Introduction

The generalised meaning of probability is the mathematical measurement of uncertainty.

For example, if we toss a coin, ahead is likely to occur, but may not occur. When a die is thrown, it may or may not show the number 5. Thus, the uncertainty principle leads to the concept of probability or chance.

The theory of probability, based on set theory, was developed by $A.\ N.\ Kalmogorov$ in 1933. Also many other mathematicians as
$Jacob\ Bernoulli,\ De'moivre,\ Laplce,\ Poisson,\ A.\ Markov,\ Cauchy$ etc.

In this chapter, we will study the axiomatic approach of probability, which requires the knowledge of some terms $i.e.$ Random experiment, Outcomes, Sample space, events etc.

Random Experiment

When we perform some experiments, since the outcomes are uncertain, such experiments are called a random experiment.

Ex: tossing a coin, throwing a die etc.

If a die is thrown once, any one of the six numbers $i.e.,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ may turn up, not sure which number will come up.

Thus an experiment is called random experiment or probabilistic experiment if it satisfies the following conditions.

(i) It has more than one possible outcomes.

(ii) It is not possible to predict the outcomes in advance.

Some Fundamental Definitions

Outcomes: A possible result of a random experiment is called its outcome.

For example, if we throw a die, its outcomes are $1,\ 2,\ 3,\ 4,\ 5,\ or\ 6$.

Sample Space: The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.

Example: Consider the random experiment of tossing a coin. The possible outcomes are H and T. Thus we define $E =$ getting head (H) on the upper face and $F =$ getting tail (T) on the upper face.

Then E and F are elementary events associated with the random experiment of tossing of a coin. The sample space associated with this experiment is given by $S = \{E, F\}$

Example: Consider the experiment of tossing two coins together or a coin twice. In this experiment, the possible outcomes are HH, HT, TH, TT.

The sample space is given by $S = \{HH, HT, TH, TT\}$

Example: Consider the random experiment in which two dice are tossed together or a die is tossed twice. If we define $E_{ij} =$ getting number i on the upper face of the first die and number j on the upper face of the second die, where $i = 1, 2, \ldots, 6$ and $j = 1, 2, \ldots, 6$.

The sample space associated with it is given by

$S = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (2, 6), \ldots, (6, 1), (6, 2), \ldots, (6, 6)\}$

Example: From a group of 2 boys and 3 girls, two children are selected. Find the sample space associated with this random experiment.

Sol: Let the two boys be taken as B_1 and B_2 and three girls be taken as G_1, G_2 and G_3. There are 5 children, out of which two children can be chosen in $5_{C_2} = 10$ ways. The sample space S associated with this random experiment is given by

$S = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$

Example: Consider the experiment in which a coin is tossed repeatedly until a tail comes up. Describe the sample space.

Sol: In the experiment, the tail may come up on the first toss or the 2nd toss or the 3rd toss and so on till tail is obtained. Thus a sample space associated with this experiment is

$S = \{T, HT, HHT, HHHT, \ldots\}$

Example: A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Sol: Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

$S = \{HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, TI, T2, T3$, T4, T5, T6}

Here HB_i means head on the coin and B_i is drawn, HW_i means head on the coin and ball W_i. Similarly, Ti means tail on the coin and the number i on the die.

Sample Point: Each element of the sample space is called a sample point.

The number of sample points or possible outcomes in S is demoted by $n(S)$.

Events and Types of Events

Event: Any subset E of a sample space S is called an event.$i.e.$ $E \subseteq S$.

For example, tossing two coins, the subset $E = \{HH, HT, TH\}$ is the event that at least one head occurs and the event of drawing an ace from a deck is $F = \{$Ace of Heart, Ace of Club, Ace of Diamond, Ace of Spade $\}$

The occurrence of an Event:

An event associated with a random experiment is said to occur if any one of the elementary events associated with it is an outcome of the experiment.$e.g.$ Suppose a die is thrown and let A be an event of getting an odd number. Then $A = \{1, 3, 5\}$

Now, if the outcome of the experiment is 3. Then, we can say that event A has occurred. Suppose, in another trial, the outcome is 4, then we say that A has not occurred.

Types of Events

Events can be classified into various types based on their elements.

Sure Event: The whole sample space S is also a subset of S, so it represents an event. Since every outcome of an experiment carried out a member of S, therefore the event represented by S is called a sure or a certain event.

Ex: (i) Sun rises in the East, is a sure event.

(ii) On tossing a coin, either head or tail will occur, is a sure event.

Impossible Event: The empty set ϕ is also a subset of the sample space S, so it represents an event. Since there is no element in the empty set, therefore it can not occurs. Thus ϕ is called an impossible event.

Ex: On throwing a die, the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Let $E =$ Event of getting a number less than 1.

Then $E = \phi$. So, E is called an impossible event.

Simple Event: If an event has only one sample point then it is called a simple event or elementary event.

Ex: Let a die is thrown, then sample space $S = \{1, 2, 3, 4, 5, 6\}$

Again, let $A =$ Event of getting 4 $= \{ 4 \}$. So A is a simple event.

Compound Events: If an event has more than one sample point, then it is called a compound event.

Ex: Let E be the event of getting a prime number on rolling of a die.

So, $E = \{ 2,\ 3,\ 5\}$

Then, E is a compound event.

Example: Two dice are rolled. Let A, B and C be the events of getting a sum of 2, a sum of 3 and a sum of 4 respectively.

(i) Which events are elementary events?

(ii) Which events are compound events?

Sol: On throwing of two dice, we get the sample space $S = \{(1,\ 1),\ (1,\ 2),\ \ldots (6,\ 6)\}$

Now as A is the event of getting a sum of 2.

So, $A = \{(1,\ 1)\}$

Similarly, $B = \{(1,\ 2),\ (2,\ 1)\}$

and $C = \{(1,\ 3),\ (3,\ 1),\ (2,\ 2)\}$

(i) Since A consists of a single sample point, therefore it is an elementary event.

(ii) Since both A and B contain more than one sample points, therefore each of them is a compound event

Algebra of Events

Since the events are sets, the statements concerning events can be written using set notations. Let A and B be two events associated with an experiment whose sample space is S. Then we have

Complimentary Event: For the event E, the complement of E is the event consisting of all outcomes which do not correspond to the occurrence of E. The complement of E is denoted by E' or $\overline{E}$ or E^c. It is also called the event ' not E' with relation to a sample space S.

$E' = S - E = \{\omega : \omega \in S$ and $\omega \notin A\}$

Example: In tossing three coins, the sample space is

$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let $E =$ the event of getting only one head $= \{THT, TTH, HTT\}$

Then $E' = \{ HHH, HHT, HTH, THH, TTT\}$

The event A or B

If A and B are two events associated with a sample space, the event ‘ A or B’ is same as the event $A \cup B$ and contains all those sample points which are either in A or in B or both.

Thus, A or $B = A \cup B = \{ \omega : \omega \in A$ or $\omega \in B\}$

The event A and B

If A and B are two events, then the set $A \cap B$ denotes, the event ‘ and B’, $A \cap B$ contains all those sample points which are common to both A and B.

Therefore, event A and $B = A \cap B = \{\omega : \omega \in A$ and $\omega \in B\}$

The event ‘A but not B’

The event ‘ but not B' is the same as the event $A - B = A \cap B'$ and it contains all those elements which are in A but not in B.

Thus $A - B = \{ \omega : \omega \in A$ and $\omega \notin B\}$

Example: Consider the experiment of rolling a die. Let A be the event ‘ getting a prime number’, B be the event ‘ getting an odd number’. Write the sets representing the events

(i) A or B (ii) A and B (iii) A but not B (iv) not A.

Sol: Here, the sample space $S = \{ 1, 2, 3, 4, 5, 6\}$

So, $A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$

Then (i) A or $B = A \cup B = \{1, 2, 3, 5\}$

(ii) A and $B = A \cap B = \{3, 5\}$

(iii) A but not $B = A - B = \{ 2 \}$

(iv) not $A = A' = \{1, 4, 6\}$

Equally Likely Events

A given number of events are said to be equally likely if none of them is expected to occur in preference to the other. For example in a throw of a die, all six possible outcomes $1, 2, 3, 4, 5, 6$ are equally likely $i.e.$ We have no reason to except any one of these in preference to the others.

If a coin is unbiased ($i.e.$, perfectly homogeneous and is uniform in both faces), there is no reason to expect that, for instance, the head will appear more often than tails or vice versa.

In this case, $S = \{H, T\}$. The outcomes head and tail are therefore equally likely to occur.

Thus, if an experiment is repeated a large number of times, then half of the times heads will occur and half of the times tails will occur.

Hence, we can say that such sample spaces are uniform spaces or equiprobable spaces.

Mutually Exclusive Events

Two events are said to be mutually exclusive or disjoint if the occurrence of any one of them excludes the occurrence of another event, $i.e.$ If they cannot occur simultaneously.

If E_1 and E_2 are any two events defined on a sample space S and $E_1 \cap E_2 = \phi$, then the events E_1 and E_2 are said to be mutually exclusive.

For example, let E_1 =Event of getting a total of 11, if a die is rolled twice

E_2 = Event of getting a total of 10, if a die is rolled twice.

Here $E_1 = \{(5, 6), (6, 5)\}$ and $E_2 = \{(4, 6), (5, 5), (6, 4)\}$

$\Rightarrow \quad E_1 \cap E_2 = \phi$

$\therefore$ E_1 and E_2 are mutually exclusive events.

Exhaustive Events

Events are said to be exhaustive when they include all possible outcomes of a random experiment. In other words, events $E_1, E_2, \ldots, E_n$ are said to be exhaustive if at least one of them necessarily occurs whenever the experiment is performed.

Thus, the events $E_1, E_2, \ldots, E_n$ are called exhaustive events of a sample space S,

if $E_1 \cup E_2 \cup \ldots \cup E_n = S$

Example: Consider the experiment of throwing a die. We have $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

Let us define the following events:

E_1: a number less than 4 appears.

E_2: a number greater than 2 but less than 5 appears.

E_3: a number greater 4 appears.

Then, $E_1 = \{1,\ 2,\ 3\},\ \ E_2 = \{3,\ 4\},\ \ E_3 = \{5,\ 6\}$

We get $E_1 \cup E_2 \cup E_3 = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = S$

Thus, the events E_1, E_2 and E_3 are exhaustive events.

Example: In tossing a coin, the event of getting head and tail are exhaustive $i.e.,\ E_1 = \{H\},\ E_2 = \{\ T\ \}$

and $E_1 \cup E_2 = \{\ H,\ T\} = S$

Mutually Exclusive and Exhaustive Events

If there are two or more mutually exclusive events such that outcomes of all such events together comprise the sample space, the events are called mutually exclusive and exhaustive events.

Let S be the sample space associated with a random experiment. A set of events $E_1, E_2,\ \ \ldots, E_n$ is said to form a set of the mutually exclusive and exhaustive system of events if

$(i) E_1 \cup E_2 \cup \ldots \cup E_n = S$

$(ii) E_i \cap E_j = \phi$ for $i \neq j$

The elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive events.

Example: State which of the following events are mutually exclusive:

(i) A be the event of drawing a black card and B be the event of drawing an ace, in a draw of a card from a well-shuffled deck of 52 cards.

$(ii) A$ be the event of getting a sum less than 5 and B be the event of getting the sum greater than 10, in a single throw of two dice.

Sol: (i) A and B are not mutually exclusive $i.e.$, $A \cap B \neq \phi$, since the black ace is favourable to both the events.

(ii) $S = \{ (x, y): x, y = 1, 2, 3, 4, 5, 6\}$. There are 36 elements in the sample space S.

Here $A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$ and

$B = \{(5, 6), (6, 5), (6, 6)\}$

$\therefore A \cap B = \phi$

Thus, A and B are mutually exclusive.

Example: A coin is tossed three times, consider the following events.

A : 'No head appears', B: ' Exactly one head appears' and C: ' At least two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Sol: The sample space of the experiment is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

and $A = \{TTT\}$, $B = \{HTT, THT, TTH\}$, $C = \{HHT, HTH, THH, HHH\}$

Now $A \cup B \cup C = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\} = S$

Therefore A, B and C are exhaustive events.

Also, $\cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint $i.e.$, they are mutually exclusive. Hence A, B and C form a set of mutually exclusive and exhaustive events.

Axiomatic Approach to Probability

In this approach, for a given sample space associated with a random experiment, the probability is considered as a function which assigns a non – negative real number $P(A)$ to every event A. This non – negative real number is called the probability of the event A.

Probability Function:

Let $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be the sample space associated with a random experiment. Then a function P which assigns every event $A \subseteq S$ to a unique non – negative real number $P(A)$ is called the probability function, if the following axioms hold:

$Axiom\ 1: 0 \leq P(\omega_i) \leq 1$ for all $\omega_i \in S$

$Axiom\ 2$: $P(S) = 1\ i.e., P(\omega_1) + P(\omega_2) + \ \dots + P(\omega_n) = 1$.

$Axiom\ 3$: For any event $A \subseteq S$, $P(A) = \sum_{\omega_k \in A} P(\omega_k)$, the number $P(\omega_k)$ is called the probability of an elementary event ω_k.

Probability of Equally Likely Outcomes

Let the sample space of an experiment is $S = \{\omega_1, \omega_2, \ \dots, \omega_n\}$.

Also, let all the outcomes are equally likely.

$i.e.\ P(\omega_i) = p$, for all $\omega_i \in S$, where $0 \leq p \leq 1$.

Since, by the axiomatic approach to probability, $\sum_{i=1}^{n} P(\omega_i) = 1$

so, $p + p + \ \dots + p\ (n\ times) = 1$

$\Rightarrow\ np = 1$

$\Rightarrow\ p = \frac{1}{n}$

If A is an event such that m elementary events are favourable to A. Then,

$$P(A) = \sum_{\omega_k \in A} P(\omega_k)$$

$$\Rightarrow\ P(A) = \frac{1}{n} + \frac{1}{n} + \ \dots + \frac{1}{n}\ (m - times)$$

$$P(A) = \frac{m}{n} = \frac{Favourable\ number\ of\ elemetary\ events}{Total\ number\ of\ elementary\ events}$$

Probability of an Event

If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$.

Thus $P(A) = \frac{m}{n} = \frac{n(A)}{n(S)}$

$0 \leq m \leq n$. Therefore $0 \leq \frac{m}{n} \leq 1 \Rightarrow 0 \leq P(A) \leq 1$

If $P(A) = 1$, then A is called a certain event and A is called an impossible event, if $P(A) = 0$.

The number of elementary events which will ensure the non –occurrence of A $i.e.,$ which ensure the occurrence of A' is $n - m$. Therefore,

$P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$

Thus we get

(i) For any event A, $P(A) \geq 0$.

$(ii) P(S) = 1$

$(iii) P(\phi) = 0$

(iv) $P(A) + P(A') = 1$

(v) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

(vi) The odds in favour of occurrence of the event A are defined by $m:(n-m)$ $i.e.,$ $P(A):P(A')$ and the odds against the occurrence of A are defined by $(n-m): m$ $i.e.,$ $P(A'):P(A)$.

Example: Find the probability of getting head in a toss of an unbiased coin.

Sol: The sample space associated with the random experiment is $S = \{H, T\}$

So, $n(S) = 2$.

Let A be the event of getting ahead. So, $A = \{H\}$

$\Rightarrow$ $n(A) = 1$

Hence, the required probability $P(A) = \frac{n(A)}{n(S)} = \frac{1}{2}$

Example: Three coins are tossed once. Find the probability of getting :

(i) all heads (ii) at least two heads (iii) at most two heads (iv) no heads (v) exactly one tail

(vi) exactly two tails (vii) a head-on first coin.

Sol: Let S be the sample space associated with the random experiment of tossing three coins.

Then, $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

So, the total number of elementary events $= 8$ $i.e..n(S) = 8$.

(i) Let A be the event of 'getting all heads'. Then $A = \{HHH\}$

$\Rightarrow$ The favourable number of elementary event $= 1$ $i.e.$, $n(A) = 1$

So, the required probability $P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$

(ii) Let B be the event of ' getting at least two heads'.

Then, $B = \{HHH, HHT, HTH, THH\} \Rightarrow n(B) = 4$

So, the required probability $= P(B) = \frac{n(B)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(iii) Let C be the event of 'getting at most two heads'.

So, $C = \{HHT, HTH, THH, HTT, THT, TTH, TTT\} \Rightarrow n(C) = 7$

Thus, the required probability $= P(C) = \frac{n(C)}{n(S)} = \frac{7}{8}$

(iv) Let D be the event of ' getting no heads'. So, $D = \{TTT\} \Rightarrow n(D) = 1$

Thus, the required probability $= P(D) = \frac{n(D)}{n(S)} = \frac{1}{8}$

(v) Let E be the event of 'getting exactly one tail'.

So, $E = \{HHT, HTH, THH\} \Rightarrow n(E) = 3$

Thus, the required probability $= P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

(vi) Let F be the event of ' getting exactly 2 tails'.

So, $F = \{HTT, THT, TTH\} \Rightarrow n(F) = 3$

Thus, the required probability $= P(F) = \frac{n(F)}{n(S)} = \frac{3}{8}$

(vii) Let G be the event of ' getting a head-on first coin'.

So, $G = \{HHH, HHT, HTH, HTT\} \Rightarrow n(G) = 4$

Thus, the required probability $= P(G) = \frac{n(G)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

Example: Two dice are thrown simultaneously. Find the probability of getting:

(i) an even as the sum (ii) a total of at least 10 (iii) a doublet of even number

(iv) a multiple of 2 on one dice and a multiple of 3 on the other dice

(v) a multiple of 3 as the sum.

Sol: When two dice are thrown together the sample space associated with the random experiment is given by $S = \{(1,\ 1),\ (1,\ 2),\ \dots,\ (6,\ 6)\}$

$i, e.,\ n(S) = 36$

(i) Let A be the event ' getting an even number as the sum'.

Then, $A = \{(1,\ 1),\ (1,\ 3),\ (1,\ 5),\ (2,\ 2),\ (2,\ 4),\ (2,\ 6),\ (3,\ 1),\ (3,\ 3),\ (3,\ 5),\ (4,\ 2),\ (4,\ 4),$

$(4,\ 6),\ (5,\ 1),\ (5{,}3)(5,\ 5),\ (6,\ 2),\ (6,\ 4),\ (6,\ 6)\}$

$\Rightarrow\ \ n(A) = 18.$

So, the required probability $= P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$

(ii) Let B be the event ' getting a total of at least 10'.

Then, $B = \{(4,\ 6),\ (5,\ 5),\ (5,\ 6),\ (6,\ 4),\ (6,\ 5),\ (6,\ 6)\}$

Favourable number of events $= 6\ i.e.,\ n(B) = 6$

So, the required probability $= P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

(iii) Let C be the event 'getting a doublet of even number'.

Then $C = \{(2,\ 2),\ (4,\ 4),\ (6,\ 6)\} \Rightarrow n(C) = 3$

So, the required probability $= P(C) = \frac{n(C)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

(iv) Let D be the event ' getting a multiple of 2 on one die and a multiple of 3 on the other'.

Then, D $= \{(2,\ 3),\ (2,\ 6),\ (4{,}3),\ (4,\ 6),\ (6,\ 3),\ (6,\ 6),\ (3,\ 2),\ (3,\ 4),\ (3,\ 6),\ (6,\ 2),\ (6,\ 4)\}$

$\Rightarrow n(D) = 11$

So, the required probability $= P(D) = \frac{n(D)}{n(S)} = \frac{11}{36}$

(v) Let E be the event ' getting a multiple of 3 as the sum'.

Then $E = \{(1,\ 2),\ (1,\ 5),\ (2,\ 1),\ (2,\ 4),\ (3,\ 3),\ (3,\ 6),\ (4,\ 2),\ (4,\ 5),\ (5,\ 1),$

$(5,\ 4),\ (6,\ 3),\ (6,\ 6)\}$

$\Rightarrow\ \ n(E) = 12$

So, the required probability $= P(E) = \frac{n(E)}{n(S)} = \frac{12}{36} = \frac{1}{3}$

Example: Suppose each child born is equally likely to be a boy or a girl. Consider the family with exactly three children.

(a) List the eight elements in the sample space whose outcomes are all possible gender of three children.

(b) Write each of the following events as a set and find its probability:

(i) The event that exactly one child is a girl.

(ii) The event that at least two children are girls.

(iii) The event that no child is a girl.

Sol: (a) All possible genders of three children are: BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG

So, the sample space S is given by $S =$ {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }.

(b) (i) Let A denote the event ``Exactly one child is a girl''. Then $A = \{BBG,\ BGB,\ GBB\}$

The favourable number of elementary events to A is 3.

Hence, $P(A) = \frac{3}{8}$.

(ii) Let B denote the event that at least two children are girls. Then $B = \{BGG,\ GBG,\ GGB,\ GGG\}$

The favourable number of elementary events to B is 4.

Hence, $P(B) = \frac{4}{8} = \frac{1}{2}$

(iii) Let C denote the event: ``No child is a girl ''. Then,

$C = \{BBB\}$ and a favourable number of elementary events to C is 1.

Hence, $P(C) = \frac{1}{8}$.

Example: Find the probability that a leap year, selected at random, will contain 53 Sundays.

Sol: In a leap year there are 365 days.

365 days = 52 weeks and 2 days.

Thus, a leap year has always 52 Sundays. The remaining 2 days can be;

(i) Sunday and Monday, (ii) Monday and Tuesday, (iii) Tuesday and Wednesday,

(iv) Wednesday and Thursday, (v) Thursday and Friday, (vi) Friday and Saturday,

(vii) Saturday and Sunday.

If S is the sample space associated with this experiment, then S consists of the above seven points.

$\therefore$ Total number of elementary events $= 7$.

Let A be the event that a leap year has 53 Sundays. So that a leap year, selected at random, should contain 53 Sundays, one of the `over' days must be a Sunday. This can be in any one of the following two ways:

(i) Sunday and Monday (ii) Saturday and Sunday

$\therefore$ The favourable number of elementary events $= 2$.

Hence, the required probability $= \frac{2}{7}$.

Example: A coin is tossed. If the head comes up, a die is thrown but if the tail comes up, the coin is tossed again. Find the probability of obtaining:

(i) two tails (ii) head and number 6 (iii) head and even number.

Sol: The sample space S associated with the given random experiment is

$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,H), (T,T)\}$

It has 8 elements.

$\therefore$ Total number of elementary events $= 8$

(i) If the outcome is (T,T), then we say that two balls are obtained.

$\therefore$ Favourable number of elementary events $= 1$

Hence, the required probability $= \frac{1}{8}$

(ii) Head and the number 6 is obtained in only one way $i.e.$ When the outcome is $(H,6)$

$\therefore$ Favourable number of elementary events $= 1$

Hence, the required probability $= \frac{1}{8}$

(iii) Head and an even number can be obtained in any one of the following ways:

$(H, 2), (H, 4), (H, 6)$

Favourable number of elementary events $= 3$

Hence, the required probability $= \frac{3}{8}$.

Example: One card is drawn from a well-shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond (ii), not an ace (iii) a black card $(i.e.,$ a club or a spade) (iv), not a diamond

(v) not a black card

Sol: When a card is drawn from a well-shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is diamond'

The number of elements in set A is 13.

Therefore, $P(A) = \frac{13}{52} = \frac{1}{4}$

$i.e.$ Probability of diamond card $= \frac{1}{4}$

(ii) We assume that the event 'Card drawn is an ace' is B

Therefore 'Card drawn is not an ace' should be B'

We know that $P(B') = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event 'card drawn is black card'

Therefore, the number of elements in the set C=26

$$P(C) = \frac{26}{52} = \frac{1}{2}$$

Thus, the Probability of a black card $= \frac{1}{2}$

(iv) We assume in (i) above that A is the event 'the card drawn is diamond'

So, the event 'the card drawn is not a diamond' may be denoted as A' or `not A'

Now $P(not\ A) = 1 - \frac{1}{4} = \frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C' or `not C'

We know that $P(not\ C) = 1 - P(C) = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, the probability of not a black card $= \frac{1}{2}$

Example: An urn contains 9 red, 7 white and 4 black balls. If two balls are drawn at random, find the probability that:

(i) both the balls are red (ii) one ball is white

(iii) the balls are of the same colour (iv) one is white and other red

Sol: There are 20 balls in the bag out of which 2 balls can be drawn in 20_{C_2} ways. So, the total number of elementary events $= 20_{C_2} = 190$.

(i) There are 9 red balls out of which 2 balls can be drawn in 9_{C_2} ways.

$\therefore$ Favourable number of elementary events $= 9_{C_2} = 36$.

So, required probability $= \frac{36}{190} = \frac{18}{95}$.

(ii) There are 7 white balls out of which one white can be drawn in 7_{C_1} ways. One ball from the remaining 13 balls can be drawn in 13_{C_1} ways. Therefore, one white and one other colour ball can be drawn in $7_{C_1} \times 13_{C_1}$ ways.

So, favourable number of elementary events $= 7_{C_1} \times 13_{C_1}$.

Hence, the required probability $= \frac{7_{C_1} \times 13_{C_1}}{20_{C_2}} = \frac{91}{190}$.

(iii) Two balls drawn are of the same colour means that either both are red or both are white or both are black. Out of 9 red balls two red balls can be drawn in 9_{C_2} ways. Similarly, two white balls can be drawn from 7 white balls in 7_{C_2} ways and two black balls from 4 black balls in 4_{C_2} ways. Therefore,

The number of ways of drawing 2 balls of the same colour $= 9_{C_2} + 7_{C_2} + 4_{C_2} = 36 + 21 + 6 = 63$

$\therefore$ The favourable number of elementary events $= 63$.

So, required probability $= \frac{63}{190}$.

(iv) Out of 7 white balls, one white ball can be drawn in 7_{C_1} ways and out of 9 red balls one red ball can be drawn in 9_{C_1} ways. Therefore,

One white and one red ball can be drawn in $7_{C_1} \times 9_{C_1}$ ways.

So, favourable number of elementary events $= 7_{C_1} \times 9_{C_1} = 63$.

So, required probability $= \frac{63}{190}$.

Example: If the letters of the word $ASSASSINATION$ are arranged at random. Find the probability that

(i) Four $S's$ come consecutively in the word. (ii) Two $I's$ and two $N's$ come together.

(iii) All $A's$ are not coming together. (iv) No two $A's$ are coming together.

Sol: There are 13 letters in the word $ASSASSINATION$ out of which there are $3A's\ 4S's\ 2I's\ 2N's$, one O and one T. These 13 letters can be arranged in a row in $\frac{13!}{3!\,4!\,2!\,1!\,1!\,1!}$ ways.

(i) Considering $4S's$ as one letter there are 10 letters ($3A's\ 4S's\ 2I's\ 2N's$, one O and one T and one letter formed by $4S's$). These 10 letters can be arranged in $\frac{10!}{3!2!\,2!\,1!\,1!\,1!}$ ways.

$\therefore P(4s's \text{ come consecutively}) = \frac{\frac{10!}{3!2!\,2!\,1!\,1!\,1!}}{\frac{13!}{3!\,4!\,2!\,1!\,1!\,1!}} = \frac{4!\times 10!}{13!} = \frac{2}{143}$

(ii) Two I's and two N's can be put together in $\frac{4!}{2!\,2!}$ ways. Considering these 4 letters as one, there are 10 letters which can be arranged in a row in $\frac{10!}{3!\,4!}$ ways.

$\therefore$ Number of arrangements in which two I's and N's come together $= \frac{10!}{3!\,4!} \times \frac{4!}{2!\,2!} = \frac{10!}{3!\,2!\,2!}$

Hence, P(Two I's and two N's come together) $= \frac{\frac{10!}{3!\,2!\,2!}}{\frac{13!}{3!\,4!\,2!\,2!}} = \frac{2}{143}$

(iii) Considering all $A's$ as one letter, there are 11 letters are which can be arranged in a row in $\frac{11!}{4!\,2!\,2!}$ ways.

$\therefore\ P$(All A's come together) $= \dfrac{\frac{11!}{4!\,2!\,2!}}{\frac{13!}{3!\,4!\,2!\,2!}} = \dfrac{1}{26}$

Hence, P(All A's are not coming together) $= 1 - \dfrac{1}{26} = \dfrac{25}{26}$

(iv) Other than 3 $A's$ there are 10 letters ($4S's\ 2I's\ 2N's$, one O and one T). These are 10 letters can be arranged in a row in $\frac{10!}{4!\,2!\,2!}$ ways. In each arrangement of these 10 letters, there are 11 places which can be filled by 3 $A's$ in 11_{C_3}.

$\therefore$ Number of arrangements in which no two $A's$ come together $= \dfrac{10!}{4!\,2!\,2!} \times 11_{C_3}$

$$= \frac{10!}{4!\,2!\,2!} \times \frac{11!}{8!\,3!}$$

Hence, P(No two $A's$ are coming together) $= \dfrac{\frac{10!}{4!\,2!\,2!} \times \frac{11!}{8!\,3!}}{\frac{13!}{3!\,4!\,2!\,2!}} = \dfrac{15}{26}$.

Example: A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Sol: The total number of persons $= 2 + 2 = 4$. Out of these four people, two can be selected in 4_{C_2} ways.

(a) No men in the committee of two means there will be two women in the committee.

Out of two women, two can be selected in $2_{C_2} = 1$ way.

Therefore, P(no man) $= \dfrac{2_{C_2}}{4_{C_2}} = \dfrac{1\times2\times1}{4\times3} = \dfrac{1}{6}$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in 2_{C_1} ways. Together they can be selected in $2_{C_1} \times 2_{C_1}$ ways.

Therefore, P(One man) $= \dfrac{2_{C_1} \times 2_{C_1}}{4_{C_2}} = \dfrac{2\times2}{2\times3} = \dfrac{2}{3}$

(c) Two men can be selected in 2_{C_2} way.

Hence, $P(Two\ men) = \dfrac{2_{C_2}}{4_{C_2}} = \dfrac{1}{4_{C_2}} = \dfrac{1}{6}$

Example: Find the probability that when a hand of 7 cards is drawn from a well-shuffled deck of 52 cards, it contains (i) all Kings (ii) 3Kings (iii) at least 3 kings.

Sol: Total number of possible hands $= 52_{C_7}$

(i) Number of hands with 4 Kings $= 4_{C_4} \times 48_{C_3}$ (other 3 cards must be chosen from the rest 48 cards)

Hence, P(a hand will have 4 Kings) $= \frac{4_{C_4} \times 48_{C_3}}{52_{C_7}} = \frac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards $= 4_{C_3} \times 48_{C_4}$

Therefore, P(3 Kings) $= \frac{4_{C_3} \times 48_{C_4}}{52_{C_7}} = \frac{9}{1547}$

(iii) P(at least 3 King) = P(3 King or 4 kings)

=P(3 Kings) + P(4 Kings) $= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$

Addition Theorem of Probability

THEOREM 1 $(Addition\ Theorem\ for\ two\ events)$ If A and B are two events associated with a random experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

COROLLARY If A and B are mutually exclusive events, then $P(A \cap B) = 0$

$\therefore\ P(A \cup B) = P(A) + P(B)$

This is the addition theorem for mutually exclusive events.

THEOREM 2 $(Addition\ Theorem\ for\ three\ events)$ If A, B, C are three events associated with a random experiment, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

THEOREM 3 Let A and B be two events associated with a random experiment. Then

$(i)\ P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$(ii)\ P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$(iii)\ P(A \cap \bar{B}) \cup P(\bar{A} \cap B) = P(A) + P(B) - 2P(A \cap B)$

THEOREM 4 For any two events A and B, prove that

$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

Example: A and B are two mutually events of an experiment. If $P('not\ A') = 0.65,\ \ P(A \cup B) = 0.65$ and $P(B) = p,\ find\ the\ value\ of\ p.$

Sol: By addition theorem for mutually exclusive events, we have

$P(A \cup B) = P(A) + P(B)$

$\Rightarrow P(A \cup B) = 1 - P('not\ A') + P(B)$

$\Rightarrow \quad 0.65 = 1 - 0.65 + p \ \Rightarrow p = 0.30$

Example: A and B are two non-mutually exclusive events. If $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$, find the values of $P(A \cap B)$ and $P(A \cap \bar{B})$.

Sol: We have $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$

By addition theorem, we have

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \ \frac{1}{2} = \frac{1}{4} + \frac{2}{5} - P(A \cap B)$

$\Rightarrow \ P(A \cap B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{2} = \frac{3}{20}$

$\Rightarrow \ P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{1}{4} - \frac{3}{20} = \frac{1}{10}$

Example: If E and F are two events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E\ and\ F) = \frac{1}{8}$, find

$(i)\ P(E\ or\ F)$ $(ii)\ P(not\ E\ and\ F)$.

Sol: We have,

$P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E\ and\ F) = \frac{1}{8}$

$(i) \quad P(E\ or\ F) = P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$

$(ii)\ P(not\ E\ and\ not\ F) = P(\bar{E} \cap \bar{F})$

$= P(\overline{E \cup F}) = 1 - P(E \cup F) = 1 - \{P(E) + P(F) - P(E \cap F)\}$

$= 1 - \left\{\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right\} = 1 - \frac{5}{8} = \frac{3}{8}$

Example: A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be

(i) red,(ii) yellow, (iii) blue, (iv) not blue, (v) either red or yellow

Sol: There are 9 discs in all so the total number of possible outcomes is 9.

Let the events A, B, C be defined as

A: `the disc drawn is red'

B: `the disc drawn is yellow'

C: `the disc drawn is blue'

(i) The number of red discs $= 4,\ i.e.,\ n(A) = 4$

Hence, $P(A) = \frac{4}{9}$

(ii) The number of yellow discs $= 2,\ i.e.,\ n(B) = 2$

Therefore, $P(B) = \frac{2}{9}$

(iii) The number of blue discs $= 3,\ i.e., n(C) = 3$

Therefore, $P(C) = \frac{3}{9} = \frac{1}{3}$

(iv) The event `not blue' is `not C'. We know that $P(not\ C) = 1 - P(C)$

Therefore, $P(not\ C) = 1 - \frac{1}{3} = \frac{2}{3}$

(v) The event `either red or yellow' may be described by the set `$A\ or\ C$'

Since A and C are mutually exclusive events, we have

$$P(A\ or\ C) = P(A \cup C) = P(A) + P(C) = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

Example: Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) At least one of them will not qualify the examination and

(c) Only one of them will qualify for the examination.

Sol: Let E and F denote the events that Anil and Ashima will qualify the examination respectively. Given that

$P(E) = 0.05,\ P(F) = 0.10$ and $P(E \cap F) = 0.02$

Then, (a) The event `both Anil and Ashima will not qualify the examination' may be expressed ad $E' \cap F'$.

Since E' is `not E', $i.e.$, Anil will not qualify the examination and F' is `not F', $i.e.$, Ashima will not qualify the examination.

Also, $E' \cap F' = (E \cup F)'$ (by De Morgan's Law)

Now, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $P(E \cup F) = 0.05 + 010 - 0.02 = 0.13$

Therefore, $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P(at least one of them will qualify)

$= 1 - P(both\ of\ them\ will\ qualify) \quad = 1 - 0.02 = 0.98$

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) $i.e.$, $E \cap F'$ or $' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) $= P(E \cap F'$ or $E' \cap F) = P(E \cap F') + \mathrm{P}(E' \cap F)$

$= P(E) - P(E \cap F) + P(F) - P(E \cap F)$

$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$