Introduction to Imaginary Numbers

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 05
CHAPTER NAME : COMPLEX NUMBERS AND QUADRATIC EQUATIONS

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Introduction:

The equations of the form $x^2 + 1 = 0, x^2 + 4 = 0$ etc. are not solvable in R

$i.e.$ There is no real number whose square is a negative real number.

Need for Complex Numbers

Quadratic equations of the form $ax^2 + bx + c = 0$, where $a, b, c \in R, a \neq 0$ and $b^2 - 4ac < 0$

whose solution is not possible in the set of real numbers.

It was in the 16^{th} century that the Italian Mathematicians **Cardano** and **Bombelli** started a serious discussion on extending the number system to include square roots of negative numbers.

Contd.....

Consider the equation

$x^2 + 1 = 0$

$\Rightarrow \ x^2 = -1$

$\Rightarrow x = \sqrt{-1} = i$

In 1977, the Swiss Mathematician **Euler** was the first mathematician to introduce the symbol $i(iota)$ for the square root of -1

$i.e.$ a solution of $x^2 + 1 = 0$ is i with the property $i^2 = -1$.

He also called this symbol as the **imaginary unit.**

Integral powers of IOTA (i)

We have $i = \sqrt{-1}$

$i^2 = -1, \qquad i^3 = i^2.i = (-1)i = -i, \qquad i^4 = (i^2)^2 = (-1)^2 = 1$

To compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r.

Let m be the quotient when n is divided by 4. Then, $n = 4m + r$ where $0 \le r < 4$

$\Rightarrow \quad i^n = i^{4m+r} = (i^4)^m i^r = i^r$

Now, $i^{4n} = (i^4)^n = 1^n = 1$

$i^{4n+1} = i^{4n}.i = 1.i = i$

$i^{4n+2} = i^{4n}.i^2 = 1 \times (-1) = -1$

$i^{4n+3} = i^{4n}.i^3 = 1 \times (-i) = -i$ where $n \in N$.

Note: $i^0 = 1$.

Example: Evaluate the following:

$(i)\ i^{135} = ?$

$(ii)\ i^{457} = ?$

$(iii)\ i^{-998} = ?$

Example: Show that

$(i)\ \left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^{2} = -4$

$(ii)\ \ i^{n} + i^{n+1} + i^{n+2} + i^{n+3} = 0$, for all $n \in N$.

Example: Evaluate the following.

$(i)\ 1 + i^{10} + i^{100} + i^{1000}$

$(ii)\ i.i^2.i^3.i^4.\ldots.i^{1000}$

$$(iii)\ \frac{i^{582} + i^{584} + i^{586} + i^{588} + i^{590}}{i^{592} + i^{594} + i^{596} + i^{598} + i^{600}}$$

$$(iv)\ \sum_{n=1}^{13} (i^n + i^{n+1})$$

(v) If n is an odd positive integer, then prove that $i^n + i^{2n} + i^{3n} + i^{4n} = 0$.

Imaginary Quantities

If $x^2 + 4 = 0 \Rightarrow x = \sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4}\sqrt{-1} = \pm 2i$

The product of a real number and an imaginary unit is called an imaginary number.

Ex: $2i,\ -3i,\ \frac{7}{4}i,\ \sqrt{2}i$ are imaginary numbers.

The square of a real number is always non-negative, but the square of an imaginary number is always negative.

Ex: $(2i)^2 = 4i^2 = -4$

$(-\sqrt{7}i)^2 = 7i^2 = -7$

Note:

- For any positive real number a, we have $\sqrt{-a} = \sqrt{(-1) \times a} = i\sqrt{a}$
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is not true when both a and b negative real numbers.

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Standard Form of Complex Numbers and Equality of Complex Numbers

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Introduction

Consider an equation: $x^2 - 4x + 13 = 0$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i = 2 + 3i,\ 2 + (-3)i$$

Definition:

Any number which can be expressed as in the form $x + iy$, where $x, y \in R$ is called a complex number.

Or, the sum of a real number and an imaginary number is called a complex number.

The set of all complex numbers is denoted by C.

$i.e. C = \{z = x + iy : x, y \in R\}$

Contd….

Let $z = x + iy \in C$

Here 'x ' is called the real part and 'y'is called the imaginary part of z

$i.e. Re\, z = x$ and $Im\, z = y$

If $y = 0$, then $z = x$, which is purely real.

If $x = 0$, then $z = iy$, which is purely imaginary.

Since a real number 'x ' can be written as $x + i0$,

so every real number is a complex number.

Hence $R \subset C$.

Also, every imaginary number is a complex number.

Equality of Complex Numbers:

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$

$i.e. Re\,(z_1) = Re\,(z_2)$ and $Im(z_1) = Im(z_2)$

Example: If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y.

Example: Find x and y if $(x + y) + 3i = -7 + (x - y)i$

Example: Which is greater: 3+2i or 2+3i ?

Note:

- Complex numbers are neither positive nor negative.
- Complex numbers cannot be compared.

Examples

Example:

1.Find the values of x and y if $(x+iy)\,(2-3i)=4+i$

2. $\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$

3. If $a+ib=\frac{c+i}{c-i}$, where c is real, prove that $a^2+b^2=1$ and $\frac{b}{a}=\frac{2c}{c^2-1}$

4. Find the smallest positive integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

Assignments

1. What is the smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$?

2. Find the real value of 'a' for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real.

3. (iv) If $(x+iy)^{\frac{1}{3}} = a + ib, x, y, a, b \in R$, then show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$

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Algebra of Complex Numbers

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Algebra of Complex Numbers

1. Closure Law: Let $z_1, z_2 \in C$ such that $z_1 = a + ib$ and $z_2 = c + id$

Addition: $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d) \in C$

Subtraction: $z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d) \in C$

Multiplication: $z_1 . z_2 = (a + ib).(c + id) = ac + iad + ibc - bd$

$= (ac - bd) + i(ad + bc) \in C$

Division: $\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac-iad+ibc+bd}{c^2+d^2}$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} \in C$$

Since the sum, difference, product, and quotient of any two complex numbers is a complex number ,

Hence the set of complex numbers is closed under addition, subtraction, multiplication, and division.

CONTD….

2. Commutative Laws:

If $z_1, z_2 \in C$ then $z_1 + z_2 = z_2 + z_1$ and $z_1.z_2 = z_2.z_1$

3. Associative Laws:

If $z_1, z_2, z_3 \in C$ then $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1.(z_2.z_3) = (z_1.z_2).z_3$

4. Distributive Laws:

If $z_1, z_2, z_3 \in C$ then $z_1(z_2 + z_3) = (z_1z_2 + z_1z_3)$

5. Existence of Additive Identity:

If $z \in C$, then there exists $0 = 0 + i0 \in C$ such that $z + 0 = z = 0 + z$

So $0 + i0$ is called the additive inverse.

CONTD…

6. Existence of Additive Inverse:

If $z \in C$, then there exists $-z \in C$ such that $z + (-z) = 0 = (-z) + z$

So, $-z$ is called the additive inverse of z.

7. Existence of Multiplicative Identity:

If $z \in C$, then there exists $1 = 1 + i0 \in C$ such that $z\,.1 = z = 1.z$.

So, $1 + i0$ is called the multiplicative identity.

8. Existence of Multiplicative Inverse:

If $z \neq 0 \in C$, then there exists $\frac{1}{z} \in C$ such that $z.\frac{1}{z} = 1 = \frac{1}{z}.z$. So $\frac{1}{z} = z^{-1}$ is called the multiplicative inverse or reciprocal of z.

Examples

Example:

1. Express the complex numbers in the form $x + iy$: $(-3i)\left(\frac{1}{9}i + 2\right)$

2. Find the additive and multiplicative inverse of the complex number $z = (2 + \sqrt{3}i)^2$

3. If $x = -5 + 2\sqrt{-4}$, then find the value of $x^4 - 9x^3 + 35x^2 - x + 4$.

Assignments

1. Express in the standard form of $(1+i)^4$

2. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

3. Show that $(3+i)^{-2} + (3-i)^{-2} = \frac{4}{25}$

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Conjugate and Modulus of Complex Numbers

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Conjugate of a Complex Number:

If $z = x + iy$, then the conjugate of z denoted by $\bar{z}$ and is defined by $\overline{z} = x - iy$.

It follows from the definition that the conjugate of a complex number is obtained by replacing i by $-i$.

For example, if $x = 3 + 4i$, then $\overline{z} = 3 - 4i$.

Example: Find the conjugate of the following complex numbers.

$(i) z = -3 + 4i \Rightarrow \overline{z} = -3 - 4i$

$(ii)\ z = -\sqrt{7} - \sqrt{3}i \Rightarrow \overline{z} = -\sqrt{7} + \sqrt{3}i$

$(iii)\ z = -\frac{1}{2}i + 5 \Rightarrow \overline{z} = 5 + \frac{1}{2}i$

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Properties:

(i) The conjugate of a real number is the number itself.

Let $z = x = x + i0 \Rightarrow \overline{z} = x - i0 = x$. So $z = \overline{z}$.

(ii) The double conjugate of a complex number is the number itself.

Let $z = x + iy \Rightarrow \overline{z} = x - iy \Rightarrow \overline{\overline{z}} = x + iy$. So, $\overline{\overline{z}} = z$.

$$(iii)\ \mathrm{z} + \overline{z} = 2\ Re(z)$$

$$(iv)\ z - \overline{z} = 2\ i\ Im(z)$$

(v) If $z + \overline{z} = 0$, then z is purely imaginary.

(vi) If $z - \overline{z} = 0$, then z is purely real.

(vii) $z\,.\,\overline{z} = \ (Rez)^2 + (Imz)^2$

$(viii)$ If z_1, z_2 are two complex numbers, then

$(a)\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$(b)\ \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

$(c)\ \overline{z_1 . z_2} = \overline{z_1}\ .\ \overline{z_2}$

$(d)\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}},\ z_2 \neq 0.$

Example: Express the following complex numbers in the form $x + iy$.

$(i)\ \frac{1}{3-4i}$

Sol: $z = \frac{1}{3-4i} = \frac{(3+4i)}{(3-4i)(3+4i)} = \frac{3+4i}{9+16} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}\ i$

$(ii)\ \frac{1+2i}{2+i}$

Sol: $z = \frac{1+2i}{2+i} = \frac{(1+2i)(2-i)}{(2+i)(2-i)} = \frac{2-i+4i+2}{4+1} = \frac{4+3i}{5} = \frac{4}{5} + \frac{3}{5}i$

$(iii)\ \frac{(3-2i)(2+3i)}{(2+5i)(5-2i)}$

Sol: $z = \frac{(3-2i)(2+3i)}{(2+5i)(5-2i)} = \frac{6+9i-4i+6}{10-4i+25i+10} = \frac{12+5i}{20+21i} = \frac{(12+5i)(20-21i)}{(20+21i)(20-21i)}$

$= \frac{240-252i+100i+105}{400+441} = \frac{345-152i}{841} = \frac{345}{841} - \frac{152}{841}i$

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$(iv)\quad \left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

Sol: $z = \left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

$= \left(\frac{1+i+3-6i}{1+i-2i+2}\right)\left(\frac{3+4i}{2-4i}\right) = \left(\frac{4-5i}{3-i}\right)\left(\frac{3+4i}{2-4i}\right)$

$= \frac{12+16i-15i+20}{6-12i-2i-4}$

$= \frac{32+i}{2-14i} = \frac{(32+i)(2+14i)}{(2-14i)(2+14i)}$

$= \frac{64+448i+2i-14}{4+196} = \frac{50+450i}{200}$

$= \frac{50}{200}+\frac{450}{200}i = \frac{1}{4}+\frac{9}{4}\ i$

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Example: Express the following complex numbers in the standard form. Also find their conjugate.

$(i)\ \ z = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-1-2i}{1+1} = -\frac{2i}{2} = -i$

$\Rightarrow \quad \bar{z} = i$

$(ii)\ z = \frac{(2+3i)^2}{2-i} = \frac{4-9+12i}{2-i} = \frac{-5+12i}{2-i} = \frac{(-5+12i)(2+i)}{(2-i)(2+i)} = \frac{-10-5i+24i-12}{4+1}$

$= \frac{-22+19i}{5} = -\frac{22}{5} + \frac{19}{5}i$

$\Rightarrow \quad \bar{z} = -\frac{22}{5} - \frac{19}{5}i$

$(iii)\ z = \ (\sqrt{7} + 5i)^2 = 7 - 25 + 10\sqrt{7}i = -18 + 10\sqrt{7}i$

$\Rightarrow \bar{z} = -18 - 10\sqrt{7}i = \ (\sqrt{7} - 5i)^2$

Example: Find the values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugates of each other.

Sol: Since the given complex numbers are conjugate of each other, so

$$-3 + ix^2y = \overline{(x^2 + y) + 4i} \quad \Rightarrow -3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts

$x^2 + y = -3$ ------ (1)

and $x^2y = -4$ ------ (2)

From (2), we get $y = -\frac{4}{x^2}$

Then from (1), we get $x^2 - \frac{4}{x^2} = -3 \Rightarrow x^4 - 4 = -3x^2 \Rightarrow x^4 + 3x^2 - 4 = 0$

$\Rightarrow \quad (x^2 - 1)(x^2 + 4) = 0 \Rightarrow x^2 - 1 = 0$ or $x^2 + 4 = 0.$

Since x is real so $x^2 + 4 \neq 0.$ Thus $x^2 - 1 = 0 \Rightarrow x = \pm 1$

When $x = \pm 1, y = -\frac{4}{1} = -4$

Example: If $z_1 = 2 + 3i,\ z_2 = 3 - 4i$, then prove the following

$(i)\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Sol: L.H.S. $= \overline{z_1 + z_2} = \overline{(2 + 3i) + (3 - 4i)} = \overline{5 - i} = 5 + i$

R.H.S.$= \overline{z_1} + \overline{z_2} = \overline{2 + 3i} + \overline{3 - 4i} = 2 - 3i + 3 + 4i = 5 + i$

$\therefore$ L.H.S. = R.H.S.

$(ii)\ \overline{z_1 . z_2} = \overline{z_1} \ . \ \overline{z_2}$

Sol: L.H.S. $= \overline{z_1 . z_2} = \overline{(2+3i)(3-4i)} = \overline{6-8i+9i+12}$

$= \overline{18+i} = 18 - i$

R.H.S. $= \overline{z_1} \ . \ \overline{z_2} = \left(\overline{2+3i}\right)\left(\overline{3-4i}\right) = (2-3i)(3+4i)$

$= 6 + 8i - 9i + 12 = 18 - i$

$\therefore$ L.H.S. $=$ R.H.S.

$(iii)\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Sol: L.H.S. $= \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{2+3i}{3-4i}\right)} = \overline{\left(\frac{(2+3i)(3+4i)}{9+16}\right)} = \overline{\left(\frac{6+8i+9i-12}{25}\right)} = \overline{\left(\frac{-6+17i}{25}\right)}$

$= \overline{\left(-\frac{6}{25} + \frac{17}{25}i\right)} = -\frac{6}{25} - \frac{17}{25}i$

R.H.S. $= \frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{2+3i}}{\overline{3-4i}} = \frac{2-3i}{3+4i} = \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i-9i-12}{9+16} = \frac{-6-17i}{25} = -\frac{6}{25} - \frac{17}{25}i$

$\therefore$ L.H.S. $=$ R.H.S.

Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{z} = \sqrt{x^2 + y^2} = \sqrt{(Re\ z)^2 + (Im\ z)^2}$$

For example, if $z = 3 - 4i$, then $|z| = \sqrt{3^2 + (-4)^2} = 5$

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Properties of Modulus

Let z, z_1, z_2 are complex numbers. Then

(i) $|z| = 0 \Rightarrow Re z = 0,\ Im\, z = 0.$

(ii) $|z| = |\overline{z}| = |-z|$

(iii) $z.\overline{z} = |z|^2$

$(iv)|z_1 . z_2| = |z_1||z_2|$

(v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0.$

$(vi)|z_1 + z_2| \leq\ |z_1| + |z_2|$ (Triangle Inequality)

$(vii)|z_1 + z_2|^2 =\ |z_1|^2 + |z_2|^2 + 2\ Re(z_1.\overline{z_2})$

$(viii)\ |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\ Re(z_1.\overline{z_2})$

$(ix)\ |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(\ |z_1|^2 + |z_2|^2)$

$(x)|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(\ |z_1|^2 + |z_2|^2)$

Note:

$(i)\ |z_1 - z_2| \leq |z_1| + |z_2|$

$(ii)\ |z_1 - z_2| \geq \left||z_1| - |z_2|\right|$

$(iii)\ |z_1 - z_2| \geq |z_1| - |z_2|$

Example: If $z = x + iy$, show that $|x| + |y| \leq \sqrt{2}|z|$.

Sol: We have $z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

Now $(|x| - |y|)^2 \geq 0$

$\Rightarrow \quad |x|^2 + |y|^2 - 2|x||y| \geq 0$

$\Rightarrow \quad |x|^2 + |y|^2 \geq 2|x||y|$

$\Rightarrow \quad x^2 + y^2 \geq 2|x||y|$

$\Rightarrow \quad 2(x^2 + y^2) \geq x^2 + y^2 + 2|x||y|$

$\Rightarrow \quad 2|z|^2 \geq (|x| + |y|)^2$

$\Rightarrow \quad |x| + |y| \leq \sqrt{2}|z|$.

Example: If $z = x + iy$ and $|2z - 1| = |z + 1|$, show that $x^2 + y^2 = 2x$

Sol: We have $z + 1 = (x + iy) + 1 = (x + 1) + iy$

Also $2z - 1 = 2(x + iy) - 1 = (2x - 1) + i(2y)$

$\therefore\ |2z - 1| = |z + 1|$

$\Rightarrow\ \sqrt{(2x-1)^2 + (2y)^2} = \sqrt{(x+1)^2 + y^2}$

$\Rightarrow\ (2x-1)^2 + 4y^2 = (x+1)^2 + y^2$

$\Rightarrow\ (2x-1)^2 - (x+1)^2 = y^2 - 4y^2$

$\Rightarrow\ 3x^2 - 6x + 3y^2 = 0 \Rightarrow x^2 - 2x + y^2 = 0 \Rightarrow x^2 + y^2 = 2x$

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Example: Find x if $(1-i)^x = 2^x$.

Sol: We have $(1-i)^x = 2^x$

Taking modulus of both sides we get $|(1-i)^x| = |2^x|$

$\Rightarrow \quad |1-i|^x = |2|^x$

$\Rightarrow \quad \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x \Rightarrow \left(\sqrt{2}\right)^x = 2^x$

$\Rightarrow \quad \Rightarrow 2^{\frac{x}{2}} - 2^x = 0 \Rightarrow 2^{\frac{x}{2}}\left(1 - 2^{\frac{x}{2}}\right) = 0 \Rightarrow 1 - 2^{\frac{x}{2}} = 0$ [$since\ 2^{\frac{x}{2}} \neq 0$]

$\Rightarrow \quad 2^{\frac{x}{2}} = 1 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$

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Geometrical and Polar Representation of Complex Numbers

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Geometrical Representation of a Complex Number

We know that a real number can be represented geometrically on the number line.

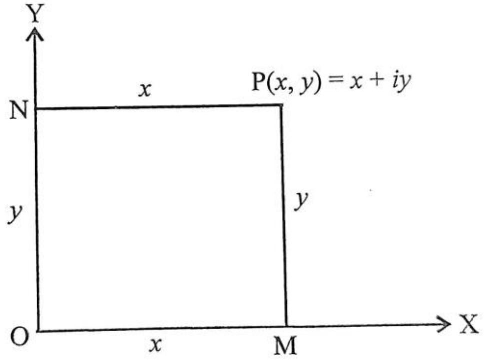
A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane.

To represent $z = x + iy$ geometrically , plot a point whose x and y coordinates are respectively real and imaginary parts of z.

This point $P(x, y)$ represents the complex number $z = x + iy$.

$x - axis$ is known as the real axis

$y - axis$ is known as the imaginary axis.



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Conversely, if $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the affix of the point P.

For every complex number $z = x + iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or **Gaussian plane**. The point P, plotted on the Argand plane is called the Argand diagram.

Modulus and Argument

Let $z = x + iy$ be a complex number

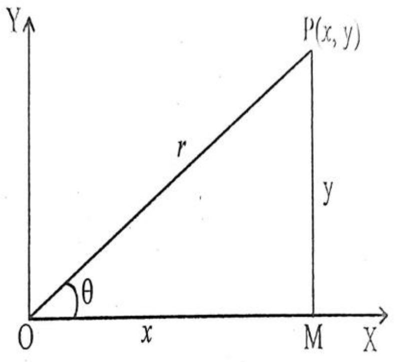
The length of the line segment OP is called the **modulus** of z i.e $|z|$

Here, $OM = x, MP = y$

In the right-angled triangle OMP,

$$|OP| = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2} = r(say)$$

Thus, if $= x + iy$, then $|z| = r = \sqrt{(Rez)^2 + (Imz)^2}$



Geometrically, $|z|$ is the distance of z from the origin.
The angle θ which OP makes with the positive direction of $x - axis$ in an anticlockwise sense is called the **argument** or **amplitude** of z and is denoted by $\arg(z)$ or $amp(z)$.

In the right-angled triangle $OMP, tan\theta = \frac{y}{x}$. So $\arg z = \theta$.

Continued….

The unique value of θ such that $-\pi < \theta \leq \pi$ is called the principal value of the amplitude or principal argument.

The argument of z depends upon the quadrant in which the point P lies.

Techniques to determine the principal argument

Let $z = x + iy$

Step I: Find the acute angle α given by $\tan \alpha = \left|\frac{y}{x}\right|$

Step II: If

(i) $x > 0, y = 0$, then $\theta = 0$.

$(ii)x > 0, y > 0$, then $\theta = \alpha$

Continued……

$(iii)\ x = 0, y > 0$, then $\theta = \frac{\pi}{2}$

$(iv) x < 0, y > 0$, then $\theta = \pi - \alpha$

$(v)\ x < 0, y = 0$, then $\theta = \pi$

$(vi) x < 0, y < 0$, then $\theta = -(\pi - \alpha)$

$(vii)\ x = 0, y < 0$, then $\theta = -\frac{\pi}{2}$

$(viii) x > 0, y < 0$, then $\theta = -\alpha$

Example

Find the modulus and argument of the following complex number :

$1 + i\sqrt{3}$

Polar Form of a Complex Number

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane.

In $\Delta\, POM$, we have $os\theta = \frac{OM}{OP} = \frac{x}{r}$, $sin\theta = \frac{PM}{OP} = \frac{y}{r}$

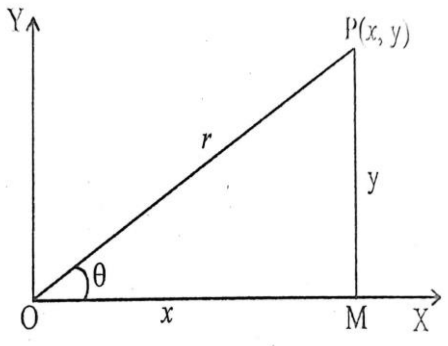
$\therefore z = x + iy \Rightarrow z = r\, cos\theta + i\, r\, sin\theta$

$= r\, (cos\theta + i\, sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z.

Note: $z = r\, e^{i\theta}$ is called exponential form,

where $e^{i\theta} = \cos\theta + i\, sin\theta$



Multiplication of a Complex Number by IOTA

Let $z = x + iy = r(cos\theta + i\ sin\theta)$

Then $r = |z|$ and $\arg(z) = \theta$

Now $iz = i\ r(cos\theta + i\ sin\theta) = r(-sin\theta + i\ cos\theta)$

$$= r\left\{\cos\left(\frac{\pi}{2} + \ \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right\}$$

Thus iz is a complex number such that

$|iz| = r = |z|$ and $\arg(iz) = \frac{\pi}{2} + \ \theta = \frac{\pi}{2} + \arg(z)$

So, the multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

Examples..

Write the following complex numbers in the polar form:

$(i)\ \sqrt{\mathbf{3}} + \boldsymbol{i}$ (ii) $-\mathbf{2}\boldsymbol{i}$

Properties of Modulus

(i) If $z = x + iy, then\ |z| = \sqrt{x^2 + y^2}$

ii) z $\bar{z} = |z|^2$

iii) $|z_1 . z_2| = |z_1||z_2|$

iv) $\left|\frac{z1}{z2}\right| = \frac{|z1|}{|z2|}$

Properties of Argument

$(i)\ \arg(z_1.z_2) = \arg z_1 + \arg z_2$

$(ii)\ \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\ , z_2 \neq 0$

$(iii)\ \arg z^n = n \arg z$

$(iv)\ \arg\left(\frac{z}{\bar{z}}\right) = 2 \arg z$

$(v)\ \arg \bar{z} = -\arg z$

$(vi)\ \arg(-z) = \arg\big((-1)z\big) = \arg(-1) + argz = \pi + argz$

$(vii)\ \arg(iz) = \arg(i) + argz = \frac{\pi}{2} + argz$

Assignments

1. Write the complex numbers in the polar form: $\mathbf{1-i}$

2. Write the complex number $\frac{1+i}{1-i}$

3. If z_1, z_2 are two complex numbers, then prove the following.

$|z_1.z_2| = |z_1||z_2|$

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Solutions of Quadratic Equation in set of Complex Numbers

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 05
CHAPTER NAME : COMPLEX NUMBERS AND QUADRATIC EQUATIONS

Solutions of Quadratic Equation in the set of Complex Numbers

The equation $ax^2 + bx + c = 0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x.

If $b^2 - 4ac < 0$, then the solution is given in the set of complex numbers.

Complex roots of an equation with real coefficients always occur in conjugate pairs. However, this may not be true in the case of equations with complex coefficients.

Fundamental Theorem of Algebra

We know that every polynomial equation $f(x) = 0$ has at least one root, real or imaginary(complex).

The theorem states that " A polynomial equation of degree n has n roots.

Quadratic Equations with real Coefficients

Complex roots of an equation with real coefficients always occur in conjugate pairs like $2 + 3i$ and $2 - 3i$.

However, this may not be true in the case of equations with complex coefficients.

For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i .

Example: Solve each of the following equations.

(i) $\mathbf{4x^2 + 9 = 0}$

(ii) $\mathbf{x^2 - 4x + 13 = 0}$

Quadratic Equations with Complex Coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$ … (1)

where a, b, c are complex numbers and $a \neq 0$.

So the roots are complex numbers.

Since the order relation is not defined in case of complex numbers, therefore, we cannot assign positive or negative sign to the discriminant $D = b^2 - 4ac$.

However, equation (1) has complex roots which are equal, if $D = b^2 - 4ac = 0$ and unequal roots if $D = b^2 - 4ac \neq 0$.

Examples

(1) Solve the following equation: $\boldsymbol{x^2 - 5ix - 6 = 0}$

(2) One root of the equation $\boldsymbol{ax^2 - 3x + 1 = 0}$ is 2+i, find the value of a .

(3) Solve $ix^2 - 4x - 4i = 0$

Assignments

Solve each of the following equations.

(1) $\sqrt{5}\,x^2 + x + \sqrt{5} = 0$

(2) $ix^2 - 4x - 2i = 0$

(3) $x^2 - (7 - i)x + (18 - i) = 0$

(4) ***Find the quadratic equation with real coefficients whose one root is (1-i).***

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