Measuring of Angles in Radians and Degrees

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
CHAPTER NAME : TRIGONOMETRIC FUNCTIONS

Introduction:

The word trigonometry is derived from the Greek words “**trigon**” and “**metron**” means the measurement of triangles. It is the branch of mathematics that deals with the measurement of the sides and the angles of a triangle and the problems allied with angles.

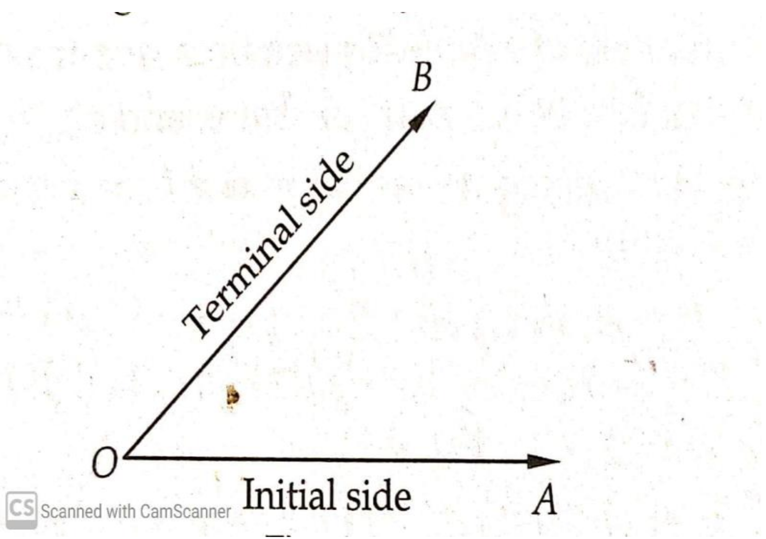
The Hindu mathematicians Aryabhatta, Varahamihira, Brahmagupta, and Bhaskara have a lot of contributions to Trigonometry.

Currently, Trigonometry is used in many areas such as the science of seismology, designing electrical circuits, describing the shape of an atom, predicting the height of the tide in the ocean, analyzing musical tones, and studying the occurrence of sunspots, forecasting fluctuations in the stock market.

Angles:

An angle is considered as the figure obtained by rotating a given ray about its endpoint.

The revolving ray is called the generating line of the angle. The initial position OA is called the initial side and the final position OB is called the terminal side. The endpoint O is called the vertex of the angle.



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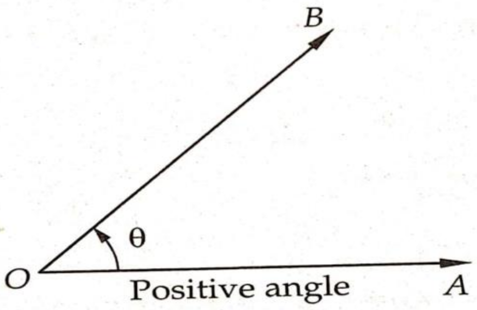
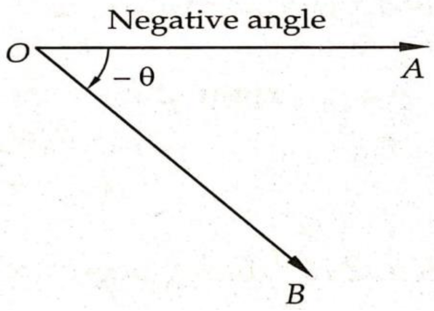
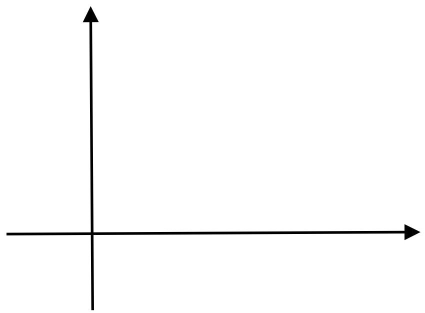
Measure of an Angle: The measure of an angle is the amount of rotation performed to get the initial side.

Sense of an Angle: The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.





Right Angle: If the revolving ray starting from its initial position to final position describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.



Systems of Measurement of Angles:

There are three systems for measuring angles

(i) **Sexagesimal or English System or Degree measure:**

In this system, the unit of measurement is degree.

1 right angle $= 90$ degrees$(90°)$

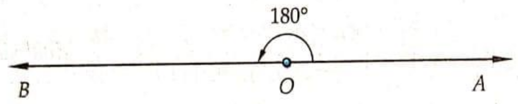
$1° = 60$ minutes$(60')$

$1' = 60$ seconds$(60'')$

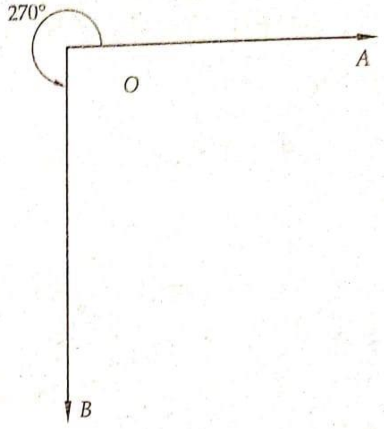
If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one degree.

So 1 complete rotation $= 360°$

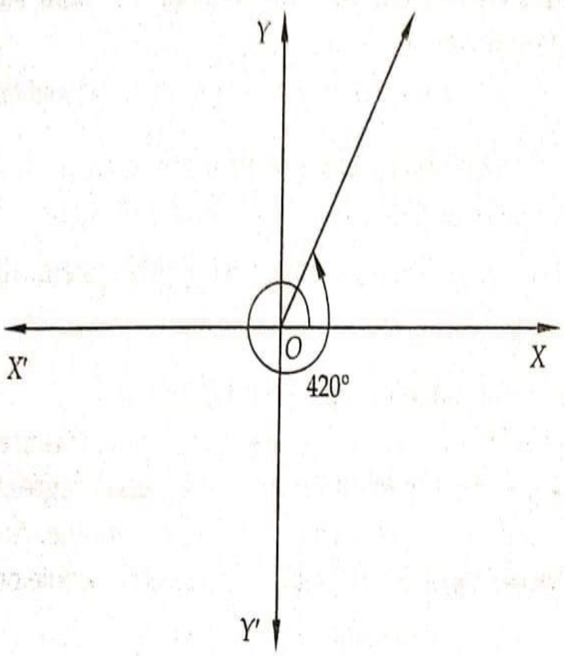
The angles of measures $180°, 270°, 420°, -420°, -30°$ are shown in the following figures.



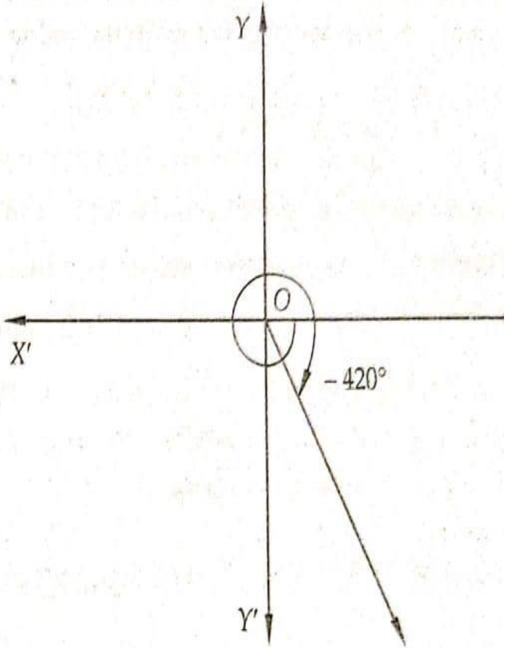
(i) Angle of measure 180°



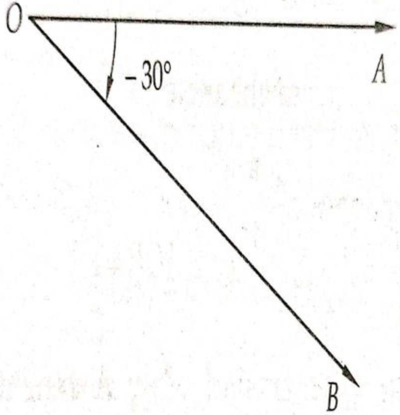
(ii) Angle of measure 270°



(iii) Angle of measure 420°



(iv) Angle of measure – 420°



(v) Angle of measure – 30°

(ii) **Centesimal or French System:**

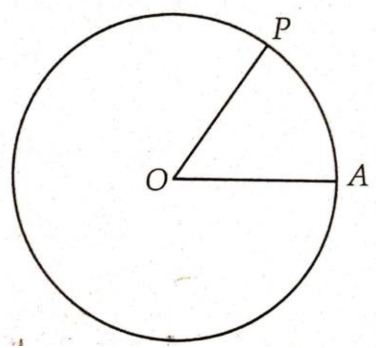
In this system, 1 right angle $= 100$ grades

1 grade $= 100$ minutes $(100')$

1 minute $= 100$ seconds $(100'')$

(iii) **Circular System or Radian Measure:**

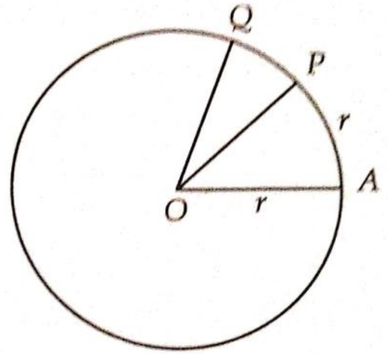
One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



Remember:

- Radian is a constant angle.
- The number of radians in angle subtended by an arc of a circle at the centre is equal to $\frac{arc}{radius}$.

Proof: Consider a circle with centre O and radius r.



Let $\angle AOQ = \theta^c$ and let arc $AQ = l$. Let P be a point on the arc AQ such that arc $AP = r$. Then $\angle AOP = 1^c$.

Since angles at the centre of a circle are proportional to the arcs subtending them, so $\frac{\angle AOQ}{\angle AOP} = \frac{arc\ AQ}{arc\ AP}$

$$\Rightarrow \angle AOQ = \left(\frac{arc\ AQ}{arc\ AP} \times 1\right)^c$$

$$\Rightarrow \theta = \frac{l}{r}\ radians$$

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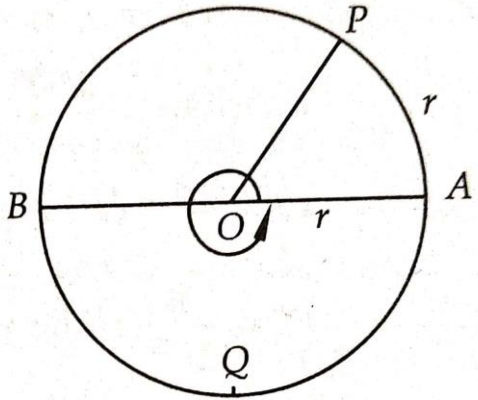
Relations between Degrees and Radians:

Consider a circle with centre O and radius r. Let A be a point on the circle. Join OA and cut off an arc AP of length equal to the radius of the circle. Then $\angle AOP = 1^c$. Produce AO to meet the circle at B.

$\therefore \angle AOB =$ a straight angle $=$ 2 right angles



We know that the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \frac{\angle AOP}{\angle AOB} = \frac{arc\ AP}{arc\ APB} \Rightarrow \frac{\angle AOP}{2\ right\ angles} = \frac{r}{\pi r}$$

$$\Rightarrow \quad \angle AOP = \frac{2\ right\ angles}{\pi}$$

$$\Rightarrow \quad 1^c = \frac{180°}{\pi}$$

Hence $one\ radian = \frac{180°}{\pi} \Rightarrow \boldsymbol{\pi\ radians = 180°}$.

Some Points to Remember:

- 2π radians $= 360°$
- All integral multiples of $\frac{\pi}{2}$ are called quadrant angles.
- When an angle is expressed in radians, the word radian is generally omitted.
- We have $1\ radian = 57°\ 16'22''(approx)$ and $1° = 0.01746\ radians$
- Radian measures of angles and real numbers can be considered as one and the same.
- Relation between the three systems of measurement of an angle is $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$
- The angle between two consecutive digits of a clock is $30°(= \frac{\pi}{6} radians)$
- The hour hand rotates through an angle of $30°$ in one hour $i.e. \left(\frac{1}{2}\right)°$ in one minute.
- The minute hand rotates through an angle of $6°$ in one minute.

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Conversion of Degrees into Radians and vice – versa:

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

The relation between degree measure and radian measure of some common angles are tabulated below.

Degree	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

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Example: Find the radian measure corresponding to the following degree measures:

(i) $340°$

Sol: $340° = (340 \times \frac{\pi}{180})^{c} = \left(\frac{17\pi}{9}\right)^{c}$

(ii) $40°\,20'$

Sol: $40°20' = \left(40 + \frac{1}{3}\right)° = \left(\frac{121}{3}\right)° = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^{c} = \left(\frac{121\pi}{540}\right)^{c}$

(iii) $5°37'30''$

Sol: We have $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

So, $37'30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)° = \left(\frac{5}{8}\right)°$

$\therefore\ 5°37'30'' = \left(5\frac{5}{8}\right)° = \left(\frac{45}{8}\right)° = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^{c} = \left(\frac{\pi}{32}\right)^{c}$

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Example: Find the degree measure corresponding to the following radian measures:

$(i)\ \left(\frac{2\pi}{15}\right)^c$

Sol: $\left(\frac{2\pi}{15}\right)^c = \left(\frac{2\pi}{15} \times \frac{180}{\pi}\right)^\circ = 24^\circ$

$(ii)\ \left(\frac{\pi}{8}\right)^c$

Sol: $\left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = \left(\frac{45}{2}\right)^\circ = \left(22\frac{1}{2}\right)^\circ = 22^\circ \left(\frac{1}{2} \times 60\right)^\circ = 22^\circ\, 30'$

$(iii)\ 6^c$

Sol: $6^c = \left(6 \times \frac{180}{\pi}\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90 \times 7 \times 6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\ \frac{7}{11}\right)^\circ$

$= 343^\circ \left(\frac{7}{11} \times 60\right)^\circ = 343^\circ \left(\frac{420}{11}\right)^\circ = 343^\circ \left(38\frac{2}{11}\right)^\circ = 343^\circ\, 38' \left(\frac{2}{11} \times 60\right)'' = 343^\circ 38' 11''$

Example: Find the length of an of a circle of radius 5 cm subtending a central angle measuring $15°$.

Sol: Let l be the length of the arc subtending an angle θ at the centre of a circle of radius r.

Here $r = 5\ cm$ and $\theta = 15° = \left(15\ \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$\therefore\ \theta = \frac{l}{r} \Rightarrow \frac{\pi}{12} = \frac{l}{5} \Rightarrow l = \frac{5\pi}{12}\ cm.$

Example: The angles of a triangle are in the ratio $3 : 4 : 5$, find the smallest angle in degrees and the greatest angle in radians.

Sol: Let the three angles be $3x, 4x$ and $5x$ degrees. Then

$3x + 4x + 5x = 180° \Rightarrow x = 15°$

So, the smallest angle $= 3x = 45°$ and

the greatest angle $= 5x = 75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}\ radians$

Example: The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes? (Use $\pi = \frac{22}{7}$)

Sol: The minute hand of a watch completes one revolution in 60 minutes.

Therefore, the angle traced by a minute hand in 60 minutes $= 360° = 2\pi\ rad$

So, angle traced by the minute hand in 18 minutes $= 2\pi\ \times \frac{18}{60}\ rad = \frac{3\pi}{5}\ rad$

Let the distance moved by the tip in 18 minutes be l, then

$l = r\ \theta = 35\ \times \frac{3\pi}{5} = 21\pi = 21 \times \frac{22}{7} = 66\ cm.$

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Problems on Measurement of Angles

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
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Example: Find the angle between the minute hand of a clock and the hour hand when the time is $7:20\ AM$.

Sol: We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

So, angle traced by the hour hand in 12 hours $= 360°$

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min.

$i.e. \frac{22}{3}\ hrs = \left(\frac{360}{12} \times \frac{22}{3}\right)^{\circ} = 220°$

Also, the angle traced by the minute hand in 60 min= $360°$.

$\Rightarrow$ Angle traced by the minute hand in 20 min= $\left(\frac{360}{60} \times 20\right)^{\circ} = 120°$

Hence, the required angle between two hands= $220° - 120° = 100°$

Example: A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 meters when it has traced out 72° at the center, find the length of the rope.

Sol: Let the post be at point P and let PA be the length of the rope in a tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72°$ and $arc\ AB = 88\ m$.

Let r be the length of the rope $i.e. PA = r$ meters.

Here $\theta = 72° = \left(72\ \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $l = 88\ m$

So, $\theta = \frac{l}{r} \Rightarrow \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70\ metres$

Example: The moon's distance from the earth is 360,000 kms and its diameter subtends an angle of $31'$ at the eye of the observer. Find the diameter of the moon.

Sol: Let AB be the diameter of the moon and let

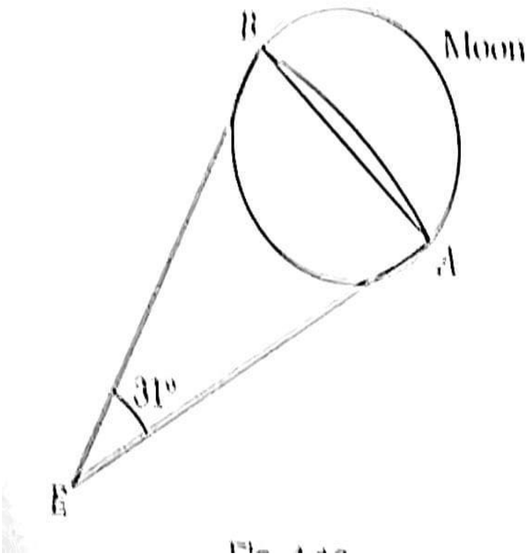
E be the eye of the observer. Since the

Distance between the earth and the moon is

quite large, so we take diameter AB as $arc\ AB$.

Let d be the diameter of the moon.

Then $d = arc\ AB$.



We have, $\theta = 31' = \left(\frac{31}{60}\right)^{\circ} = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^{c}$

Also, $r = 360000\ kms$

$\theta = \frac{arc}{radius} \Rightarrow \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$

$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right)\ kms = 3247.62\ kms$

Hence, the diameter of the moon is $3247.62\ kms$.

Example: The wheel of a railway carriage is 40 cm in diameter and makes 6 revolutions in a second; how fast is the train going?

Sol: Diameter of the wheel $= 40\ cm \Rightarrow$ Radius of the wheel $= 20\ cm$

Circumference of the wheel $= 2\pi r = 2\pi \times 20 = 40\pi\ cm$

Number of revolutions made in 1 second $= 6$

So, distance covered in 1 second $= 40\pi \times 6 = 240\ \pi\ cm$

$\therefore$ Speed of the train $= 240\pi\ cm/sec.$

Example: A circular wire of radius 3 cm is cut and bent to lie along the circumference of a hoop whose radius is 48 cm. Find the angle in degrees which is subtended at the centre of the hoop.

Sol: Given that circular wire is of radius 3 cm, so when it is cut then its length $= 2\pi \times 3 = 6\pi\ cm$.

Again, it is being placed along a circular hoop of radius 48 cm.

Here $l = 6\pi\ cm$ is the length of the arc and $r = 48\ cm$ is the radius of the circle.

Therefore, the angle θ, in radian, subtended by the arc at the centre of the circle is given by

$$\theta = \frac{arc}{Radius} = \frac{6\pi}{48} = \frac{\pi}{8} = 22.5°$$

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Trigonometric Functions

SUBJECT : MATHEMATICS
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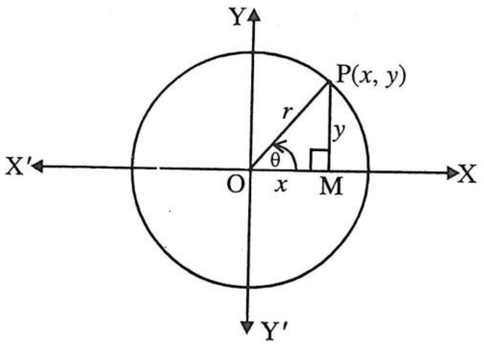
Let $X'OX$ and $Y'OY$ be the coordinate axes and let a revolving line starts from OX in the anticlockwise direction and trace out an angle $\angle XOP = \theta$.

From P, draw $PM \perp OX$.

Let in the right-angled triangle $POM, OM = x, PM = y$ and $OP = r$.



There are six possible ratios among three sides of a triangle.

These six ratios are called trigonometrical ratios and defined as follows:

$$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$$

$$cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$tan\theta = \frac{PM}{OM} = \frac{y}{x}$$

$$cot\theta = \frac{OM}{PM} = \frac{x}{y}$$

$$sec\theta = \frac{OP}{PM} = \frac{r}{x}$$

$$cosec\ \theta = \frac{OP}{PM} = \frac{r}{y}$$

Hence, the trigonometrical ratios of an angle are the numerical quantities. Each one of them represents the ratio of the length of one side to another side of a right-angled triangle. The functions $sin\theta, cos\theta, tan\theta, cot\theta, sec\theta$, and $cosec\theta$ are called trigonometric functions.

We define $cosec\theta = \frac{1}{sin\theta}, sec\theta = \frac{1}{cos\theta},\ cot\theta = \frac{1}{tan\theta}$

Also, $tan\theta = \frac{sin\theta}{cos\theta}$ and $cot\theta = \frac{cos\theta}{sin\theta}$

Values of Trigonometric Functions:

Angle / Trigonometric Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

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Trigonometric Identities:

An equation involving trigonometric functions which is true for all these values of the variable for which the function is defined is called a trigonometric identity.

We have the following three identities among trigonometrical ratios:

$(i)\ sin^2\theta + cos^2\theta = 1$

$(ii)\ sec^2\theta - tan^2\theta = 1$

$(iii)\ cosec^2\theta - cot^2\theta = 1$

These are called “Pythagorean Identities”.

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Signs of the Trigonometric Ratios / Functions

The signs of six trigonometric ratios depend on the quadrant in which the terminal side of the angle lies. The length $OP = r$ always positive.

Thus $sin\theta = \frac{y}{r}$ has the sign of y, $cos\theta = \frac{x}{r}$ has a sign of x. The sign of $tan\theta$ depends on the signs of x and y and similarly, the signs of other trigonometric ratios are decided by the signs of x and/or y.

In the first quadrant: We have $x > 0$ and $y > 0$.

So, $sin\theta = \frac{y}{r} > 0$ and $cos\theta > 0$.

Thus in the first quadrant, all trigonometric ratios are positive.

In the second quadrant: We have $x < 0$ and $y > 0$.

So, $sin\theta = \frac{y}{r} > 0$ and $cos\theta = \frac{x}{r} < 0$

Thus in the second quadrant sine and cosecant functions are positive and all others are negative.

In the third quadrant: We have $x < 0$ and $y < 0$.

So, $sin\theta = \frac{y}{r} < 0$ and $cos\theta = \frac{x}{r} < 0$

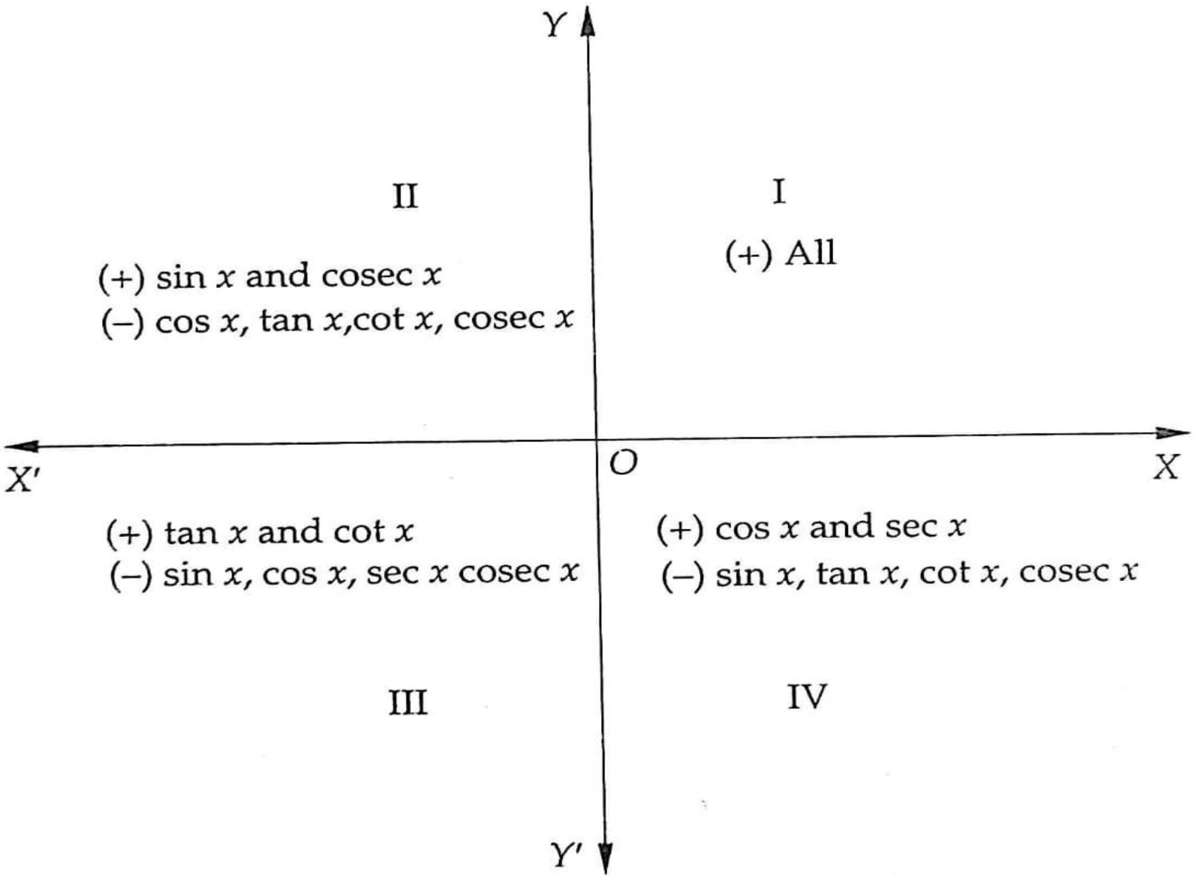
Thus, in the third quadrant, all trigonometric functions are negative except tangent and cotangent.

In the fourth quadrant: We have $x > 0$ and $y < 0$.

So, $sin\theta = \frac{y}{r} < 0$ and $cos\theta = \frac{x}{r} > 0$

Thus, in the fourth quadrant, all trigonometric functions are negative except cosine and secant.

The above rule is known as $ASTC$ Rule.


Y
II
I
(+) All
(+) sin x and cosec x
(–) cos x, tan x,cot x, cosec x
X'
O
X
(+) tan x and cot x
(–) sin x, cos x, sec x cosec x
(+) cos x and sec x
(–) sin x, tan x, cot x, cosec x
III
IV
Y'

Note:

Since $sin^2\theta + cos^2\theta = 1$, so, $|sin\theta| \leq 1$ and $|cos\theta| \leq 1$

$\Rightarrow \quad -1 \leq sin\theta \leq 1$ and $-1 \leq cos\theta \leq 1$

Also, $0 \leq sin^2\theta \leq 1, \; 0 \leq cos^2\theta \leq 1$

Since, $cosec\theta = \frac{1}{sin\theta}$, therefore $cosec\theta \geq 1$ or $cosec\theta \leq -1$

Since, $c\theta = \frac{1}{cos\theta}$, therefore $sec\theta \geq 1$ or $sec\theta \leq -1$

Domain and Range of Trigonometric Functions

$sin: R \to [-1, 1]$

$cos: R \to [-1, 1]$

$tan: R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\} \to R$

$cot: R - \{n\pi, n \in Z\} \to R$

$sec:\ R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\} \to (-\infty, -1] \cup [1, \infty)$

$cosec:\ R - \{n\pi, n \in Z\} \to (-\infty, -1] \cup [1, \infty)$

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Variations in Values of Trigonometric Functions in different quadrants:

Trigonometric function	*I quadrant*	*II quadrant*	*III quadrant*	*IV quadrant*
sine	increase from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cosine	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0	increases from 0 to 1
tangent	increases from 0 to ∞	increases from – ∞ to 0	increases from 0 to ∞	increases from – ∞ to 0
cotangent	decreases from ∞ to 0	decreases from 0 to – ∞	decreases from ∞ to 0	decreases from 0 to – ∞
secant	increases from 1 to ∞	increases from – ∞ to 1	decreases from –1 to – ∞	decreases from ∞ to 1
cosecant	decreases from ∞ to 1	decreases from 1 to ∞	decreases from – ∞ to –1	decreases from –1 to – ∞

Example: Find $sinx$ and $tanx$, if $cosx = -\frac{12}{13}$ and x lies in the third quadrant.

Sol: We know that $sin^2x + cos^2x = 1 \Rightarrow sinx = \pm\sqrt{1 - cos^2x}$

In the third quadrant $sinx$ is negative.

So, $sinx = -\sqrt{1 - cos^2x} \Rightarrow sinx = -\sqrt{1 - \left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$

and $tanx = \frac{sinx}{cosx} \Rightarrow tanx = \frac{-5}{13} \Big/ \frac{-12}{13} = \frac{5}{12}$

Example: Find all other trigonometric ratios, if $sinx = -\frac{2\sqrt{6}}{5}$ and x lies in quadrant III.

Sol: We know that $sin^2x + cos^2x = 1 \Rightarrow \text{cos}x = \pm\sqrt{1 - sin^2x}$

In the third quadrant $cosx$ is negative.

$$\therefore cosx = -\sqrt{1 - sin^2x} \Rightarrow cosx = -\sqrt{1 - \frac{24}{25}} = -\frac{1}{5}$$

In the third quadrant $tanx$ is positive.

So, $tanx = \frac{sinx}{cosx} \Rightarrow tanx = \frac{-2\sqrt{6}}{5} \Big/ \frac{-1}{5} = 2\sqrt{6}$

Now, $cosecx = \frac{1}{sinx} = -\frac{5}{2\sqrt{6}}$ $\quad secx = \frac{1}{cosx} = -5$ $\quad cotx = \frac{1}{tanx} = \frac{1}{2\sqrt{6}}$

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Example: If $secx = \sqrt{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of $\frac{1+tanx+cosecx}{1+cotx - cosecx}$.

Sol: We have $secx = \sqrt{2} \Rightarrow cosx = \frac{1}{secx} = \frac{1}{\sqrt{2}}$

It is given that x lies in the fourth quadrant in which $sinx$ is negative.

So, $sinx = -\sqrt{1 - cos^2x} = -\sqrt{1 - \frac{1}{2}} = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad cosecx = \frac{1}{sinx} = -\sqrt{2}$

and $tanx = \frac{sinx}{cosx} = -\frac{1}{\sqrt{2}} \times \sqrt{2} = -1 \Rightarrow cotx = -1$

$\therefore \frac{1+tanx+cosecx}{1+cotx - cosecx} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} = -1$

Example: If x is any non – zero real number, show that $cos\theta$ and $sin\theta$ can never be equal to $x + \frac{1}{x}$.

Sol: We have the following cases:

The case I: When $x > 0$.

In this case, we have $x + \frac{1}{x} = (\sqrt{x}\,)^2 + \frac{1}{(\sqrt{x}\,)^2} - 2\sqrt{x}\,\frac{1}{\sqrt{x}} + 2\sqrt{x}\,\frac{1}{\sqrt{x}}$

$\Rightarrow \quad x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \geq 2$

Case II: When $x < 0$. Let $x = -y$. Then $y > 0$

In this case $x + \frac{1}{x} = -y - \frac{1}{y} = -\left(y + \frac{1}{y}\right)$

But, $y + \frac{1}{y} \geq 2 \quad \Rightarrow -\left(y + \frac{1}{y}\right) \leq -2 \quad \Rightarrow x + \frac{1}{x} \leq -2$

$\therefore x+\frac{1}{x} \geq 2$ for $x>0$ and $x+\frac{1}{x} \leq -2$ for $x<0$

But, $-1 \leq sin\theta \leq 1$ and $-1 \leq cos\theta \leq 1$ for all θ.

Hence $sin\theta$ and $cos\theta$ cannot equal to $x+\frac{1}{x}$ for any non – zero x.

Example: Prove the following. $sec^2\theta + cosec^2\theta \geq 4.$

Sol: We have $sec^2\theta + cosec^2\theta = 1 + tan^2\theta + 1 + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta - 2\, tan\theta\, cot\theta + 2\, tan\theta\, cot\theta$

$= 2 + (tan\theta - cot\theta)^2 + 2$

$= 4 + (tan\theta - cot\theta)^2 \geq 4$ [since $(tan\theta - cot\theta)^2 \geq 0$]

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Trigonometric Functions of Complementary and Supplementary Angles

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
CHAPTER NAME : TRIGONOMETRIC FUNCTIONS

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Values of Trigonometric Functions at Allied Angles

Two angles are said to be allied when their sum or difference is either zero or multiples of $\frac{\pi}{2}$.

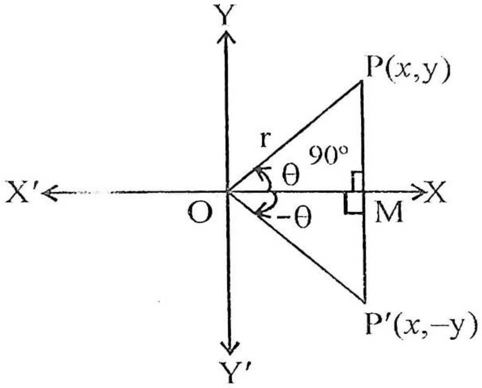
The angles allied to θ are $-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, \frac{3\pi}{2} \pm \theta, 2\pi \pm \theta$, etc.

Trigonometrical Ratios of $(-\boldsymbol{\theta})$

Suppose a revolving line OP makes an angle $+\theta$ with OX by revolving in the anticlockwise direction and angle $(-\theta)$ by revolving in clockwise directions (position OP').

Let $\angle POM = \theta$. In $\Delta\, POM, OP = r, OM = x, PM = y$.

Now produce PM to P' such that $PM = MP'$. Join OP'.



Hence $\Delta OMP \equiv \Delta OMP'$

$\Rightarrow$ $MP = MP'$ and $OP = OP'$

If we consider the sign of these distances, then $P'M = -PM$

Also, we have $\angle P'OM = -\theta$, $OM = x, OP' = r$ and $P'M = -y$

In $\Delta P'OM, \sin(-\theta) = \frac{P'M}{OP'} = -\frac{y}{r} = -\frac{PM}{OP} = -sin\theta$

$\cos(-\theta) = \frac{OM}{OP'} = \frac{x}{r} = \frac{OM}{OP} = cos\theta$, $\tan(-\theta) = \frac{P'M}{OM} = -\frac{y}{x} = -\frac{PM}{OM} = -tan\theta$

Similarly, $cosec(-\theta) = -cosec\theta$, $\sec(-\theta) = sec\theta$ and $\cot(-\theta) = -cot\theta$

Trigonometrical Ratios of $\frac{\pi}{2} - \theta$

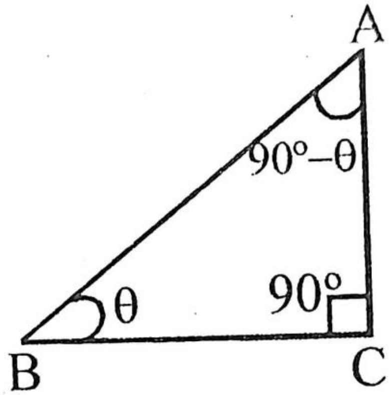
Let $\angle ACB = \frac{\pi}{2}, \angle ABC = \theta$

$\Rightarrow \angle BAC = \frac{\pi}{2} - \theta$, which is the complementary angle of θ.

In $\Delta ABC, \sin\left(\frac{\pi}{2} - \theta\right) = \frac{BC}{AB} = cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{AC}{AB} = sin\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{BC}{AC} = cot\theta$



Trigonometrical Ratios of $\frac{\pi}{2} + \theta$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2} - (-\theta)\right) = \cos(-\theta) = cos\theta$$

$$cos\left(\frac{\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2} - (-\theta)\right) = \sin(-\theta) = -sin\theta$$

$$\tan\left(\frac{\pi}{2} + \ \theta\right) = \tan\left(\frac{\pi}{2} - (-\theta)\right) = \mathrm{co}t(-\theta) = -cot\theta$$

Trigonometrical Ratios of $\boldsymbol{\pi \pm \theta}$

$$\sin(\pi - \theta) = \sin\left(\frac{\pi}{2} + \overline{\frac{\pi}{2} - \theta}\right) = \cos\left(\frac{\pi}{2} - \theta\right) = sin\theta$$

Similarly, $\cos(\pi - \theta) = -cos\theta$ and $\tan(\pi - \theta) = -tan\theta$

Also, $\sin(\pi + \theta) = \sin\left(\frac{\pi}{2} + \overline{\frac{\pi}{2} + \theta}\right) = \cos\left(\frac{\pi}{2} + \theta\right) = -sin\theta$

Similarly, $\cos(\pi + \theta) = -cos\theta$ and $\tan(\pi + \theta) = tan\theta$

Trigonometrical Ratios of $\frac{3\pi}{2} \pm \theta$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = \sin\left(\pi + \overline{\frac{\pi}{2} - \theta}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -cos\theta$$

Similarly, $\cos\left(\frac{3\pi}{2} - \theta\right) = -sin\theta$ and $\tan\left(\frac{3\pi}{2} - \theta\right) = cot\theta$

Also, $\sin\left(\frac{3\pi}{2} + \theta\right) = \sin\left(\pi + \overline{\frac{\pi}{2} + \theta}\right) = -\sin\left(\frac{\pi}{2} + \theta\right) = -cos\theta$

Similarly, $\cos\left(\frac{3\pi}{2} + \theta\right) = sin\theta$ and $\tan\left(\frac{3\pi}{2} + \theta\right) = -cot\theta$

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Trigonometrical Ratios of $2\pi \pm \theta$

$\sin(2\pi - \theta) = \sin(\pi + \overline{\pi - \theta}) = -\sin(\pi - \theta) = -sin\theta$

Similarly, $\cos(2\pi - \theta) = cos\theta$ and $\tan(2\pi - \theta) = tan\theta$

Again, $\sin(2\pi + \theta) = \sin(2\pi - (-\theta)) = -\sin(-\theta) = sin\theta$

Similarly, $\cos(2\pi + \theta) = cos\theta$ and $\tan(2\pi + \theta) =$

To remember the above results we adopt the following technique.

$$(i)\ \sin\left(n\frac{\pi}{2}+\theta\right)=\begin{cases}(-1)^{\frac{n-1}{2}}\ cos\theta, if\ n\ be\ an\ odd\ integer\\ (-1)^{\frac{n}{2}}\ sin\theta\ , if\ n\ be\ an\ even\ integer\end{cases}$$

$$(ii)\ \cos\left(n\frac{\pi}{2}+\theta\right)=\begin{cases}(-1)^{\frac{n+1}{2}}\ sin\theta, if\ n\ be\ an\ odd\ integer\\ (-1)^{\frac{n}{2}}\ cos\theta\ , if\ n\ be\ an\ even\ integer\end{cases}$$

$$(iii)\ \tan\left(n\frac{\pi}{2}+\theta\right)=\begin{cases}-\ cot\theta, if\ n\ be\ an\ odd\ integer\\ tan\theta, if\ n\ be\ an\ even\ integer\end{cases}$$

Periodic Functions:

A function $f(x)$ is said to be a periodic function if there exists a positive real number T such that $f(x+T)=f(x)$ for all x. Here T is called the period of f.

Periodicity of Trigonometric Functions:

(i) $sinx, cosx, secx, cosecx$ are all periodic with pe

(ii) $tanx$ and $cotx$ are periodic with period π.

(iii) $sin^n x, cos^n x, sec^n x, cosec^n x$ are periodic with period 2π and π, according to n is odd or even.

(iv) $tan^n x, cot^n x$ are periodic with period π for n being odd or even.

$(v) |sinx|, |cosx|, |secx|, |cosecx|, |tanx|, |cotx|$ are periodic with period π.

Even and Odd Functions

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x and is said to be odd if $f(-x) = -f(x)$ for all x.

sine and *tangent* are odd functions, where *cosine* is an even function.

Example: Evaluate the following:

$(i)\ \sin\dfrac{31\pi}{3}$

Sol: $\sin\frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$(ii)\ \cos\frac{7\pi}{6}$

Sol: $\cos\frac{7\pi}{6} = \cos\left(2\times\frac{\pi}{2} + \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

$(iii)\ \sin\left(-\dfrac{25\pi}{4}\right)$

Sol: $\sin\left(-\frac{25\pi}{4}\right) = -\sin\left(\frac{25\pi}{4}\right) = -\sin\left(12\times\frac{\pi}{2} + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

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$(iv)\ \tan\left(\frac{19\pi}{3}\right)$

Sol: $\tan\left(\frac{19\pi}{3}\right) = \tan\left(\ 12 \times \frac{\pi}{2} + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \ \sqrt{3}$

$(v)\ \cot\left(-\frac{15\pi}{4}\right)$

Sol: $\cot\left(-\frac{15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot\left(7 \times \frac{\pi}{2} + \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

Example: Evaluate the following:

$(i)\ \tan 480°$

Sol: $\tan 480° = \tan\frac{8\pi}{3} = \tan\left(\ 5 \times \frac{\pi}{2} + \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -\sqrt{3}$

$(ii)\ \cos(-1710°)$ **Sol:** $\cos 1710° = \cos\frac{19\pi}{2} = 0$

Example: Prove that $cos510° \, cos330° + sin390° \, cos120° = -1$

Sol: $LHS = cos510° \, cos330° + sin390° \, cos120°$

$= \cos(5 \times 90° + 60°) \cos(3 \times 90° + 60°) + \sin(4 \times 90° + 30°) \cos(1 \times 90° + 30°)$

$= (-1)^3 \, sin60°.(-1)^2 \, sin60° + (-1)^2 \, sin30° \, (-1)^1 \, sin30°$

$= -sin^2 \, 60° - sin^2 \, 30° = -\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = -\frac{3}{4} - \frac{1}{4} = -1 = RHS$

Example: Prove that $\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = cot^2 x$

Sol: $LHS = \frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \frac{-cosx \; cosx}{sinx \, (-sinx)} = \frac{cos^2 x}{sin^2 x} = cot^2 x = RHS.$

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Example: Prove that $\frac{\cos(2\pi+x)cosec(2\pi+x)\tan(\frac{\pi}{2}+x)}{\sec\left(\frac{\pi}{2}+x\right)cosx\cot(\pi+x)} = 1.$

Sol: $LHS = \frac{\cos(2\pi+x)cosec(2\pi+x)\tan(\frac{\pi}{2}+x)}{\sec\left(\frac{\pi}{2}+x\right)cosx\cot(\pi+x)} = \frac{cosx\ cosecx\ (-cotx)}{-\ cosecx\ cosx\ cotx} = 1 = RHS$

Example: Find the value of $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°}$

Sol: $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°} = \frac{\cot(90° - 36°)}{\tan 36°} + \frac{tan20°}{\cot(90° - 20°)} = \frac{\tan 36°}{tan36°} + \frac{tan20°}{tan20°} = 1 + 1 = 2$

Example: State the sign of $sin201° + cos201°$

Sol: Since $201°$ lies in 3rd quadrant, so $sin201° < 0$ and $cos201° < 0$

Thus $sin201° + cos201° < 0$

Example: Find the value of $sin1° . sin2°. sin3° \ldots . \sin 200°$.

Sol: $sin1° . sin2°. sin3° \ldots . \sin 200° = 0$, since $sin180° = 0$

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Trigonometric Functions of Compound Angles

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
CHAPTER NAME : TRIGONOMETRIC FUNCTIONS

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Trigonometric Ratios of Compound Angles

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example, if A, B, C are three angles then $A \pm B, A + B + C, A - B + C$, etc. are compound angles.

Trigonometric Ratios of Sum and Difference of two angles:

Let the rotating line start from initial line OX and trace out $\angle XOS = \angle A$ in the anticlockwise direction and let the rotating line further rotates to trace out $\angle SOP = \angle B$.

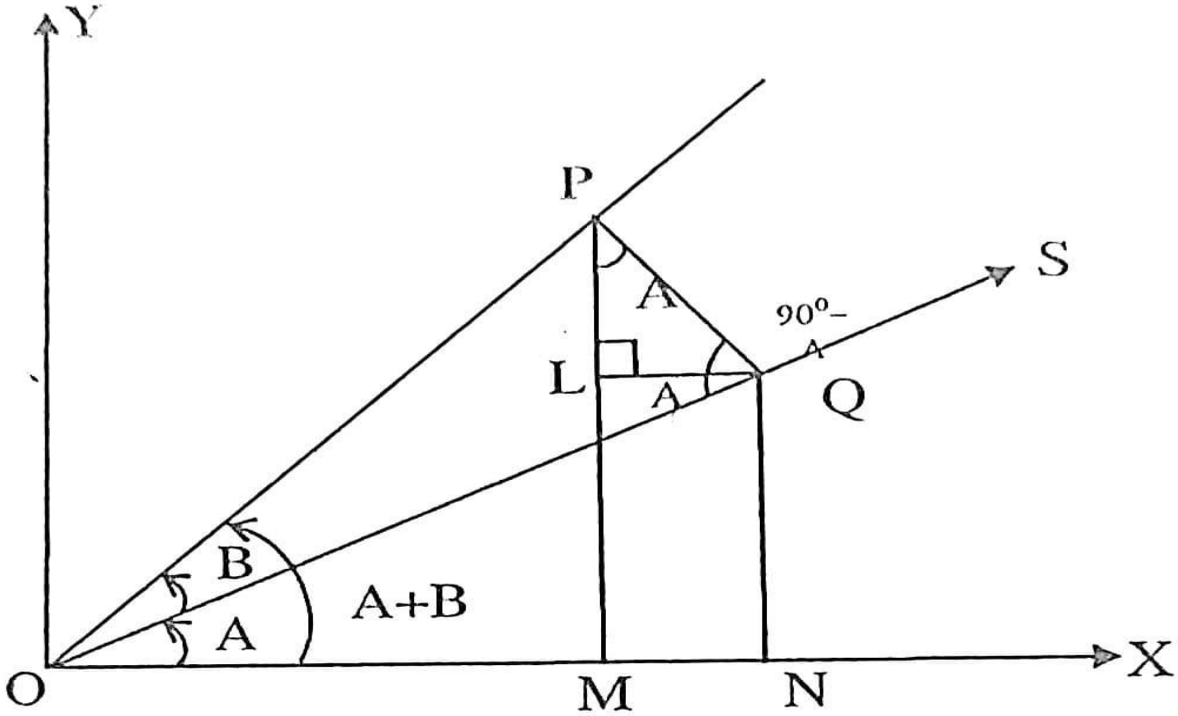
So that $\angle XOP = \angle (A + B)$

Draw $PM \perp OX$ and $PQ \perp OS$. Again draw $QL \perp PM$ and $QN \perp OX$.

Then $\angle QPL = 180° - 90° - \angle PQL = 90° - (90° - A) = A$

$\therefore \sin(A + B) = sin \angle POM = \frac{PM}{OP} = \frac{PL+LM}{OP} = \frac{PL+QN}{OP} = \frac{PL}{OP} + \frac{QN}{OP}$

$= \frac{PL}{PQ} . \frac{PQ}{OP} + \frac{QN}{OQ} . \frac{OQ}{OP} = cosA\ sinB + sinA . cosB = sinA\ cosB + cosA\ sinB$ .... (1)


Y
P
S
A
90°–
A
L
A
Q
B
A+B
A
O
M
N
X

$$\cos(A+B) = \frac{OM}{OP} = \frac{ON - MN}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OP} - \frac{LQ}{OP}$$

$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{LQ}{PQ} \cdot \frac{PQ}{OP} = cosA\ cosB\ - sinA\ sinB \ \ldots. (2)$$

$$\tan(A+B) = \frac{PM}{OM} = \frac{PL + LM}{OM} = \frac{QN + PL}{ON - MN} = \frac{QN + PL}{ON - LQ}$$

$$= \frac{\frac{QN}{ON} + \frac{PL}{ON}}{1 - \frac{LQ}{ON}} = \frac{\frac{QN}{ON} + \frac{PL}{ON}}{1 - \frac{LQ}{PL} \cdot \frac{PL}{ON}} = \frac{\tan A + tanB}{1 - tanA.tanB} \ \ldots. (3)$$

Replacing B by $(-B)$ in $(1), (2)$ and (3) we have

$$\sin(A-B) = sinA\cos(-B) + cosA\sin(-B) = sinA\,cosB - cosA\sin B$$

$$\cos(A-B) = cosA\cos(-B) - sinA\sin(-B) = cosA\,cosB + sinAsinB$$

$$\tan(A-B) = \frac{tanA + \tan(-B)}{1 - tanA\tan(-B)} = \frac{tanA\ - tanB}{1 + tanA\,tanB}$$

Similarly, it can be proved that

$\cot(A+B) = \frac{cotA\,cotB\,-1}{cotB+cotA}$ and $\cot(A-B) = \frac{cotA\,cotB+1}{cotB-cotA}$

More Useful Result:

$(i)\sin(A+B)\sin(A-B) = sin^2A - sin^2B = cos^2B - cos^2A$

$(ii)\cos(A+B)\cos(A-B) = cos^2A - sin^2B = cos^2B - sin^2A$

$(iii)\sin(A+B+C) = sinA\ cosB\ cosC + cosA\ sinB\ cosC + cosA\ cosB\ sinC - sinAsinBsinC$

$(iv)\cos(A+B+C) = cosA\ cosBcosC\ - cosA\ sinB\ sinC - sinA\ cosBsinC - sinAsinBcosC$

$$(v)\tan(A+B+C) = \frac{tanA + tanB + tanC - tanA\ tanB\ tanC}{1 - tanA\ tanB - tanB\ tanC\ - tanC\ tanA}$$

Example: Find the value of $\sin 15°$.

Sol: We have $\sin 15° = \sin(45° - 30°)$

$= \sin 45°\, cos30° - cos45°\, sin30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Example: Prove that $\frac{\sin(x+y)}{\sin(x-y)} = \frac{tanx+tany}{tanx - tany}$

Sol: $LHS = \frac{\sin(x+y)}{\sin(x-y)} = \frac{sinx\ cosy+cosx\ siny}{sinx\ cosy - cosx \sin y}$

$= \frac{\frac{sinx\ cosy}{cosx\ cosy} + \frac{cosx\ siny}{cosx\ cosy}}{\frac{sinx\ cosy}{cosx\ cosy} - \frac{cosx\ siny}{cosx\ cosy}} = \frac{tanx+tany}{tanx - tany} = RHS$

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Example: Prove that $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\, cosx$

Sol: $LHS = \cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right)$

$= (\cos\frac{\pi}{4}\; cosx\; - \sin\frac{\pi}{4}\; sinx) + (\cos\frac{\pi}{4}\; cosx + \sin\frac{\pi}{4}\; sinx)$

$= \; 2\cos\frac{\pi}{4}\; cosx = 2\; \times \frac{1}{\sqrt{2}} \times cosx = \sqrt{2}\, cosx\; = RHS$

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Example: Show that $tan3x\ tan2x\ tanx = tan3x - tan2x - tanx$

Sol: We know that $3x = 2x + x$

So, $\tan 3x = \tan(2x + x)$

$\Rightarrow \quad \tan 3x = \frac{tan2x + tanx}{1 - \tan 2x\ tanx}$

$\Rightarrow \quad tan3x\ - tan3x\ tan2x\ tanx = tan2x + tanx$

$\Rightarrow \quad \tan 3x\ - tan2x - tanx = tan3x\ tan2x\ tanx$

Example: Prove that $\tan 75° + \cot 75° = 4$.

Sol: $LHS = \tan 75° + \cot 75° = \left(2 + \sqrt{3}\,\right) + \frac{1}{2+\sqrt{3}} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4 = RHS$

Example: If $tanA = \frac{1}{2}$ and $tanB = \frac{1}{3}$, show that $cos2A = \sin 2B$

Sol: We have $\tan(A + B) = \frac{tanA+tanB}{1-tanA\ tanB} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = {}^{\frac{5}{6}}\!/_{\frac{5}{6}} = 1$

$\Rightarrow \quad A + B = \frac{\pi}{4} \Rightarrow A = \frac{\pi}{4} - B$

Now $\cos 2A = \cos 2\left(\frac{\pi}{4} - B\right) = \cos\left(\frac{\pi}{2} - 2B\right) = \sin 2B$

Example: If $tanA - tanB = x$ and $cotB - \cot A = y$, prove that $\cot(A-B) = \frac{1}{x} + \frac{1}{y}$

Sol: We have $\cot B - \cot A = y$

$$\Rightarrow \quad \frac{1}{tanB} - \frac{1}{tanA} = y \Rightarrow \frac{tanA - tanB}{tanA\ tanB} = y$$

$$\Rightarrow \quad \frac{x}{tanA\ tanB} = y \Rightarrow tanA\ tanB = \frac{x}{y}$$

Now $\cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+tanA\ tanB}{tanA-tanB} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$

Example: If $A + B = \frac{\pi}{4}$, prove that $(1 + tanA)(1 + tanB) = 2$

Sol: We have $A + B = \frac{\pi}{4}$

$\Rightarrow \quad \tan(A + B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A + tanB}{1 - tanA\ tanB} = 1$

$\Rightarrow \quad tanA + tanB = 1 - tanA\ tanB$

$\Rightarrow \quad tanA + tanB + tanA\ tanB = 1$

$\Rightarrow \quad 1 + tanA + tanB + tanA\ tanB = 2$

$\Rightarrow \quad (1 + tanA) + tanB(1 + tanA) = 2$

$\Rightarrow \quad (1 + tanA)(1 + tanB) = 2$

Example: If angle θ is divided into two parts such that the tangent of one part is k times the tangent of other and φ is their difference, then show that $sin\theta = \frac{k+1}{k-1}\ sin\varphi$

Sol: Let $= \ \alpha + \beta$. Then $tan\alpha = k\ tan\beta$

$$\Rightarrow \quad \frac{tan\alpha}{tan\beta} = \frac{k}{1}$$

$$\Rightarrow \quad \frac{tan\alpha + tan\beta}{tan\alpha - tan\beta} = \frac{k+1}{k-1}$$ (Applying componendo and dividendo)

$$\Rightarrow \quad \frac{sin\alpha\ cos\beta + cos\alpha\ sin\beta}{sin\alpha\ cos\beta - cos\alpha\ sin\beta} = \frac{k+1}{k-1} \Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{k+1}{k-1}$$

$$\Rightarrow \quad \frac{\sin\theta}{\sin\varphi} = \frac{k+1}{k-1}$$

$$\Rightarrow \quad sin\theta = \frac{k+1}{k-1}\ sin\varphi$$

Maximum and Minimum Values of Trigonometrical Expressions

The maximum and minimum values of $a\ sin\theta + b\ cos\theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$

Example: Find the maximum and minimum values of $7\ cos\theta + 24\ sin\theta$.

Sol: Here $a = 24$ and $b = 7$

So,$\sqrt{a^2 + b^2} = \sqrt{24^2 + 7^2} = 25$

Thus the maximum and minimum values of $7\ cos\theta + 24\ sin\theta$ are 25 and -25 respectively.

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Transformation Formulae

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
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We have $\sin(A+B) = sinA\ cosB + cosA\ sinB \ldots (i)$

$\sin(A-B) = sinA\ cosB\ - cosA\ sinB \ldots (ii)$

$\cos(A+B) = cosA\ cosB\ - sinA\ sinB \ldots (iii)$

$\cos(A-B) = cosA\ cosB + sinA\ sinB \ldots (iv)$

Adding (i) and (ii), we obtain

$\sin(A+B) + \sin(A-B) = 2\ sinA\ cosB$

Subtracting (ii) from (i), we obtain

$\sin(A+B) - \sin(A-B) = 2\ cosA\ sinB$

Adding (iii) and (iv), we obtain

$\cos(A+B) + \cos(A-B) = 2\ cosA\ .\ cosB$

Subtracting (iii) from (iv), we obtain

$\cos(A-B) - \cos(A+B) = 2\ sinA\ sinB$

In above formulae $A > B$

Formulae to transform the sum or difference into products

Let $A + B = C$ and $A - B = D$. Then, $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

Substituting the values of A, B, C and D in the above formulae, we get

$$\sin C + sinD = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$sinC - sinD = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$cosC + cosD = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$cosD - cosC = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Or, $\cos C - cosD = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Or, $cosC - cosD = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Example: Convert each of the following products into the sum or difference of sines and cosines.

$(i) 2\sin 5\theta\ cos\theta$

Sol: $2\sin 5\theta\ cos\theta = \sin(5\theta+\theta) + \sin(5\theta-\theta) = \sin 6\theta + \sin 4\theta$

$(ii)\ cos75° \cos 15°$

Sol: $cos75° \cos 15° = \frac{1}{2}(\ 2cos75° \cos 15°) = \frac{1}{2}\{\ \cos(75° + 15°) + \cos(75°\ - 15°)\}$

$$= \frac{1}{2}(\cos 90° + \cos 60°)$$

Example: Express each of the following as a product:

$(i)\ \sin 6\theta\ - \sin 2\theta$

Sol: $\sin 6\theta\ - \sin 2\theta = 2\sin\left(\frac{6\theta-2\theta}{2}\right)\ cos\left(\frac{6\theta+2\theta}{2}\right) = 2\sin 2\theta \cos 4\theta$

$(ii)\ \cos 6\theta\ - \cos 8\theta$

Sol: $\cos 6\theta\ - \cos 8\theta = 2\sin\left(\frac{8\theta+6\theta}{2}\right)\sin\left(\frac{8\theta-6\theta}{2}\right) = 2\ sin7\theta \sin\theta$

Example: Prove that $\frac{\cos 7x+\cos 5x}{sin7x -\sin 5x} = cotx$

Sol: $LHS = \frac{\cos 7x+\cos 5x}{sin7x -\sin 5x} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{cosx}{sinx} = cotx = RHS$

Example: Prove that $\frac{\sin 5x-2\sin 3x+sinx}{\cos 5x -\cos x} = tanx$

Sol: $LHS = \frac{\sin 5x-2\sin 3x+sinx}{\cos 5x -\cos x} = \frac{sin5x+sinx-2\,sin3x}{\cos 5x-cosx}$

$= \frac{2\,sin3x\cos 2x\ -2\,sin3x}{-2\sin 3x\ sin2x} = -\frac{sin3x(\ cos2x\ -1)}{sin3x\ sin2x}$

$= \frac{1-\cos 2x}{sin2x} = \frac{2\,sin^2x}{2\,sinx\ cosx} = tanx = RHS$

Example: Prove that $cos18° - sin18° = \sqrt{2}\, sin27°$

Sol: $LHS = cos18° - sin18°$

$= \cos(90° - 72°) - sin18°$

$= sin72° - sin18°$

$$= 2\sin\left(\frac{72° - 18°}{2}\right)\cos\left(\frac{72° + 18°}{2}\right)$$

$= 2\sin 27° \cos 45°$

$$= 2\, sin27° \frac{1}{\sqrt{2}}$$

$= \sqrt{2}\, sin27° = RHS$

Example: Prove that $sin10° \; sin30° \; sin50° \; sin70° = \frac{1}{16}$

Sol: $LHS = sin10° \; sin30° \; sin50° \; sin70°$

$= sin30° \; (sin10° \; sin50°) sin70°$

$= \frac{1}{2} \; (sin50° \; sin10°) \sin 70°$

$= \frac{1}{2} \times \frac{1}{2} (\; 2 \; sin50° \; sin10°) sin70°$

$= \frac{1}{4} \; [\{\cos(50° - 10°) - \cos(50° + 10°)\} sin70°]$

$= \frac{1}{4} \{(cos40° - \cos 60°) \sin 70°\}$

$= \frac{1}{4} \; (\; sin70° \cos 40° - sin70° \cos 60°)$

$$= \frac{1}{4}\left(sin70° \, cos40° \; - \frac{1}{2} \; sin70° \right)$$

$$= \frac{1}{8} \; (\, 2sin70° \, cos40° - sin70°)$$

$$= \frac{1}{8} \; \{\sin(70° + 40°) + \sin(70° \; - 40°) - sin70°\}$$

$$= \frac{1}{8} \; (\, sin110° + \sin 30° - sin70)$$

$$= \; \frac{1}{8} \left\{\sin(180° - 70°) + \frac{1}{2} \; - sin70°\right\}$$

$$= \frac{1}{8} \left(sin70° + \frac{1}{2} \; - sin70° \right) = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = RHS$$

Example: Prove that $\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right) = 0$

Sol: $LHS = \sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right)$

$$= sinx + 2\sin\left(\frac{x + \frac{2\pi}{3} + x + \frac{4\pi}{3}}{2}\right)\cos\left(\frac{x + \frac{2\pi}{3} - x - \frac{4\pi}{3}}{2}\right)$$

$$= sinx + 2\sin(\pi + x)\cos\left(-\frac{\pi}{3}\right)$$

$$= sinx + 2\,(-sinx)\cos\frac{\pi}{3}$$

$$= sinx\; - 2\; sinx\frac{.1}{2}$$

$$= sinx\; - sinx$$

$$= 0 = RHS$$

Example: Prove that $\frac{\sin 5A - sin7A + sin8A - sin4A}{cos4A + cos7A - cos5A - cos8A} = \cot 6A$

Sol: $LHS = \frac{\sin 5A - sin7A + sin8A - sin4A}{cos4A + cos7A - cos5A - cos8A}$

$$= \frac{sin8A - sin4A - (sin7A - sin5A)}{cos4A - cos8A - (cos5A - cos7A)}$$

$$= \frac{2\cos 6A\ sin2A - 2cos6A\ sinA}{2\ sin6A\ sin2A - 2\ sin6A\ sinA}$$

$$= \frac{cos6A\ (sin2A - sinA)}{sin6A(sin2A - sinA)}$$

$$= \frac{\cos 6A}{\sin 6A} = \cot 6A = RHS$$

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THANKING YOU

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