -2a)$.

2. Let the vertex of the parabola is at the origin.If the focus is $F(0, a), i.e$

   the parabola opens upwards, then the equation of the parabola is $x^2 = 4ay$.

   The equation of the axis is $x = 0$.

   The equation of the directrix is $y = -a$.

   The end points of the latus rectum is $L(-2a, a), R(2a, a)$.

3. Let the vertex of the parabola is at the origin. If the focus is at $F(0. -a), i.e.$

   the parabola opens downwards, then the equation of the parabola is $x^2 = -4ay$.

   The equation of the axis is $x = 0$.

   The equation of the directrix is $y = a, i.e\; y - a = 0$.

   The endpoints of the latus rectum are at $L(-2a, -a), R(2a, -a)$.

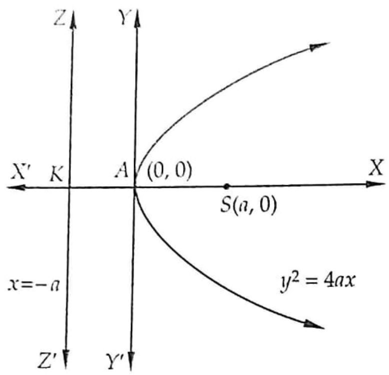


Fig. 25.12

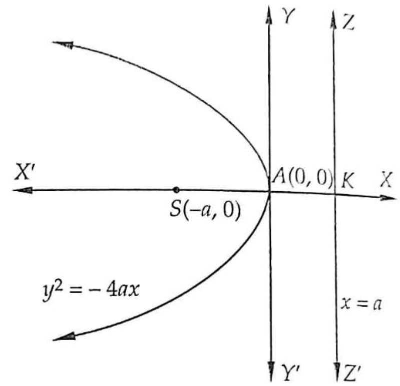
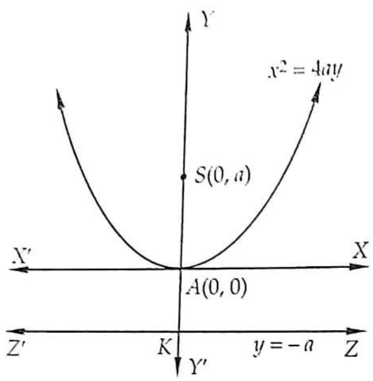
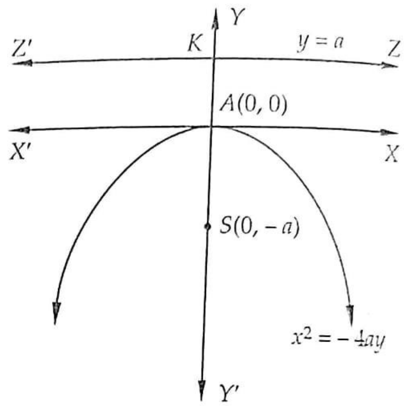


Fig. 25.13





General Equation of a Parabola

1. If the vertex is at (h, k) and axis is parallel to the x-axis, then the equation of the parabola is

   $(y - k)^2 = \pm 4a(x - h)$.

2. If the vertex is at (h, k) and axis is parallel to the y-axis, then the equation of the parabola is

   $(x - h)^2 = \pm 4a(y - k)$.

   The general equation of a parabola can be written either of the following forms:

   $x^2 + ax + by + c = 0$ (Axis parallel to y-axis) or, $y^2 + dy + ex + f = 0$ (Axis parallel to x-axis)

Thus, the general equation of a parabola is either quadratic in x and linear in y or quadratic in y and linear in x.

Parametric Representation:

The parametric representation of the equation of the parabola: $y^2 = 4ax$ are $x = at^2, y = 2at$, where t is a parameter.

Formation of the Equation of a Parabola from given conditions

Example: Find the vertex, focus, equation of the axis, equation of the directrix, and length of the latus rectum, endpoints of the latus rectum of the parabola $y^2 = -8x$.

Solution: The given parabola is $y^2 = -8x$

$$= -4 \cdot 2x$$

Here, $a = 2$

Then, the vertex is $(0,0)$. The focus is at $(-a, 0), i.e.\ (-2,0)$

End points of latus rectum are at $(-a, 2a)$ and $(-a, -2a)$ $i.e.\ (-2,\ 4)$ and $(-2, -4)$

Equation of axis is $y = 0$. Equation of the directrix is $x - a = 0$, $i.e.\ x - 2 = 0$

The length of latus rectum is $4a = 8$.

Example: Find the equation of parabola which is symmetric about the x-axis and passing through points $(2, -3)$.

Solution: Since the parabola is symmetric about the y-axis and vertex is at the origin. So the equation is of the form $x^2 = 4ay$. Since, this parabola passes through point $(2, -3)$.

So, we have, $4 = 4a(-3)$

$$\Rightarrow a = -\frac{1}{3}$$

$\therefore$ The required equation of the parabola is $x^2 = 4\left(-\frac{1}{3}\right)y$

$$\Rightarrow 3x^2 + 4y = 0.$$

Example: A double ordinate of the parabola $y^2 = 4ax$ is of length $8a$. Prove that the lines from the vertex to its ends are at right angles.

Solution: Let AB is a double ordinate of length $8a$ of the parabola $y^2 = 4ax$.

So, $AC = 4a$, $CB = 4a$. Let $OC = x_1$.

Then, coordinates of A are $(x_1, 4a)$. Since, A lies on the parabola,

So, $(4a)^2 = 4a(x_1)$

$\Rightarrow 16a^2 = 4a(x_1)$

$\Rightarrow \ x_1 = 4a$

Thus, the coordinates of A are $(4a, 4a)$ and B are $(4a, -4a)$.

Now, slope of $OA = \frac{4a-0}{4a-0} = 1$

and slope of $OB = -\frac{4a-0}{4a-0} = -1.$

Since the product of slopes of OA and OB is -1, OA and OB are perpendicular to each other.

Example: Find the equation of the parabola with focus $(2,\ 0)$ and directrix $x = -2$.

Solution: Since the focus $(2,\ 0)$ lies on the $x - axis$, the $x - axis$ itself is the axis of the parabola.

Hence the equation of the parabola is of the form either $y^2 = 4ax$ or $y^2 = -4ax$.

Since the directrix is $x = -2$ and the focus is $(2,\ 0)$, the parabola is to be of the form $y^2 = 4ax$ with $a = -2$.

Hence the required equation is $y^2 = 4(2)x = 8x$.

Example: Find the coordinates of the focus, axis of the parabola, the equation of the directrix, and length of the latus rectum of the parabola $x^2 = -9y$.

Solution: Given, equation of a parabola is $x^2 = -9y$, which is of the form $x^2 = -4ay$

$i.e.,$ Focus lies in the negative direction of $the\ y - axis$.

So, focus $= (0,\ -a) = \left(0,\ -\frac{9}{4}\right)$

Axis $= y - axis$ and its equation are $x = 0$.

Directrix, $y = a \Rightarrow y = \frac{9}{4} \Rightarrow 4y - 9 = 0$

Length of latus rectum $= 4a = 9$

Example: Find the equation of a parabola when the vertex is at $(0,\ 0)$ and focus is at $(0,\ -4)$.

Sol: Here, the vertex is at $(0,\ 0)$ and focus is at $(0,\ -4)$, which lies on $the\ y - axis$.

So, the $y - axis$ is the axis of the parabola.

Thus, the equation of the parabola is of the form $x^2 = -4ay$

$\Rightarrow\ x^2 = -4(4)y$ [since $a = 4$]

$\Rightarrow\ x^2 = -16y$

Example: Find the coordinates of a point on the parabola $y^2 = 8x$, whose focal distance is 4.

Sol: Given the equation of a parabola is $y^2 = 8x$.

Here, $= 2$. So, focus $= (2,\ 0)$

Now, focal distance $= 4$

We know that focal distance is a distance of any point $P(x,\ y)$ on the parabola from the focus F.

So, PF = focal distance

$\Rightarrow\ PF = \sqrt{(x-a)^2 + y^2}$

$\Rightarrow\ 4 = \sqrt{(x-2)^2 + 8x}$

$\Rightarrow\ 16 = x^2 - 4x + 4 + 8x$

$\Rightarrow\ (x+2)^2 = 16$

$\Rightarrow\ x + 2 = 4$

$\Rightarrow\ x = 2$ Then, $y^2 = 8 \times 2 = 16 \Rightarrow y = \pm 4$

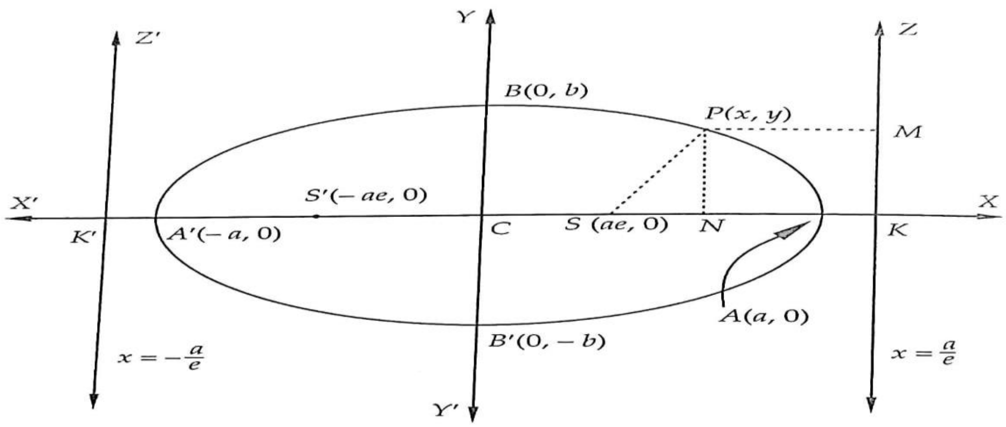
Hence coordinates of the points are $(2,\ 4)$ and $(2,\ -4)$

Equation of an Ellipse in different forms

The locus of a point which moves on the plane in such a way that the sum of its distances from two fixed points remains constant is called an ellipse.

Each of the fixed points is called a focus.

Equation of an Ellipse



Consider an ellipse whose foci are along the x-axis and the mid-point of the line segment joining the foci is at the origin. If $|FF'| = 2c$,then the foci are at $F(c, 0)$ and $F'(-c, 0)$.

If $P(x, y)$ be any point on the ellipse, then $|PF'|+|PF|= constant$

$$= 2a \ \ (say)$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow (x+c)^2+y^2=4a^2+(x-c)^2+y^2-4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow (x+c)^2-(x-c)^2-4a^2=-4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow \ 4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow \ cx-a^2=-a\sqrt{(x-c)^2+y^2} \qquad ....(1)$$

$$\Rightarrow \ c^2x^2+a^4-2a^2cx=a^2\{(x-c)^2+y^2\}$$

$$\Rightarrow \ c^2x^2+a^4-2a^2cx=a^2x^2+a^2c^2-2a^2cx+a^2y^2$$

$$\Rightarrow \ a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$\Rightarrow \ (a^2+c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

$$\Rightarrow \ b^2x^2+a^2y^2=a^2b^2$$

$\Rightarrow \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the required equation of the ellipse. .... (2)

Properties

1. The ellipse(2) meets the x-axis at $A'(-a,0)$ and $A(a,0)$. These two points are called vertices.

   The line segment $A'A$ is called the major x-axis and its length is 2a.

   The equation of the major axis is y=0.

2. The ellipse(2) meets the y-axis at $B(0,b)$ and $B'(0,-b)$.

   The line segment BB' is called the minor axis and its length is 2b.

   The equation of the minor axis is x=0.

3. The length of the major axis is greater than the minor axis and foci lie on the major axis.

4. The point of intersection of the major and minor axis is called the center. Here the center is at $(0,0)$.

5. The chords through foci and perpendicular to the major axis are called latera recta.

6. The end-points of the latera recta are at $L\left(c,\frac{b^2}{a}\right), R(c,-\frac{b^2}{a})$

$$L'\left(-c\,,\ \frac{b^2}{a}\right), R'(-c,-\frac{b^2}{a}).$$

   The length of each latus rectum is $\frac{2b^2}{a}$.

   The equation of the latera recta is $x=\pm c$.

7. From equation (1), we get

$$cx-a^2=-a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow \sqrt{(x-c)^2+y^2}=a-\frac{cx}{a}$$

$$=\frac{c}{a}\left(\frac{a^2}{c}-x\right)$$

$\Rightarrow \frac{\sqrt{(x-c)^2+y^2}}{\left(\frac{a^2}{c}-x\right)}=\frac{c}{a}$, the ratio of the distance of $P(x,y)$ from the focus $(c,0)$ and the distance of $P(x,y)$ from the line $x=\frac{a^2}{c}$ is a directrix w.r.t. the focus$(c,0)$.Similarly, $x=-\frac{a^2}{c}$ s a directrix w.r.t. the focus $(-c,0)$. Thus the equation of the directrices are $x=\pm\frac{a^2}{c}$.

8. Eccentricity, $e=\frac{c}{a}<1$.

9. If the foci are along the y-axis at $F(0,c),\ F'(0,-c)$, then the equation of the ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

The vertices are at $A(0,a), A'(0,-a)$.

The endpoints of the minor axis are at $B(b,0),\ B'(-b,0)$.

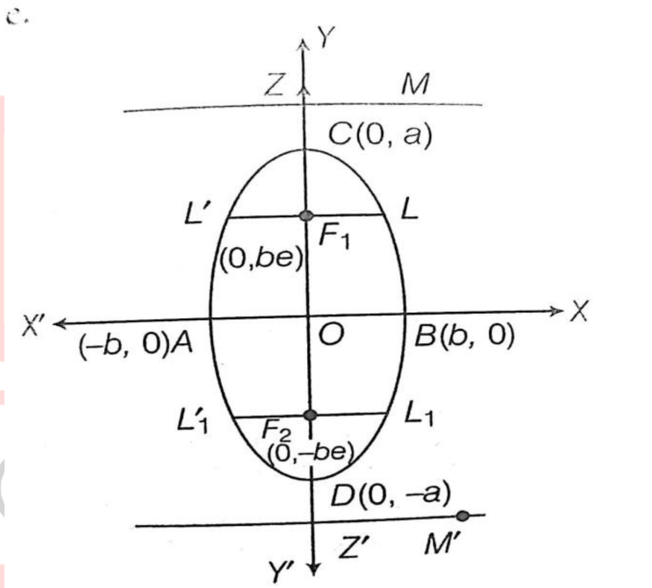
The endpoints of the latera recta are at $\left(\pm\frac{b^2}{a},\pm c\right)$.

The equation of the major axis is $x=0$. The equation of the minor axis is $y=0$.

The equations of the latera recta are $y=\pm c$.

The equations of the directrices are $y=\pm\frac{a^2}{c}$.

The center is at $(0,0)$.



10. If $a=b$, then the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1, i.e.\ x^2+y^2=a^2$, which represent a circle and known as auxiliary circle of the ellipse.

General Equation of Ellipse:

1. If the center is at (h,k) and the major axis is parallel to the x-axis, the equation of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

2. If the center is at (h,k) and the major axis is parallel to the y-axis, the equation of the ellipse is

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1.$$

Parametric Representation:

The parametric forms of the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $x=a\cos\theta$, $y=b\sin\theta$ where θ is a parameter and $0\leq\theta<2\pi$.

Formation of the Equation of an Ellipse from given conditions

Example: Find the coordinates of the centre, foci, ends of the major and minor axis, endpoints of latera recta, equation of major axis, minor axis, directrices, length of each latus rectum, and eccentricities of the ellipse $9x^2+4y^2=36$.

Sol. The given of ellipse can be written as $\frac{x^2}{4}+\frac{y^2}{9}=1$

Here, $a^2=9,\ b^2=4$

$\Rightarrow\ a=3\ ,\ b=2$

We have, $b^2=a^2-c^2$

$\Rightarrow\ c^2=a^2-b^2=9-4$

$\Rightarrow c^2=5$

$\Rightarrow\ c=\sqrt{5}$

The major axis of the given ellipse is on the y-axis.

The center is at $(0,0)$.

The foci are $(0,c)$ and $(0,-c)$ $i.e.$ $\left(0,\sqrt{5}\right)$ and $(0,\ -\sqrt{5})$.

The end-points of the major axis are $(0,a)$ and $(0,-a)$ $i.e.,$ $(0,\ 3)$ and $(0,-3)$.

The endpoints of the minor axis are $(b,0)$ and $(-b,0)$ $i.e.,$ $(2,0)$ and $(-2,0)$.

The endpoints of the latera recta are $\left(\pm\frac{b^2}{a},\pm c\right)$ $i.e.,\left(\pm\frac{4}{3},\pm\sqrt{5}\right)$.

The equation of the major axis is $x=0$. The equation of minor axis $y=0$.

The equation of the directrices is $y=\pm\frac{a^2}{c}=\pm\frac{9}{\sqrt{5}}$.

Length of latus rectum is $\frac{2b^2}{a}=2\times\frac{4}{3}=\frac{8}{3}$.

Eccentricity, $e=\frac{c}{a}=\frac{\sqrt{5}}{3}$.

Example: Find the equation of ellipse whose vertices are $(\pm 13,0)$ and foci are at $(\pm 5,0)$.

Sol. Since, the vertices are on the x-axis, so the equation of ellipse is of the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Here, $a = 13\,, c = 5$. We have $b^2 = a^2 - c^2$

$$= 169 - 25 = 144$$

Therefore, the equation of ellipse is $\frac{x^2}{169}+\frac{y^2}{144}=1$.

Example: Find the equation of an ellipse with a major axis along the x-axis and passing through the points $(4,3)$ and $(-1,4)$.

Sol. Since the major axis is along the x-axis, so the equation of the ellipse is of the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ....(1)$$

Since the ellipse(1) passing through the points $(4,3)$ and $(-1,4)$, so

$$\frac{16}{a^2}+\frac{9}{b^2}=1 \quad ....(2) \text{ and } \quad \frac{1}{a^2}+\frac{16}{b^2}=1 \quad ....(3)$$

Eq.$(2) \times 1 \;\rightarrow\; \frac{16}{a^2}+\frac{9}{b^2}=1$

Eq.$(3) \times 16 \rightarrow \frac{1}{a^2}+\frac{256}{b^2}=1$

On Subtracting: $-\frac{247}{b^2}=-15$

$\Rightarrow\; b^2=\frac{247}{15}$

Eq.$(2) \times 16 \;\rightarrow\; \frac{256}{a^2}+\frac{144}{b^2}=16$

Eq.$(3) \times 9 \;\rightarrow\; \frac{9}{a^2}+\frac{144}{b^2}=9$

On subtracting: $\frac{247}{a^2}=7$

$\Rightarrow a^2=\frac{247}{7}$

Thus, the required equation of ellipse is: $\frac{x^2}{247/7}+\frac{y^2}{247/15}=1$

$\Rightarrow\ \frac{7x^2}{247}+\frac{15y^2}{247}=1$

$\Rightarrow\ 7x^2+15y^2=247$

Example: If the eccentricity of the ellipse is $\frac{5}{8}$ and the distance between the foci is 10, then find the latus rectum of the ellipse.

Sol. Given that, $e=\frac{5}{8}$ and $2c=10$

$\Rightarrow \frac{c}{a}=\frac{5}{8}$ and $c=5$

$\Rightarrow a=8$

We have, $b^2=a^2-c^2$

$=64-25=39$

Hence, the length of the latus rectum is $\frac{2b^2}{a}=\frac{2\times 39}{8}=\frac{39}{4}$.

Example: If the length of the latus rectum of the ellipse is equal to half of the minor axis, find its eccentricity.

Sol. Atq, latus rectum $=\frac{1}{2}(minor\ axis)$

$\Rightarrow\ \frac{2b^2}{a}=\frac{1}{2}(2b)$

$\Rightarrow\ \frac{2b}{a}=1$

$\Rightarrow\ a=2b$

Now, eccentricity, $e=\frac{c}{a}$

$\Rightarrow e^2=\frac{c^2}{a^2}=\frac{a^2-b^2}{a^2}$

$=\frac{4b^2-b^2}{4b^2}=\frac{3b^2}{4b^2}=\frac{3}{4}$

$\Rightarrow e=\frac{\sqrt{3}}{2}$

Example: Find the equation of the ellipse whose axes are along the coordinate axes, vertices are $(0,\pm 10)$ and eccentricity, $e=\frac{4}{5}$.

Sol. Since the vertices are on the y-axis, so the equation of the ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

Here, $a=10$ and $\frac{c}{a}=\frac{4}{5}$

$\Rightarrow c = 8$

We have, $b^2 = a^2 - c^2$

$= 100 - 36 = 36$

Therefore, the required equation of the ellipse is $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

Example: Find the equation of the ellipse, whose length of the major axis is 20 and foci are $(0, \ \pm 5)$.

Sol: Since the foci are on the $y - axis$, the major axis is along the $y - axis$.

So, the equation of the ellipse is of the form of $\frac{x^2}{b^2} + \ \frac{y^2}{a^2} = 1$.

Given that $a =$ semi –major axis $= \frac{20}{2} = 10$ and $c = 5$

We have $b^2 = \ a^2 - c^2 = 100 - 25 = 75$.

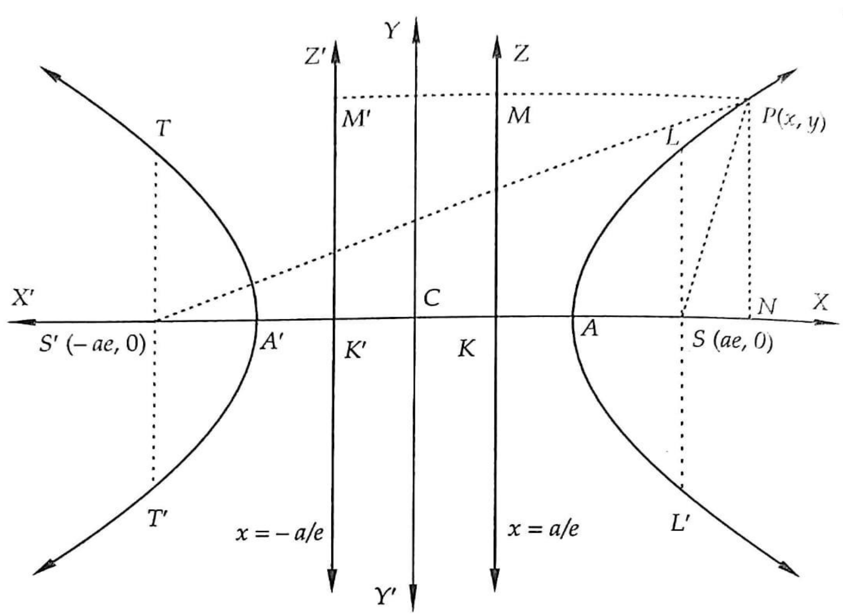
Therefore, the equation of the ellipse is $\frac{x^2}{75} + \frac{y^2}{100} = 1$.

Equation of a Hyperbola in different forms

The locus of a point which moves on the plane in such a way that the difference of its distance from two fixed points remains constant is called a hyperbola.

Each of the fixed points is called a focus.

Equation of a Hyperbola:



Let the foci of the hyperbola are along the x-axis at $F(c, 0)$, and $F'(-c, 0)$.

If $P(x, y)$ be any point on the hyperbola, then $|PF'|-|PF|= constant$

$\Rightarrow\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$

After simplification, we get $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ....(1) $/\because b^2=c^2-a^2$

Properties:

1. The hyperbola equation (1) meets the x-axis at $A(a, 0)$, and $A'(-a, 0)$. These two points are called vertices. The line segment $A-A'$ is called the transverse axis and its length is 2a.

The equation of the transverse axis is $y=0$.

2. The hyperbola (1) meets the y-axis at imaginary points $B(0, b)$ and $B'(0, -\mathrm{b})$. The line segment BB' is called the conjugate axis and its length is $2b$.

The equation of the conjugate axis is $x=0$.

3. There is no restriction for the relative values of a and b. The foci always lie on the transverse axis.
4. The point of intersection of the transverse and conjugate axis is called the center and here center is $O(0,0)$.
5. The chords LR and $L'R'$ are called latera recta. The end-points

of the latera recta are at $\left(\pm c, \pm\frac{b^2}{a}\right)$.

The length of each latus rectum is $\frac{2b^2}{a}$. The equations of the latera recta are $x=\pm c$.

7. Eccentricity, $e=\frac{c}{a}>1$.
8. If the foci are on the y-axis at $(0, c)$, and $(0, -c)$, then the equation of the hyperbola is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

The vertices are at $A(0, a)$, $A'(0, -a)$.

The endpoints of the conjugate axis are at $\mathrm{B}'(-b, 0)$ and $B(b, 0)$.

The end-points of the latera recta are at $\left(\pm\frac{b^2}{a}, \pm c\right)$.

The equation of the transverse axis is $x=0$ and the equation of the conjugate axis is y

The equation of latera recta is $y=\pm c$.

The equation of directrices is $y=\pm\frac{a^2}{c}$.

Rectangular Hyperbola:

If $a = b$, then the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, becomes $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, $i.e.\ x^2 - y^2 = a^2$,which is the equation of a rectangular or an **equilateral** hyperbola.

The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

Conjugate Hyperbolas:

The hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$, $i.e.$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are conjugate of each other. If e_1 and e_2 are the eccentricities of the conjugate hyperbolas, then $\frac{1}{e_1} + \frac{1}{e_2} = 1$.

General Equation of a Hyperbola

If the center is at (h, k) and the transverse axis is parallel to the x-axis, the equation of the hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

2. If the center is at (h, k) and the transverse axis is parallel to the y-axis, then the equation of the hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

Parametric Representation:

The parametric representation form equations of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are:

$x = a \sec\theta$ and $y = b \tan\theta$, or

$x = a \cosh\theta$ and $y = b \sinh\theta$,

where θ is a parameter and $0 \le \theta < 2\pi$.

Formation of the Equation of a Hyperbola from given conditions

Example: Find the center, foci, endpoints of the transverse and conjugate axis, endpoints of latera recta, equations of the transverse and conjugate axis, equations of directrixes, length of transverse axis, conjugate axis, latus rectum and eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

Sol. The given equation of hyperbola can be written as:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Here, transverse axis is along x-axis. Also, $a^2 = 16$, $b^2 = 9$

$$\Rightarrow a = 4\ ,\ \ b = 3$$

We have, $c^2 = a^2 + b^2 = 16 + 9 = 25$

$\Rightarrow c = 5$

The centre is at $(0,0)$.

Foci are at $(\pm c, 0)$ $i.e.$ $(\pm 5,0)$.

Endpoints of the transverse axis are at $(\pm a, 0)$ $i.e.$ $(\pm 4,0)$

Endpoints of the conjugate axis are at $(0, \pm b)$ $i.e.$ $(0, \pm 9)$

Endpoints of latera recta are at $\left(\pm c, \pm \frac{b^2}{a}\right)$ $i.e.$ $\left(\pm 5, \pm \frac{9}{4}\right)$.

The equation of the transverse axis is $y = 0$.

The equation of the conjugate axis is $x = 0$.

Equation of directrices are $x = \pm \frac{a^2}{c} \Rightarrow x = \pm \frac{16}{5}$.

Length of transverse axis $= 2a = 8$.

Length of conjugate axis $= 2b = 6$.

Length of latus rectum is $\frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$.

Eccentricity, $e = \frac{c}{a} = \frac{5}{4}$

Example: Find the equation of the hyperbola with foci$(0, \pm 3)$ and vertices $\left(0, \pm \frac{\sqrt{11}}{2}\right)$.

Sol.: Since foci are on the y-axis, so the equation of the hyperbola is of the form:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Here, $c = 3$, $a = \frac{\sqrt{11}}{2}$

We have, $b^2 = c^2 - a^2$

$= 9 - \frac{11}{4} = \frac{25}{4}$

Therefore, the required equation of the hyperbola is $\frac{y^2}{11/4} - \frac{x^2}{25/4} = 1$ $i.e.$ $\frac{4y^2}{11} - \frac{4x^2}{25} = 1$.

Example: Find the equation of hyperbola here foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Sol. Since the foci are on the y-axis, so the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Here, $c = 12$ and $\frac{2b^2}{a} = 36 \Rightarrow b^2 = 18a$

We have, $b^2 = c^2 - a^2$

$\Rightarrow 18a = 144 - a^2$

$\Rightarrow \ a^2 + 18a - 144 = 0$

$\Rightarrow \ a = \frac{-18 \pm \sqrt{324+576}}{2} = \frac{-18 \pm 30}{2} = -9 \pm 15$

$\Rightarrow \ a = 6, -24$

Since, a cannot be negative, so $a = 6$.

Thus, $b^2 = 18 \times 6 = 108$

Thus, the required equation of the hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$.

Example: Find the equation of hyperbola whose length of the latus rectum is 8 and eccentricity is $\frac{3}{\sqrt{5}}$.

Sol: Let the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Atq, $\frac{2b^2}{a} = 8$, $\frac{c}{a} = \frac{3}{\sqrt{5}}$

$\Rightarrow b^2 = 4a , c = \frac{3a}{\sqrt{5}}$

We have, $b^2 = c^2 - a^2$

$\Rightarrow \ 4a = \frac{9a^2}{5} - a^2$

$\Rightarrow \ 4a = \frac{4a^2}{5}$

$\Rightarrow \ a = 5$

Then, $b^2 = 4(5) = 20$

Thus, the required equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1$.

Example: Find the equation of the hyperbola with vertices at $(0, \ \pm 7)$ and $e = \frac{4}{3}$.

Sol: Since the vertices of the required hyperbola lie on $the\ y - axis$, so, let its equation be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

The coordinates of the vertices of this hyperbola are $(0, \pm b)$ and the coordinates of vertices are given as $(0, \pm 7)$. So, $b = 7$

Now, $a^2 = b^2(e^2 - 1) \Rightarrow a^2 = 49\left(\frac{16}{9} - 1\right) = \frac{343}{9}$

Therefore, the required equation of the hyperbola is $\frac{9x^2}{343} - \frac{y^2}{49} = -1$

Example: Find the equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13.

Sol: Let $2a$ and $2b$ be the transverse and conjugate axes and e be the eccentricity.

Let the center be the origin and the transverse and the conjugate axes the coordinate axes.

Then, the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have, $2b = 5$ and $2ae = 13$ $i.e.,$ $2c = 13$

$\Rightarrow b = \frac{5}{2}$ and $ae = \frac{13}{2}$

Now, $b^2 = a^2(e^2 - 1) = a^2e^2 - a^2$

$\Rightarrow \frac{25}{4} = \frac{169}{4} - a^2$

$\Rightarrow a^2 = \frac{144}{4}$

$\Rightarrow a = 6$

Therefore, the required equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$

$\Rightarrow 25x^2 - 144y^2 = 900$

Example: Find the eccentricity of the hyperbola whose length of the latus rectum is 8 and the conjugate axis is equal to half of its distance between the foci.

Sol: We have, the conjugate axis is half of the distance between foci.

$\therefore 2b = \frac{1}{2}\,2c$

$\Rightarrow 2b = c$

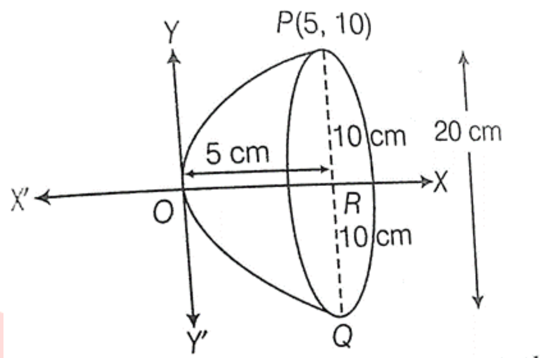
$\Rightarrow 4b^2 = c^2 = a^2 + b^2 \Rightarrow \frac{b^2}{a^2} = \frac{1}{3}$

Now, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$

Word Problems related to Conic

Example: If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Sol:



Let POQ be the parabolic reflector which is 20 cm in diameter and 5 cm deep.

Then $PQ = 20\ cm$ and $R = 5\ cm$, where R Is the midpoint of PQ.

We take OX as $x - axis$ and OY as the $y - axis$.

The equation of the parabola may be taken as $y^2 = 4ax$.

Since, the point $(5,\ 10)$ lies on the parabola

So, $10^2 = 4a\ (5) = 20\text{a}$

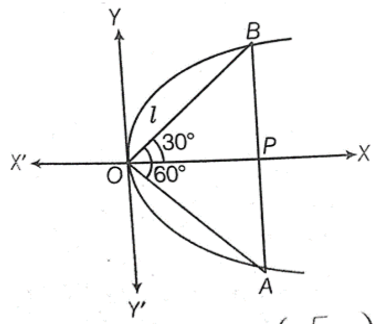
$\Rightarrow\ a = 5$

Therefore, the coordinates of the focus are $(a,\ 0) i.e.,\ (5,\ 0)$

Hence, the focus is the mid-point of the given diameter.

Example: An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Sol:



First, we draw the parabola on the positive side of the $x - axis$ and inside that draw an equilateral triangle $\Delta\, AOB$.

Let $OB = l = OA = AB$

$\Rightarrow \ \angle BOA = 60° \Rightarrow \angle BOP = 30°$

In $\Delta\, BOP,\ sin30° = \frac{PB}{OB} \Rightarrow \frac{1}{2} = \frac{PB}{l}$

$\Rightarrow PB = \frac{l}{2}$

Also, $\cos 30° = \frac{OP}{OB} \Rightarrow \frac{\sqrt{3}}{2} = \frac{OP}{l}$

$\Rightarrow \ OP = \frac{\sqrt{3}l}{2}$

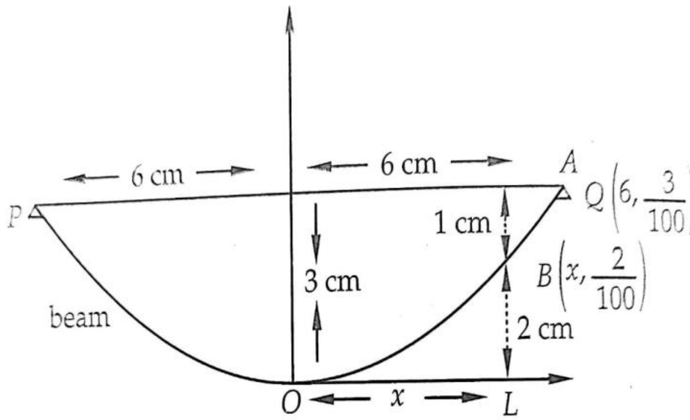
SO, coordinates of $B = (OP,\ PB) = \left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$ will satisfy $y^2 = 4ax$

$i.e.,\ \left(\frac{l}{2}\right)^2 = \frac{4a\sqrt{3}\,l}{2} \Rightarrow l = 8\sqrt{3}\,a$

Hence, the length of the side of the triangle is $8\sqrt{3}\,a$.

Example: A beam is supported at its ends by supports which are 12 meters apart. Since the load connected at its center, there is a deflection of 3 cm at the center and the deflected beam is in the shape of a parabola. How far from the center is the deflection 1 cm?

Sol:



Let O be the center of the beam in a deflected position. Taking O as the Origin, OX as $x - axis$, and OY as $the\ y - axis$. The equation representing The parabolic shape of the beam is $x^2 = 4ay$.

This passes through $Q\left(6, \frac{3}{100}\right)$.

$\therefore 36 = 4a \times \frac{3}{100} \Rightarrow a = 300\ m$

So, the equation of the curve representing the deflecting beam is $x^2 = 1200y$.

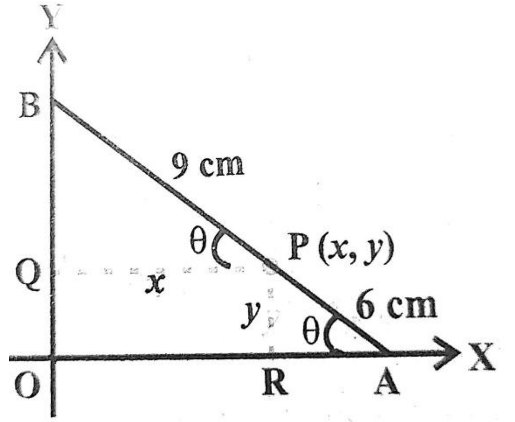
Let the deflection of the beam be $1\ cm = \frac{1}{100} m$ at the point B. Then, the coordinates of B are $\left(x, \frac{2}{100}\right)$, where $OL = x$. Since B lies on the parabola $x^2 = 1200y$

so, $x^2 = 1200 \times \frac{2}{100} = 24 \Rightarrow x = 2\sqrt{6}\ metres$.

Hence, the deflection of the beam is 1 cm at a distance of $2\sqrt{6}\ meters$ from the center O.

Example: A rod AB of length 15 cm rests in between two coordinate axes in such a way that the endpoint A lies on the $x - axis$ and the endpoint B lies on the $y - axis$. A point $P(x,\ y)$ is taken on the rod in such a way that $AP = 6cm$. Show that the locus of P is an ellipse.

Sol:



Let AB be the rod making an angle θ with OX and $P(x, y)$ the point on it such that $AP = 6cm$

Since $AB = 15\ cm$, we have $PB = 9cm$

From P draw PQ and PR perpendicular on $y - axis$ and $x - axis$ respectively.

From $\Delta\ PBQ$, $cos\theta = \frac{x}{9}$

From $\Delta\ PRA$, $sin\theta = \frac{y}{6}$

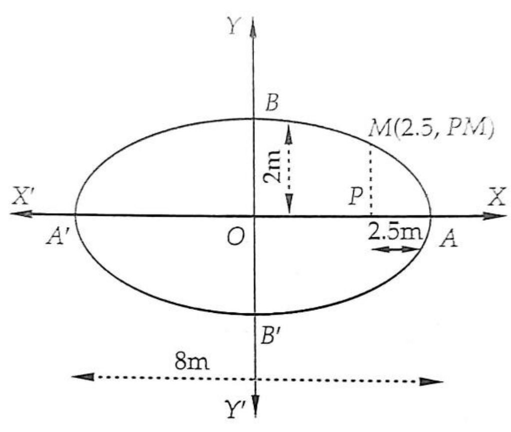
Since $sin^2\theta + cos^2\theta = 1$,

So $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

$\Rightarrow \frac{x^2}{81} + \frac{y^2}{36} = 1$. Thus the locus of P is an ellipse.

Example: An arch is in the form of a semi – ellipse. It is 8m wide and 2 m high at the center. Find the height of the arc at a point $1.5m$ from one end.

Sol:



Let ABA' be the given arc such that $AA' = 8m$ and $OB = 2m$.

Let the arc be a part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then $2a = 8 \Rightarrow a = 4$ and $b = 2$

So, the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$ .... (1)

We have, to find the height of the arc at P such that $AP = 1.5\ m$

In other words, we have to find the $y -$ coordinate at P.

Since, $OA = 4m$ and $AP = 1.5\ m$, so $Op = 2.5\ m$

Thus, the coordinates of M are $\left(\frac{5}{2}, PM\right)$

Since M lies on the ellipse (1), so $\frac{25/4}{16} + \frac{PM^2}{4} = 1$

$\Rightarrow\ PM^2 = \frac{39}{16}$

$\Rightarrow\ PM = \frac{\sqrt{39}}{4}\ m$

Hence, the height of the arc at a point 1.5 m from one end is $\frac{\sqrt{39}}{4}\ m$.