Chapter- 13

LIMIT AND DERIVATIVES

Fundamental of Limits:-

Definition of Limit:- If $f(x)$ approaches to a real number ℓ, when x approaches a (through lesser or greater values to a) i.e if $f(x) \to \ell$, when $x \to a$, then ℓ is called the limit of the function $f(x)$. In symbolic form, it can be written as $\lim_{x\to a} f(x) = \ell$.

Algebra of Limits:-

Sometimes two or more functions involving algebraic operations such as addition, subtraction, multiplication, and division are given, and then to find the limit of these functions involving algebraic operations, we use the following theorem.

Let f and g be two real functions with common domain D, such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists. Then

(a) The limit of the sum of two functions is the sum of the limits of the functions.

i.e $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) Limit of the difference between two functions is the difference between the limits of the function

i.e $\lim_{x\to a}(f-g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(c) Limit of the product of a constant and one function is the product of that constant and limit of a function,

i.e $\lim_{x\to a}[c.f(x)] = c\lim_{x\to a} f(x)$, where c is a constant.

(d) Limit of the product of two functions is the product of the limits of the function, i.e

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

(e) Limit of the quotient of two functions is the quotient of the limits of the functions, i.e

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, where \lim_{x\to a} g(x) \neq 0$$

Limits of polynomial function:-

A function f is said to be a polynomial function if $f(x)$ is a zero function or if

$f(x) = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n$, where a_i's are real numbers and $a_n \neq 0$.

Method to find the limit of a polynomial:-

To find the limit of a given polynomial, we use the algebra of limits and then put the limit and simplify. It can be understood in the following way.

We know that, $\lim\limits_{x\to a} x = a$. Then $\lim\limits_{x\to a} x^2 = \lim\limits_{x\to a}(x.x) = \lim\limits_{x\to a} x.\lim\limits_{x\to a} x = a.a = a^2$

Similarly, $\lim\limits_{x\to a} x^n = a^n$

Example:-1

Evaluate the limits $\lim\limits_{x\to 3}(4x^3 - 2x^2 - x + 1)$

Solution:-

$$\lim_{x\to 3}(4x^3 - 2x^2 - x + 1)$$

$$= 4\lim_{x\to 3} x^3 - 2\lim_{x\to 3} x^2 - \lim_{x\to 3} x + \lim_{x\to 3} 1$$

$$= 4(3)^3 - 2(3)^2 - 3 + 1 = 108 - 18 - 2 = 88$$

Limits of Rational Functions:-

A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomial functions such that $h(x) \neq 0$.

Then, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \frac{g(x)}{h(x)}$

$$= \frac{\lim\limits_{x\to a} g(x)}{\lim\limits_{x\to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h\,(a) = 0$, then there are two cases arise,

(i) $g(a) \neq 0$ (ii) $g(a) = 0$

In the first case, we say that the limit does not exist. In the second case, we can find a limit. The limit of a rational function can be found with the help of the following methods.

Direct Substitution Method:-

In this method, we substitute the point, to which the variable tends to in the given limit. If it gives us a real number, then the number so obtained is the limit of the function and if it does not give us a real number, then use other methods.

Example:-

Find the limits of the following

(i) $\lim\limits_{x\to 2} \frac{x^2-4}{x+3}$ (ii) $\lim\limits_{x\to -1} \frac{(x-1)^2+3x^2}{(x^4+1)^2}$

Solution:-

(i) $\lim\limits_{x\to 2} \frac{x^2-4}{x+3} = \frac{4-4}{2+3} = \frac{0}{5} = 0$

(ii) $\lim_{x\to -1} \frac{(x-1)^2+3x^2}{(x^4+1)^2} = \frac{(-1-1)^2+3(-1)^2}{((-1)^4+1)^2} = \frac{(-2)^2+3(1)}{(1+1)^2} = \frac{4+3}{2^2} = \frac{7}{4}$

Factorization Method:-

Let $\lim_{x\to a} \frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$ when we substitute x = a. Then we factorize $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

Method to determine the limit by using the factorization method:-

Step – I write the given limit as $\lim_{x\to a} \frac{f(x)}{g(x)}$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then go to the next step, otherwise use the direct substitution method.

Step – II Factorise f(x) and g(x), such that $(x-a)$ is a common factor, and write the given limit as

$$\lim_{x\to a} \frac{(x-a)f_1(x)}{(x-a)g_1(x)}$$

Step – III Cancel the common factor(s) then the limit obtained in step III becomes $\lim_{x\to a} \frac{f_1(x)}{g_1(x)}$

Step – IV Use a direct substitution method to obtain a limit.

Example:- 1

Evaluate $\lim_{x\to \frac{1}{2}} \frac{4x^2-1}{2x-1}$

Solution:-

On putting $x = \frac{1}{2}$, we get the form $\frac{0}{0}$. So, let us first factorize it

Consider, $\lim_{x\to \frac{1}{2}} \frac{4x^2-1}{2x-1} = \lim_{x\to \frac{1}{2}} \frac{(2x+1)(2x-1)}{(2x-1)}$

$$= \lim_{x\to \frac{1}{2}} (2x+1) = 2\left(\frac{1}{2}\right) + 1 = 2$$

Example:- 2

Evaluate $\lim_{x\to 2} \left[\frac{x^2-4}{x^3-4x^2+4x}\right]$

Solution:-

On putting $x = 2$, we get the form $\frac{0}{0}$ so, let us first factorize it.

Consider, $\lim_{x\to 2} \frac{x^2-4}{x^3-4x^2+4x} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$

$= \lim_{x\to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$ which is not defined.

$\therefore \lim\limits_{x\to 2}\left[\frac{x^2-4}{x^3-4x^2+4x}\right]$ does not exist.

Rationalization Method:-

If we get $\frac{0}{0}$ form and numerator or denominator or both have a radical sign, then we rationalize the numerators or denominator or both by multiplying them to remove $\frac{0}{0}$ the form and then find the limit by direct substitution method.

Example:- 1

Evaluate $\lim\limits_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

Solution:-

When x = 0, then the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ becomes of the form $\frac{0}{0}$. So we will be rationalizing the numerator by multiplying and dividing its conjugate i.e $\sqrt{2+x}+\sqrt{2}$

$$\therefore \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\left(\sqrt{2+x}-\sqrt{2}\right)\left(\sqrt{2+x}+\sqrt{2}\right)}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$

$$=\lim_{x\to 0}\frac{2+x-2}{x\left(\sqrt{2+x}+\sqrt{2}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

By using Some standard Limits:-

If the given limit is of the form $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}$, then we can find the limit directly by using the following theorem.

Theorem:- Let n be any positive integer. Then $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$.

Proof:- We have known that $x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+\ldots+a^{n-2}x+a^{n-1})$

By dividing both sides $(x-a)$, we get $\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+\ldots\ldots+a^{n-2}x+a^{n-1}$

Thus, $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=\lim\limits_{x\to a}(x^{n-1}+ax^{n-2}+\ldots..+a^{n-2}x+a^{n-1})$

$$=a^{n-1}+a(a^{n-2})+\ldots+a^{n-2}a+a^{n-1}$$

$$=a^{n-1}+a^{n-1}+\ldots..+a^{n-1}+a^{n-1}$$

$$=na^{n-1}$$

Example:-1

Evaluate $\lim\limits_{x\to 2}\frac{x^{10}-1024}{x-2}$

Solve:- When x = 2, the expression $\frac{x^{10}-1024}{x-2}$ becomes of the form $\frac{0}{0}$

Now, $\lim\limits_{x\to 2}\frac{x^{10}-1024}{x-2}=\lim\limits_{x\to 2}\frac{x^{10}-2^{10}}{x-2}=10\times 2^{10-1}=5120$

Example:-2

Evaluate $\lim\limits_{x\to 2}\frac{x-2}{\sqrt[3]{x}-\sqrt[3]{2}}$

Solution:-

$$\lim_{x\to 2}\frac{x-2}{x^{1/2}-2^{1/3}}=\frac{1}{\lim\limits_{x\to 2}\frac{x^{1/3}-2^{1/3}}{x-2}}=\frac{1}{\frac{1}{3}(2^{1/3-1})}=\frac{1}{\frac{1}{3}\times(2^{-2/3})}=3(2^{2/3})$$

Example:- 3

Evaluate $\lim\limits_{x\to 2}\frac{x^5-32}{x^3-8}$

Solution:-

$$\lim_{x\to 2}\frac{x^5-32}{x^3-8}=\lim_{x\to 2}\frac{x^5-2^5}{x^3-2^3}=\lim_{x\to 2}\frac{\frac{x^5-2^5}{x-2}}{\frac{x^3-2^3}{x-2}}=\lim_{x\to k2}\frac{x^5-2^5}{x-2}\div\lim_{x\to 2}\frac{x^3-2^3}{x-2}$$

$$=5\times 2^{5-1}\div 3\times 2^{3-1}$$

$$=5\times 2^4\div 3\times 2^2=\frac{5\times 2^4}{3\times 2^2}=\frac{5}{3}\times 2^2=\frac{20}{3}$$

Example:-4

Evaluate $\lim\limits_{x\to 2}\frac{(2x+4)^{1/3}-2}{x-2}$

Solve:-

Put $2x+4=y, then y\to 8 as x\to 2$

$$\therefore \lim_{x\to 2}\frac{(2x+4)^{1/3}-2}{x-2}=\lim_{y\to 8}\frac{y^{1/3}-2}{\frac{y-4}{2}-2}$$

$$=2\lim_{x\to 8}\frac{y^{1/3}-(8)^{1/3}}{y-4-4}=2\lim_{x\to 8}\frac{y^{1/3}-8^{1/3}}{y-8}$$

$$=2.\frac{1}{3}(8)^{\frac{1}{3}-1}$$

$$=2.\frac{1}{3}(2^3)^{-\frac{2}{3}}=\frac{2}{3}.(2)^{-2}=\frac{2}{3}\times\frac{1}{4}=\frac{1}{6}$$

Limits of Trigonometric Functions:-

Three important limits are

(i) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

(ii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=0$

(iii) $\lim\limits_{x\to 0}\frac{\tan x}{x}=1$

Where x is measured in radian

Method to determine the limit of trigonometric Function:-

Step – I First, check that the given variable tends to zero or not. If yes, then go to step II, otherwise put $x = a + h$ in the given function such that as $x \to a, then h \to 0$

Step – II Put the limit in a given function, if $\frac{0}{0}$ the form is obtained, then we go to the next step. Otherwise, we get the required answer.

Step-III Simplify the numerator and denominator to eliminate those factors which become 0 on putting the limit.

Step – IV Now, convert the result obtained in step III, into the form of $\frac{\sin\theta}{\theta}$ or $\frac{\tan\theta}{\theta}$.

Step – V Substitute the value of the standard limit of a trigonometric function as obtained in step IV and simplify it.

Example:- 1

Evaluate $\lim_{\theta \to 0} \theta \,cos\,e\,c\theta$

Solution:-

$$\lim_{\theta \to 0} \theta \,cos\,e\,c\theta = \lim_{\theta \to 0} \frac{\theta}{\sin\theta}$$

$$= \lim_{\theta \to 0} \frac{1}{\frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$$

Example:- 2

Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$

Solution:-

$$\lim_{x \to 0} \frac{\sin 3x}{5x} = \lim_{x \to 0} \frac{\sin 3x}{5x \times \frac{3}{3}} = \lim_{x \to 0} \frac{3}{5} . \frac{\sin 3x}{3x} = \frac{3}{5} . \lim_{x \to 0} \frac{\sin 3x}{3x}$$

$$= \frac{3}{5} \times 1 = \frac{3}{5}$$

Example:- 3

Evaluate $\lim_{x \to 0} \frac{\tan x^o}{x^o}$

Solution:-

$$\lim_{x \to 0} \frac{\tan x^o}{x^o} = \lim_{x \to 0} \frac{\tan \frac{\pi x}{180}}{\frac{\pi x}{180}} = 1$$

Example:- 4

Evaluate $\lim_{x \to 0} \left(\frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}\right)$

Solution:-

$$\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)=\lim_{x\to 0}\left(\frac{2\sin 4x\cos 3x}{2\cos 4x\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin 4x\cos 2x}{\cos 4x\sin x}\right)$$

$$=\lim_{x\to 0}\left\{\frac{\sin 4x}{4x}\times\frac{x}{\sin x}\times\cos 2x\times\frac{1}{\cos 4x}\times 4\right\}$$

$$=4\times\lim_{4x\to 0}\frac{\sin 4x}{4x}\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{2x\to 0}\cos 2x\times\frac{1}{\lim\limits_{4x\to 0}\cos 4x}=\left(4\times 1\times 1\times 1\times\frac{1}{1}\right)=4$$

Example:- 5

Evaluate $\lim_{x\to 0}\frac{1-\cos 4x}{x}$

Solution:-

$$\lim_{x\to 0}\frac{1-\cos 4x}{x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 2x}{x}\times\frac{x}{x}$$

$$=\lim_{x\to 0}2\left(\frac{\sin 2x}{2x}\right)^2\times 4x=2\times 1\times 0=0$$

Example:- 6

Evaluate $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$

Solution:- $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$

$$=\lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x\sin 2}}{x^3}$$

$$=\lim_{x\to 0}\frac{\sin 2x-\sin 2x.\cos 2x}{x^3.\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin 2x(1-\cos 2x)}{x^3.\cos 2x}$$

$$=\lim_{x\to 0}\frac{\tan 2x}{x}\times\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$$=2.\lim_{x\to 0}\frac{\tan 2x}{2x}\times 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$=2(1)\times 2(1)^2=4$$

Example:- 7

Evaluate $\lim_{x\to\pi}\frac{\sin x}{\pi-x}$

Solution:-Put $\pi-x=y, y\to 0\ as\ x\to\pi$

Therefore $\lim_{x\to\pi}\frac{\sin x}{\pi-x}$

$$= \lim_{y\to 0} \frac{\sin(\pi - y)}{y}$$

$$= \lim_{y\to 0} \frac{\sin y}{y}$$

=1

Evaluation of Trigonometric Limits by Factorisation:-

Sometimes, trigonometric limits can be evaluated by the factorization method.

Example:- 8

Evaluate $\lim_{x\to\pi/6} \frac{\cot^2 x - 3}{\cos ecx - 2}$

Solution:-

$$\lim_{x\to\pi/6} \frac{\cot^2 x - 3}{\cos e\, cx - 2}$$

$= \lim_{x\to\pi/6} \frac{\cos ec^2 x - 1 - 3}{\cos ecx - 2}$

$= \lim_{x\to\pi/6} \frac{\cos ec^2 x - 4}{\cos ecx - 2}$

$$= \lim_{x\to\pi/6} \frac{(\cos e\, cx - 2)(\cos e\, cx + 2)}{(\cos e\, cx - 2)}$$

$$= \lim_{x\to\pi/6} (\cos e\, cx + 2)$$

$$= \cos e\, c\frac{\pi}{6} + 2 = 2 + 2 = 4$$

Limits of Exponential Functions and Logarithmic Functions:-

Limits of Exponential Functions:-

A function of the form $f(x) = e^x$ is called the exponential function

To find the limit of a function involving an exponential function, we use the following theorem.

Theorem $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Method to find the limit of exponential functions:-

If the given function has an exponential term, then we convert the given theorem into the form of $\frac{e^x - 1}{x}$ and then use the theorem $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

Example:- 1

Find the value of $\lim_{x\to 0} \frac{e^{3x} - 1}{x}$

Solution:-

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{e^{3x} - 1}{x} \times \frac{3}{3}$$

$= 3\lim_{x\to 0}\frac{e^{3x}-1}{3x}$ ....(1)

Let h = 3x. Then $h \to 0 as x \to 0$

Now, from eq. (1) we get

$$3\lim_{x\to 0}\frac{e^{3x}-1}{x} = 3\lim_{h\to 0}\frac{e^{h}-1}{h} = 3\times 1 = 3$$

Example:- 2

Evaluate $\lim_{x\to 3}\frac{e^{x}-e^{3}}{x-3}$

Solution:-

We have $\lim_{x\to 3}\frac{e^{x}-e^{3}}{x-3}$

On put h = x – 3 we get

$$\lim_{x\to 3}\frac{e^{x}-e^{3}}{x-3} = \lim_{h\to 0}\frac{e^{h+3}-e^{3}}{h}$$

$$= \lim_{h\to 0}\frac{e^{h}e^{3}-e^{3}}{h}$$

$$= e^{3}\lim_{h\to 0}\frac{e^{h}-1}{h} = e^{3}\times 1 = e^{3}$$

Theorem:- $\lim_{x\to 0}\frac{log_e(1+x)}{x} = 1$

Note:- $\lim_{x\to 0}\frac{log_e(1-x)}{-x} = 1$

Corollary:-

(i) $\lim_{x\to 0}\frac{log_e(1-x)}{-x} = 1$ (ii) $\lim_{x\to 0}\frac{a^{x}-1}{x} = log\, a$

Method to find the limit of logarithmic Function:-

If the given function involves a logarithmic function, then we convert the given function into the form of $\frac{log_e(1+x)}{x}$ and then use the theorem $\lim_{x\to 0}\frac{log_e(1+x)}{x} = 1$

Example:- 3

Evaluate $\lim_{x\to 0}\frac{log_e(1+2x)}{x}$

Solve:- We have, $\lim_{x\to 0}\frac{log_e(1+2x)}{x}\times\frac{2}{2}$

$$= 2\lim_{x\to 0}\frac{log_e(1+2x)}{2x}$$

On putting h = 2x we get

$$= 2\lim_{x\to 0}\frac{log_e(1+2x)}{x} = 2\lim_{h\to 0}\frac{log_e(1+h)}{h} = 2\times 1 = 2$$

Example:- 4

Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{log(1+x)}$

Solution:-

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{log(1+x)}$$

By multiplying numerators and denominators by $\sqrt{1+x}+1$ we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{log(1+x)}\times\frac{\sqrt{1+x}+1}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0}\frac{1+x-1}{(\sqrt{1+x}+1)\,log(1+x)}$$

$$= \lim_{x\to 0}\frac{x}{(\sqrt{1+x}+1)\,log(1+x)}$$

$$= \frac{1}{(\sqrt{1+0}+1)}\lim_{x\to 0}\frac{1}{\frac{log(1+x)}{x}} = \frac{1}{1+1}\times 1 = \frac{1}{2}$$

Example:- 5

Evaluate $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$

Solution:-

$$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$$

$$= \lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}\times\{\sqrt{1+x}+1\}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}\times\lim_{x\to 0}(\sqrt{1+x}+1)$$

$$= (log\,2)\times 2 = 2\,log\,2$$

Concept of Left hand and right hand limit:-

Left-hand limit:- A real number ℓ_1 is a left-hand limit of the function $f(x)$ at $x = a$ if the values of $f(x)$ can be made as close as ℓ_1 at points close to a and on the left of a. Symbolically, it is written as $LHL = \lim_{x\to a^-} f(x) = \ell_1$.

In other words, we can say that $LHL = \lim_{x \to a^-} f(x) = \ell_1$ is the expected value of f at $x = a$ when we have the values of f near x to the left of a. This value is called the left-hand limit of $f(x)$ at a.

Right-hand limit:- A real number ℓ_2 is a right-hand limit of function $f(x)$ at $x = a$ if the values of $f(x)$ can be made as close as ℓ_2 at point close to a and on the right of a. Symbolically, it is written as $RHL = \lim_{x \to a^+} f(x) = \ell_2$. In other words, we can say that, $RHL = \lim_{x \to a^+} f(x) = \ell_2$ is the expected value of f at $x = a$, when we have the values of f near x to the right of a. This value is called the right-hand limit of $f(x)$ at a.

Existence of Limit:-

If the right-hand limit and left-hand limit coincide (i.e same), then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and denoted by $\lim_{x \to a} f(x)$.

Method to find the left-hand and right-hand limits of a function:-

With the help of the following steps, we can find the left-hand and right-hand limits of a function easily.

Step – I For left-hand limit, write the given function as $\lim_{x \to a^-} f(x)$, and for right-hand limit, write the given function as $\lim_{x \to a^+} f(x)$.

Step – II For the left-hand limit, put $x = a - h$ and change the limit $x \to a^-$ by $h \to 0$. Then, the limit obtained from step I is $\lim_{h \to 0} f(a - h)$. Similarly, for the right-hand limit, put $x = a + h$ and change the limit $x \to a^+$ by $h \to 0$. Then, the limit obtained from step I is $\lim_{h \to 0} f(a + h)$.

Step – III Now, simplify the result obtained in step – **II** i.e $\lim_{h \to 0} f(a - h)$ or $\lim_{h \to 0} f(a + h)$.

Example:- 1

Suppose the function is defined by $f(x) = \begin{cases} \frac{|x-3|}{x-3}, & if x \neq 3 \\ 0, & if x = 3 \end{cases}$

(i) Find the left hand limit of $f(x)$ at $x = 3$

(ii) Find the right hand limit of $f(x)$ at $x = 3$

Solution:-

(i) Given, $f(x) = \begin{cases} \frac{|x-3|}{x-3}, & if x \neq 3 \\ 0, & if x = 3 \end{cases}$

∴ Left-hand limit at x = 3 is

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{|x-3|}{x-3}$............................. (1)

On putting $x = 3 - h$ and changing the limit $x \to 3^{-} by h \to 0$ in Eq. (i) we get

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{h\to 0} \frac{|h|}{-h}$$

$$\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{h\to 0} \frac{h}{(-h)} = -1$$

(ii) Right-hand limit at x = 3 is

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{|x-3|}{x-3} \text{ .... (ii)}$$

On putting $x = 3 + h$ and changing the limit $x \to 3^{+}$ by $h \to 0$ in equation (ii) we get

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{h\to 0} \frac{|h|}{h}$$

$$\Rightarrow \lim_{x\to 3^+} f(x) = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Example:-2

Evaluate the left-hand and right-hand limits of the following functions at x = 2, $f(x) = \begin{cases} 2x+3, & if x \le 2 \\ x+5, & if x > 2 \end{cases}$. Does $\lim_{x\to 2} f(x)$ exist?

Solution:-

Given $f(x) = \begin{cases} 2x+3, & if x \le 2 \\ x+5, & if x > 2 \end{cases}$

LHL = $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x+3$

$$= \lim_{h\to 0}[2(2-h)+3] = 2(2-0)+3$$

[Putting $x = 2 - h$ and when $x \to 2^{-}$, then $h \to 0$]

$= 4 + 3 = 7$

RHL = $\lim_{x\to 2^+} f(x)$

$= \lim_{x\to 2^+} (x+5)$

$= \lim_{h\to 0}(2+h+5) = 2+0+5 = 7$

Putting x = 2 + handx $\to 2^{+}$, then $h \to 0$

LHL of $f(at x = 2) = RHL$ of $f(at\ x = 2)$

$\lim_{x\to 2} f(x)$ exists and it is equal to 7.

Derivative and the first principle of Derivative:-

Derivative at a point

Suppose f is a real-valued function and a is a point in its domain. Then, the derivative of f at a is defined by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ provided this limit exists.

The derivative of $f(x)$ at a is denoted by $f'(a)$.

Example:- 1

Find the derivative $f(x) = 4x + 5$ at $x = 3$.

Solution:-

Given $f(x) = 4x + 5$. We know that at x = a, $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$\therefore f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0}\frac{4(3+h)+5-(4\times 3+5)}{h}$$

$$= \lim_{h\to 0}\frac{12+4h+5-17}{h} = \lim_{h\to 0}\frac{4h}{h} = 4$$

Geometrical meaning of derivative. Consider a graph of a function $y = f(x)$:

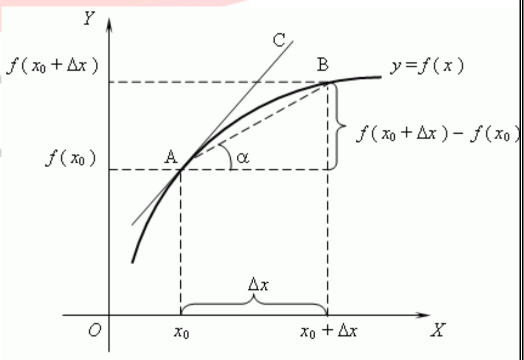

Fig. 1

From Fig.1 we see, that for any two points A and B of the function graph:

where α- a slope angle of the secant AB.
So, the difference quotient is equal to a secant slope. If to fixpoint A and move point B towards A, then Δx will unboundedly decrease and approach 0, and the secant AB will approach the tangent AC. Hence, a limit of the difference quotient is equal to a slope of a tangent at point A. Hence it follows: *a derivative of a function at a point is a slope of a tangent of this function graph at this point.*

First Principle of Derivative:-

Suppose f is a real-valued function, the function defined by $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is called the first principle of the derivative.

Thus, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Sometimes $f'(x)$ is denoted by $\frac{d}{dx}[f(x)] or if y = f(x)$ then it is denoted by $\frac{dy}{dx}$ and referred to as derivative of $f(x)$or y w.r.t x. It is also denoted by $D[f(x)]$.

Note:-

A derivative of at $x = a$ is also given by substituting $x = a$ in $f'(x)$ and it is denoted by $\frac{d}{dx}f(x)\Big|a \; or \left.\frac{df}{dx}\right|_a \; or \left(\frac{df}{dx}\right)_{x=a}$.

Example:-2

Find the derivative of $f(x) = \frac{1}{x}$ using the first principle.

Solution:-

We have $f(x) = \frac{1}{x}$

By using the first principle, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore f'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{x(x+h)}\right]$$

$$= \lim_{h\to 0}\left[\frac{-1}{x(x+h)}\right] = \frac{-1}{x^2}$$

Example:- 3

Find the derivative of e^x, using the first principle.

Solution:-

Let $f(x) = e^x$. By using the first principle of derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$\lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{(e^h-1)}{h} = e^x \times 1 = e^x$$

Example:-4

Find the derivative of the function $log\, x$, by using the first principle.

Solution:-

Let $f(x) = log\, x$

By using the first principle of derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{log(x+h)-log\, x}{h}$$

$$= \lim_{h\to 0}\frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\times\frac{1}{x}$$

$$= 1\times\frac{1}{x} = \frac{1}{x}$$

Example:-5

Find the derivative of the following function by using the first principle.

(i) sin x (ii) sec x (iii) tanx

Solution:-

(i) Let $f(x) = \sin x$

By using the first principle of derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\times\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right).\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\times 1$$

$$= \cos(x+0) = \cos x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

(ii) Let $f(x) = \sec x$

By using the first principle of derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h \times \cos x \,.\cos(x+h)}$$

$$= \lim_{h\to 0} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right).\sin\left(\frac{x-x-h}{2}\right)}{h.\cos x\,.\cos(x+h)}\right]$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h\to 0} \left[\frac{-2\sin\left(x+\frac{h}{2}\right).\left(-\sin\frac{h}{2}\right)}{h.\cos x\cos(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h)\,.\cos x}.\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \frac{\sin x}{\cos^2 x} \times 1$$

$$= \frac{\sin x}{\cos x}.\frac{1}{\cos x} = \tan x\,.\sec x$$

(iii) Let $f(x) = \tan x$

Then, by the first principle of derivative, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$[\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$$

$$= \lim_{h\to 0} \frac{\sin h}{h \lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}}$$

$$= 1.\frac{1}{\cos(x+0)\cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Hence, $f'(x) or \frac{d}{dx}(\tan x) = \sec^2 x$

Algebra of Derivative of Functions:-

Let f and g be two functions such that their derivatives are defined in a common domain. Then,

(a) The derivative of the sum of two functions is the sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

(b) The derivative of the difference between two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

(c) The derivative of the product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x).g(x)]=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x).$$

This is also known as the Leibnitz product rule of the derivative.

(d) The derivative of the quotient of two functions is given by the following quotient rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}, g(x)\neq 0$$

Note:- $\frac{d}{dx}[c.f(x)]=c\frac{d}{dx}f(x)$

Theorem:- Derivative of $f(x)=x^n \ is \ nx^{n-1}$ for any real number n.

Example:- 1

Differentiate $2x^3-4x^2+6x+8$ w.r.t x

Solution:-

Let $y=2x^3-4x^2+6x+8$

On differentiating both sides w.r.t x we get

$$\frac{dy}{dx}=\frac{d}{dx}(2x^3-4x^2+6x+8)$$

$$=2\frac{d}{dx}(x^3)-4\frac{d}{dx}(x^2)+6\frac{d}{dx}(x)+\frac{d}{dx}(8)$$

$$=2(3x^2)-4(2x)+6(1)+0$$

$$=6x^2-8x+6$$

Example:-2

If $u=7t^4-2t^3-8t-5$, then find $\frac{du}{dt} at \ t=2$.

Solution:-

We have $u=7t^4-2t^3-8t-5$

On differentiating both sides w.r.t x we get

$$\frac{du}{dt} = \frac{d}{dt}[7t^4 - 2t^3 - 8t - 5]$$
$$= 7(4t^3) - 2(3t^2) - 8(1) - 0$$
$$= 28t^3 - 6t^2 - 8$$

Now, $\left(\frac{du}{dt}\right)_{t=2} = 28(2)^3 - 6(2)^2 - 8$

$$= 224 - 24 - 8 = 192$$

Example:-3

Differentiate the following functions w.r.t x $(ax+b)(cx+d)^2$

Solution:-

Let $y = (ax+b)(cx+d)^2$

On differentiating both sides w.r.t x we get

$\frac{dy}{dx} = (ax+b)\frac{d}{dx}(cx+d)^2 + (cs+d)^2\frac{d}{dx}(ax+b)$ (using product rule of derivatives)

$$= (ax+b)\frac{d}{dx}(c^2x^2 + d^2 + 2cxd) + (cx+d)^2(a \times 1 + 0)$$
$$= (ax+b)(c^2(2x) + 0 + 2c \times 1 \times d) + (cx+d)^2 \times a$$
$$= (ax+b)(2c^2x + 2cd) + a(cx+d)^2$$
$$= (ax+b)2c(cx+d) + a(cx+d)^2$$
$$= (cx+d)[2c(ax+b) + a(cx+d)]$$
$$= (cx+d)(2acx + 2bc + acx + ad)$$
$$= (cx+d)(3ax + 2bc + ad)$$

Example:-4

Differentiate $\frac{x^2+3x-9}{x^2-9x+3}$ w.r.t. x

Solution:-

Let $y = \frac{x^2+3x-9}{x^2-9x+3}$

On differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2+3x-9}{x^2-9+3}\right]$$

$$= \frac{\left\{(x^2-9x+3)\left[\frac{d}{dx}(x^2+3x-9)\right] - (x^2+3x-9)\frac{d}{dx}(x^2-9x+3)\right\}}{(x^2-9x+3)^2}$$

$$= \frac{[(x^2-9x+3)(2x+3-0) - (x^2+3x-9) \times (2x-9+0)]}{(x^2-9x+3)^2}$$

$$= \frac{[(x^2-9x+3)(2x+3)-2x(x^2+3x-9)+9(x^2+3x-9)]}{(x^2-9x+3)^2}$$

$$= \frac{[2x(x^2-9x+3-x^2-3x+9)+(3x^2-27x+9+9x^2+27x-81)]}{(x^2-9x+3)^2}$$

$$= \frac{2x(-12x+12)+(12x^2-72)}{(x^2-9x+3)^2}$$

$$= \frac{-2x^2+24x+12x^2-72}{(x^2-9x+3)^2}$$

$$= \frac{-12x^2+24x-72}{(x^2-9x+3)^2} = \frac{-12(x^2-2x+6)}{(x^2-9x+3)^2}$$

The derivative of Trigonometric Functions:-

To find the derivative of trigonometric functions, we use the algebra of derivative and the following formulae.

(a) $\frac{d}{dx}(\sin x) = \cos x$ (b) $\frac{d}{dx}(\cos x) = -\sin x$ (c) $\frac{d}{dx}(\tan x) = \sec^2 x$

(d) $\frac{d}{dx}(\sec x) = \sec x . \tan x$ (e) $\frac{d}{dx}(\cot x) = -\cos e c^2 x$ (f) $\frac{d}{dx}(\cos e cx) = -\cos e cx . \cot x$

Example:- 1

Find the derivative of the following functions.

(a) $x^2 \cos x$ (b) $x^3 \sec x$ (c) $x \tan x$

Solution:-

(a) Let $y = x^2 \cos x$

On differentiating both sides w.r.t x we get

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x) = x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$

$= x^2(-\sin x) + \cos x (2x)$

$= 2x \cos x - x^2 \sin x$

(b) Let $y = x^3 \sec x$

On differentiating both sides w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}(x^3 \sec x)$

$$\Rightarrow \frac{dy}{dx} = x^3 \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(x^3)$$

$= x^3 . \sec x \tan x + \sec x (3x^2)$

$= x^3 . \sec x \tan x + 3x^2 \sec x$

(c) Let $y = x \tan x$

On differentiating both sides w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}(x\tan x)$

$$\frac{dy}{dx} = x\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x)$$

$= x\sec^2 x + \tan x\,(1)$

$= x\sec^2 x + \tan x$

Example:-2

If $y = \left\{\frac{1-\tan x}{1+\tan x}\right\}$, then show that $\frac{dy}{dx} = \frac{-2}{1+\sin 2x}$.

Solution:-

We have, $y = \frac{1-\tan x}{1+\tan x}$

On differentiating both sides w.r.t x we get

$$\frac{d}{dx} = \frac{(1+\tan x).\frac{d}{dx}(1-\tan x) - (1-\tan x).\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{(\cos^2 x)(1+\tan^2 x + 2\tan x)}$$

$$= \frac{-2}{(\cos^2 x)\left\{1 + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos x}\right\}} = \frac{-2}{(1+\sin 2x)}$$

DERIVATIVE OF COMPOSITION FUNCTIONS (CHAIN RULE)

To study the derivative of composition functions, we start with an illustrative example, say we want to find the derivative of f where $f(x) = (2x+1)^3$.

Now $\frac{df(x)}{dx} = \frac{d(2x+1)^3}{dx} = \frac{d(8x^3+12x^2+6x+1)}{dx} = 24x^2 + 24x + 6 = 6(2x+1)^2$

We observe that, if we take $g(x) = 2x+1$ and $h(x) = x^3$

Then $f(x) = hog(x) = (2x+1)^3 \Rightarrow \frac{df(x)}{dx} = f^/(x) = \frac{dhog(x)}{dg(x)} \times \frac{dg(x)}{dx}$

i.e. $\frac{d(2x+1)^3}{d(2x+1)} \times \frac{d(2x+1)}{dx} \Rightarrow f^/(x) = 3 \times (2x+1)^2 \times 2 = 6(2x+1)^3$

The advantage of such observation is that it simplifies the calculation of finding the derivative.

CHAIN RULE

Let f be the real-valued function which is a composition of two functions u & v. i.e. $f = uov$.

Where $u\&v$ are differentiable functions and uov is also a differentiable function?

$\Rightarrow \frac{df}{dx} = \frac{duov}{dx} = \frac{duov}{dv} \times \frac{dv}{dx}$, Provided all the derivatives in the statement exists.

PROBLEMS

Example:-1

Find $\frac{dy}{dx}$ if $y = sin(x^2 + 1)$.

Solution:- Given $y = sin(x^2 + 1) \Rightarrow \frac{dy}{dx} = \frac{d(sin(x^2+1))}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(sin(x^2+1))}{d(x^2+1)} \times \frac{d(x^2+1)}{dx} \quad \Rightarrow \frac{dy}{dx} = cos(x^2 + 1) \times 2x$

Example:2

Find $\frac{dy}{dx}$, if $y = log(\tan x)$.

Solution:- Given $y = log(\tan x) \quad \Rightarrow \frac{dy}{dx} = \frac{d(log(\tan x))}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(log(\tan x))}{d \tan x} \times \frac{d \tan x}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\tan x} \times sec^2 x$

Example:-3

Find $\frac{dy}{dx}$ if, $y = e^{\sin(x^2)}$

Solution:- Given $y = e^{\sin(x^2)} \Rightarrow \frac{dy}{dx} = \frac{d(e^{\sin(x^2)})}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(e^{\sin(x^2)})}{d \sin(x^2)} \times \frac{d \sin(x^2)}{d(x^2)} \times \frac{d(x^2)}{dx} \quad \Rightarrow \frac{dy}{dx} = e^{\sin(x^2)} \times cos(x^2) \times 2x$

Example:-4

Find $\frac{dy}{dx}$, if $y = (x^2 + x + 1)^4$

Solution:- Given $y = (x^2 + x + 1)^4 \Rightarrow \frac{dy}{dx} = \frac{d(x^2+x+1)^4}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(x^2+x+1)^4}{d(x^2+x+1)} \times \frac{d(x^2+x+1)}{dx} \Rightarrow \frac{dy}{dx} = 4(x^2 + x + 1)^3 \times (2x + 1)$

Example:-5

Find $\frac{dy}{dx}$, if $y = \frac{1}{\sqrt{a^2-x^2}}$

Solution:-Given $y = \frac{1}{\sqrt{a^2-x^2}} \Rightarrow \frac{dy}{dx} = \frac{d(a^2-x^2)^{-1/2}}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(a^2-x^2)^{-1/2}}{d(a^2-x^2)} \times \frac{d(a^2-x^2)}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{2}(a^2 - x^2)^{-3/2} \times (-2x) = \frac{x}{(a^2-x^2)^{\frac{3}{2}}}$

Example:-6

Find $\frac{dy}{dx}$, if $y = sin^3 x$

Solution:-Given $y = sin^3 x \Rightarrow \frac{dy}{dx} = \frac{d(sin^3 x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{d(sin^3 x)}{d\,sin\,x} \times \frac{d\,sin\,x}{dx} = 3\,sin^2 x \times cos\,x$

Example:-7

Find $\frac{dy}{dx}$, if $y = log(sec\,x + tan\,x)$

Solution:-Given that $y = log(sec\,x + tan\,x) \Rightarrow \frac{dy}{dx} = \frac{d(log(sec\,x+tan\,x))}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(log(sec\,x+tan\,x))}{d(sec\,x+tan\,x)} \times \frac{d(sec\,x+tan\,x)}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{sec\,x+tan\,x} \times (sec\,x\,.\,tan\,x + sec^2 x) = sec\,x$

Example:-8

Find $\frac{dy}{dx}$, if $y = e^{x\,sin\,x}$

Solution:-Given that $y = e^{x\,sin\,x} \Rightarrow \frac{dy}{dx} = \frac{d(e^{x\,sin\,x})}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(e^{x\,sin\,x})}{d(x\,sin\,x)} \times \frac{d(x\,sin\,x)}{dx} \Rightarrow \frac{dy}{dx} = e^{x\,sin\,x} \times \{x\,cos\,x + sin\,x\}$

Example:-9

Find $\frac{dy}{dx}$, if $y = \frac{sin(ax+b)}{cos(cx+d)}$

Solution:-Given that $y = \frac{sin(ax+b)}{cos(cx+d)} \Rightarrow \frac{dy}{dx} = \frac{d\left\{\frac{sin(ax+b)}{cos(cx+d)}\right\}}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{cos(cx+d) \times \frac{d(sin(ax+b))}{d(ax+b)} \times \frac{d(ax+b)}{dx} - sin(ax+b) \times \frac{d(cos(cx+d))}{d(cx+d)} \times \frac{d(cx+d)}{dx}}{cos^2(cx+d)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos(cx+d) \times cos(ax+b) \times a + sin(ax+b) \times sin(cx+d) \times c}{cos^2(cx+d)}$$

Example:-10

Find $\frac{dy}{dx}$, if $y = cos(x^3).sin(x^3)$

Solution:-Given $y = cos(x^3).sin(x^3) \Rightarrow \frac{dy}{dx} = \frac{d(cos\,x^3.sin\,x^3)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(cos\,x^3\,.\,sin\,x^3)}{dx} \Rightarrow \frac{dy}{dx} = cos\,x^3\,\frac{d(sin\,x^3)}{dx^3} \times \frac{dx^3}{dx} + sin\,x^3\,\frac{d(cos\,x^3)}{dx^3} \times \frac{dx^3}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (cos\,x^3)(cos\,x^3)3x^2 - (sin\,x^3)(sin\,x^3)3x^2 \Rightarrow \frac{dy}{dx} = (cos\,x^3)^2 3x^2 - (sin\,x^3)^2 3x^2$$