Introduction to Binomial Theorem, Pascal's Triangle

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 08
CHAPTER NAME : BINOMIAL THEOREM

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Learning Objectives:

- Students will be able to learn about binomial and Pascal's triangle.
- Students will be able to learn about binomial theorem for any positive integral index.
- Students will be able to learn the general term from the beginning and end.
- Students will be able to learn about middle term(s) of a binomial expansion.
- Students will be able to learn the equidistant terms.
- Students will be able to implement application oriented skills in their day to day life.

Introduction

An algebraic expression consisting of two terms, connected by $+$ or – sign is called a binomial expression.

- The terms like $a+b, x-3y^2, \frac{2}{x}-\frac{1}{x^2}$ etc are binomial expressions.

The binomial theorem refers to the expansion of integral power of such a binomial $i.e.$, of the form $(x+y)^n, (a+b)^n, (3x+4y)^n$ etc.

In earlier classes, we have already studied that

$(i)\ (a+b)^0 = 1$

$(ii)\ (a+b)^1 = a+b$

$(iii)(a+b)^2 = a^2 + 2ab + b^2$

$(iv)\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(v)(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

The coefficients in the above expansions follow a particular pattern as given below:

The above pattern or structure of numbers is known as Pascal’s triangle. (Blaise Pascal (1623 – 1662))

The above coefficients can be written in a combinatorial form as

$$^{0}C_{0}$$

$$^{1}C_{0} \quad ^{1}C_{1}$$

$$^{2}C_{0} \quad ^{2}C_{1} \quad ^{2}C_{2}$$

$$^{3}C_{0} \quad ^{3}C_{1} \quad ^{3}C_{2} \quad ^{3}C_{3}$$

$$^{4}C_{4} \quad ^{4}C_{1} \quad ^{4}C_{2} \quad ^{4}C_{3} \quad ^{4}C_{4}$$

$$^{5}C_{0} \quad ^{5}C_{1} \quad ^{5}C_{2} \quad ^{5}C_{3} \quad ^{5}C_{4} \quad ^{5}C_{5}$$

$$^{6}C_{0} \quad ^{6}C_{1} \quad ^{6}C_{2} \quad ^{6}C_{3} \quad ^{6}C_{4} \quad ^{6}C_{5} \quad ^{6}C_{6}$$

$$^{7}C_{0} \quad ^{7}C_{1} \quad ^{7}C_{2} \quad ^{7}C_{3} \quad ^{7}C_{4} \quad ^{7}C_{5} \quad ^{7}C_{6} \quad ^{7}C_{7}$$

$$^{8}C_{0} \quad ^{8}C_{1} \quad ^{8}C_{2} \quad ^{8}C_{3} \quad ^{8}C_{4} \quad ^{8}C_{5} \quad ^{8}C_{6} \quad ^{8}C_{7} \quad ^{8}C_{8}$$

(Pascal triangle)

For $n = 7$, the row is $7_{C_0}, 7_{C_1}, 7_{C_2}, 7_{C_3}, 7_{C_4}, 7_{C_5}, 7_{C_6}, 7_{C_7}$.

We have

$$(a+b)^7 = 7_{C_0}a^7 + 7_{C_1}a^6b + 7_{C_2}a^5b^2 + 7_{C_3}a^4b^3 + 7_{C_4}a^3b^4 + 7_{C_5}a^2b^5 + 7_{C_6}ab^6 + 7_{C_7}b^7$$

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Binomial Theorem for positive integral index

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Introduction

Binomial theorem is a mathematical formula which enables to determine the power or root of a binomial is the form of a series.

If a and b are any two real numbers, then for any positive integer n, we have

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_n\, a^0 b^n$$

$$\Rightarrow\ (a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_n\, b^n = \sum_{r=0}^{n} n_{C_r}\, a^{n-r} b^r$$

where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, … , ${}^nC_n$ are called binomial coefficients.

Some Observations

- The total number of terms in the expansion of $(a+b)^n$ is $n+1$.
- In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a+b$ $i.e., n$.
- In the successive terms of the expansion, powers of the first quantity a go on decreasing by 1 whereas the powers of the second quantity b increase by 1.
- The binomial coefficients of terms equidistant from the beginning and end are equal.

Particular Cases

- Expansion of $(a-b)^n$:

We have

$$(a-b)^n = n_{C_0}a^n - n_{C_1}a^{n-1}b + n_{C_2}a^{n-2}b^2 - \cdots + (-1)^r\, n_{C_r}\, a^{n-r}b^r + \cdots + (-1)^n n_{C_n} b^n$$

- Expansion of $(1+x)^n$:

Replacing a by 1 and b by x , we have

$$(1+x)^n = n_{C_0} + n_{C_1}x + n_{C_2}x^2 + \cdots + n_{C_r}x^r + \cdots + n_{C_n}x^n$$

- Expansion of $(1-x)^n$:

Replacing x by $-x$, we get

$$(1-x)^n = n_{C_0} - n_{C_1}x + n_{C_2}x^2 + \cdots + (-1)^r\, n_{C_r}x^r + \cdots + (-1)^n\, n_{C_n}x^n$$

Remember:

1. If n is odd, then $(a+b)^n+(a-b)^n$ and $(a+b)^n-(a-b)^n$, both have $\left(\frac{n+1}{2}\right)$ terms.

2. If n is even, then $(a+b)^n+(a-b)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(a+b)^n-(a-b)^n$ has $\frac{n}{2}$ terms.

Example: Find the number of terms in the expansion of the following:

$i)\ (2x - 3y)^9$

$(ii)\ (1 + z)^4$

$(iii)\ (1 + 5\sqrt{2}x)^9 + (1 - 5\sqrt{2}x)^9$

$(iv)\ (1 + 2x + x^2)^{20}$

Example: Expand $(x^2 + 2y)^5$ by binomial theorem.

Example: Using the binomial theorem, expand $\left(\frac{2x}{3} - \frac{3}{2x}\right)^4$.

Example: Using the binomial theorem, expand $\{(x + y)^5 + (x - y)^5\}$ and hence find the value of

$\left\{(\sqrt{2} + 1)^5 + \left(\sqrt{2} - 1\right)^5\right\}$.

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General and Middle Term(s) in Binomial Expansion

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General Term

In the expansion $(a + b)^n$, the $(r + 1)th$ term is called the general term and is denoted by T_{r+1}.

Thus $\boldsymbol{T_{r+1} = n_{C_r}\, a^{n-r} b^r}$

Example: Find the 9th term in the expansion of $\left(\frac{x}{a}+\frac{3a}{x^2}\right)^{12}$.

Example: Find the 6th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{9}$.

Example: Find the coefficient of 5th term in the expansion of $\left(\frac{x}{3}-3y\right)^{7}$.

Example: Find a, if 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal.

Middle Term(s)

There are two cases:

Case – 1: If n is even, then the total number of terms $(i.e., n+1)$ is odd. In this case, there is only one middle term which is the $\left(\frac{n}{2}+1\right) th$ term.

Case – 2: If n is odd, then the total number of terms $(i.e., n+1)$ is even. In this case, there are two middle terms, which are $\left(\frac{n+1}{2}\right)$ th term and $\left(\frac{n+3}{2}\right) th$ term.

Example: Find the middle term in the expansion of $\left(\frac{2x^2}{3} - \frac{3}{2x}\right)^{20}$.

Example: Find the middle terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^{7}$.

Example: If the middle term of $\left(\frac{1}{x} + x\, sinx\right)^{10}$ is equal to $7\frac{7}{8}$, then find the value of x.

Example: Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1\,.3\,.5\,.....(2n-1)}{n!}\;2^n x^n$, where n is a positive integer.

Example: Prove that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of the middle terms in the expansion of $(1+x)^{2n-1}$.

Example: Find the value of k for which the coefficients of the middle terms in the expansion of $(1+kx)^4$ and $(1-kx)^6$ are equal.

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Equidistant Terms and problems on term independent of x

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Equidistant Terms

In the binomial expansion of $(a+b)^n$, the $(r+1)th$ term from the end is $\{(n+1)-r\} = (n-r+1)th$ term from the beginning.

The $(r+1)th$ term from the end in the expansion of $(a+b)^n$ is same as the $(r+1)th$ term from the beginning in the expansion of $(b+a)^n$.

Example: Find the 11^{th} term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.

Example: Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6} : 1$.

Example: Find the term independent of x in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$.

Example: Find the value of a so that the term independent of x in $\left(\sqrt{x} + \frac{a}{x^2}\right)^{10}$ is 405.

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Simple Applications of Binomial Theorem

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Example: Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Example: Which is larger $(1.01)^{1000000}$ or, $10{,}000$?

Example: Using, binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

Example: Using, binomial theorem, prove that $8^n - 7n - 1$ is divisible by 49.

Example: Find the coefficient of x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, when $x \neq 0$.

Example: Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Example: Find the coefficient of x^{40} in the expansion off $(1 + 2x + x^2)^{27}$.

Example: Prove that there is no term involving x^6 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, where $x \neq 0$.

Example: The coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio $1:7:42$. Find n.

Example: If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in $A.P.$, prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

Example: If O be the sum of odd terms and E that of even terms in the expansion of $(x+a)^n$, prove that

$(i)\ O^2 - E^2 = (x^2 - a^2)^n$

$(ii)\ 4OE = (x+a)^{2n} - (x-a)^{2n}$

$(iii) 2(O^2 + E^2) = (x+a)^{2n} + (x-a)^{2n}$

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