Problems on Compound Angles and Transformation Formulae

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 03
CHAPTER NAME : TRIGONOMETRIC FUNCTIONS

Example: Prove that $cos3A + \cos 5A + cos7A + cos15A = 4\ cos4A\ cos5A \cos 6A$

Sol: $LHS = \cos 3A + cos5A + cos7A + cos15A$

$= \cos 15A + cos5A + cos7A + cos3A$

$= 2\cos\left(\frac{15A + 5A}{2}\right)\cos\left(\frac{15A - 5A}{2}\right) + 2\cos\left(\frac{7A + 3A}{2}\right)\cos\left(\frac{7A - 3A}{2}\right)$

$= 2\cos 10A \cos 5A + 2\cos 5A \cos 2A$

$= 2\cos 5A(\cos 10A + \cos 2A)$

$= 2\cos 5A \left\{2\cos\left(\frac{10A + 2A}{2}\right)\cos\left(\frac{10A - 2A}{2}\right)\right\}$

$= 2\cos 5A \,.\, 2\cos 6A \cos 4A$

$= 4\cos 4A\ cos5A \cos 6A = RHS$

Example: If $a\,sinx = b\ \sin\left(x + \frac{2\pi}{3}\right) = c\sin\left(x + \frac{4\pi}{3}\right)$, prove that $ab + bc + ca = 0$.

Sol: We have $a\,sinx = b\ \sin\left(x + \frac{2\pi}{3}\right) = c\sin\left(x + \frac{4\pi}{3}\right) = k\ (say)$

$\Rightarrow \quad \frac{k}{a} = sinx, \frac{k}{b} = \sin\left(x + \frac{2\pi}{3}\right), \frac{k}{c} = \sin\left(x + \frac{4\pi}{3}\right)$

$\Rightarrow \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = \sin x + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right)$

$\Rightarrow \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = \left\{\sin\left(x + \frac{4\pi}{3}\right) + sinx\right\} + \sin\left(x + \frac{2\pi}{3}\right)$

$\Rightarrow \quad \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = 2\sin\left(x + \frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\left(x + \frac{2\pi}{3}\right)$

$\Rightarrow \quad k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -\sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{2\pi}{3}\right)$

$\Rightarrow \quad k\left(\frac{ab+bc+ca}{abc}\right) = 0$

$\Rightarrow \quad ab + bc + ca = 0$

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Example: Prove that $\frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = \tan 54°$

Sol: $\text{RHS} = \tan 54° = \tan(45° + 9°)$

$$= \frac{\tan 45° + \tan 9°}{1 - tan 45° \tan 9°}$$

$$= \frac{1 + \frac{\sin 9°}{\cos 9°}}{1 - \frac{\sin 9°}{\cos 9°}} = \frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = LHS$$

Example: Prove that $\frac{sinx}{cos3x}+\frac{sin3x}{cos9x}+\frac{sin9x}{cos27x}=\frac{1}{2}(\,tan27x-tanx)$

Sol: $LHS=\frac{sinx}{cos3x}+\frac{sin3x}{cos9x}+\frac{sin9x}{cos27x}$

$$=\frac{1}{2}\left(\frac{2\ sinx\ cosx}{cos3x\ cosx}+\frac{2\ sin3x\ cos3x}{cos9x\ cos3x}+\ \frac{2\ sin9x\ cos9x}{cos27x\ cos9x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin(x+x)}{cos3x\ cosx}+\frac{\sin(3x+3x)}{cos9x\ cos3x}+\frac{\sin(9x+9x)}{co27x\ cos9x}\right)$$

$$=\frac{1}{2}\left(\frac{sin2x}{cos3x\ cosx}+\frac{sin6x}{cos9x\ cos3x}+\frac{sin18x}{cos27x\ cos9x}\right)$$

$$=\frac{1}{2}\left\{\frac{\sin(3x-x)}{cos3x\ cosx}+\frac{\sin(9x-3x)}{cos9x\ cos3x}+\frac{\sin(27x-9x)}{cos27x\ cos9x}\right\}$$

$$= \frac{1}{2}\left(\frac{sin3xcosx - cos3x\ sinx}{cos3xcosx} + \frac{sin9x\ cos3x\ - cos9x\ sin3x}{cos9x\ cos3x} + \frac{sin27xcos9x - cos27xsin9x}{cos27x\ cos9x}\right)$$

$$= \frac{1}{2}\{(tan3x - tanx) + (tan9x - tan3x) + (tan27x - tan9x)\}$$

$$= \frac{1}{2}\ (tan27x - tanx) = RHS$$

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Example: Prove that $cos2\alpha \cos 2\beta + \ sin^2(\alpha - \beta) - \ sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$.

Sol: $LHS = cos2\alpha \cos 2\beta + \ sin^2(\alpha - \beta) - \ sin^2(\alpha + \beta) = \cos 2(\alpha + \beta)$

$= \cos 2\alpha cos2\beta \ + sin\,(\alpha - \beta + \alpha + \beta) \sin(\alpha - \beta - \alpha - \beta)$

$= \cos 2\alpha \cos 2\beta \ - sin2\alpha \sin 2\beta$

$= \cos(2\alpha + 2\beta)$

$= \cos 2(\alpha + \ \beta) = RHS$

Example: If α and β are the solutions of the equation $a\ tanx + b\ secx = c$, show that

$\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$

Sol: We have, $a\ tanx + b\ secx = c \ldots (i)$

$\Rightarrow\ c - a\ tanx = b \sec x$

$\Rightarrow\ (c - a\ tanx)^2 = b^2 sec^2 x$

$\Rightarrow\ c^2 + a^2 tan^2 x - 2ac\ tanx = b^2(1 + tan^2 x)$

$\Rightarrow\ tan^2 x\,(a^2 - b^2) - 2ac\ tanx + (c^2 - b^2) = 0 \ldots (ii)$

It is given that α and β are the solutions of the given equation (i). Therefore $tan\alpha$ and $tan\beta$ are roots of the equation (ii).

So, $tan\alpha + tan\beta = \frac{2ac}{a^2-b^2}$ and $tan\alpha\ tan\beta = \frac{c^2-b^2}{a^2-b^2}$

Hence $\tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\ tan\beta} = \frac{2ac}{a^2-c^2}$

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Example: Prove that : $\cos A \cos\left(\frac{\pi}{3} - A\right) \cos\left(\frac{\pi}{3} + A\right) = \frac{1}{4} \cos 3A$

Sol: $LHS = \cos A \cos\left(\frac{\pi}{3} - A\right) \cos\left(\frac{\pi}{3} + A\right)$

$$= \frac{1}{2} \cos A \{2\cos\left(\frac{\pi}{3} - A\right) \cos\left(\frac{\pi}{3} + A\right)\}$$

$$= \frac{1}{2} \cos A \left[\cos\left(\frac{\pi}{3} + A + \frac{\pi}{3} - A\right) + \cos\left(\frac{\pi}{3} + A - \frac{\pi}{3} + A\right)\right]$$

$$= \frac{1}{2} cosA \left[\cos\frac{2\pi}{3} + \cos 2A\right]$$

$$= \frac{1}{2} \cos A \left\{\cos 2A - \frac{1}{2}\right\} = \frac{1}{2} \cos A \cos 2A - \frac{1}{4} cosA$$

$$= \frac{1}{4} 2 \cos 2A\, cosA - \frac{1}{4} cosA = \frac{1}{4} (\cos 3A + cosA) - \frac{1}{4} cosA$$

$$= \frac{1}{4} \cos 3A + \frac{1}{4} \cos A - \frac{1}{4} cosA = \frac{1}{4} \cos 3A = RHS$$

Note : We have $\cos A \cos\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right) = \frac{1}{4}\cos 3A$

Similarly $\sin A \sin\left(\frac{\pi}{3} - A\right)\sin\left(\frac{\pi}{3} + A\right) = \frac{1}{4}\sin 3A$

and $\tan A \tan\left(\frac{\pi}{3} - A\right)\tan\left(\frac{\pi}{3} + A\right) = \tan 3A$

Example: If three angles A, B and C are in $A.P.$, prove that $\cot B = \frac{sinA - sinC}{\cos C - \cos A}$

Sol: Since A, B and C are in $A.P.$, so $2B = A + C$

Now, $RHS = \frac{sinA - sinC}{\cos C - \cos A} = \frac{2\sin\frac{A-C}{2}\cos\frac{A+C}{2}}{2\sin\frac{A+C}{2}\sin\frac{A-C}{2}}$

$= \cot\left(\frac{A+C}{2}\right) = \cot\left(\frac{2B}{2}\right) = \cot B = LHS$

Example: If A, B and C are the angles of a triangle, then prove that

$sin^2A + sin^2B + sin^2C - 2\ cosA \cos B \cos C = 2$

Sol: $LHS = \ sin^2A + sin^2B + sin^2C - 2\ cosA \cos B \cos C$

$= \ \ 1 - cos^2A + 1 - \ cos^2B + sin^2C \ - 2 \cos A \cos B \cos C$

$= 2 \ - \ cos^2A - (cos^2B \ - \ sin^2C) - 2\ cosA \cos B \cos C$

$= 2 \ - \ cos^2A \ - \cos(B + C) \cos(B - C) - 2 \cos A \cos B \cos C$

$= 2 \ - \ cos^2A + \cos A \cos(B - C) - 2 \cos A\ cosB \cos C \ \ [\ since\ A + B + C = \ \pi\ ]$

$= 2 \ - cosA\ [\cos A \ - \cos(B - C)] \ - 2 \cos A \cos B \cos C$

$= 2 \ - \cos A\ [- \cos(B + C) - \cos(B - C)] - 2 \cos A \cos B \cos C$

$= 2 + \cos A \ \ [\cos(B + C) + \cos(B - C)] - \ \ 2\cos A \cos B \cos C$

$= 2 + \cos A\ [\ 2 \cos B \cos C] \ - \ \ 2 \cos A \cos B \cos C$

$= 2 = RHS$

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Trigonometric Functions of Multiple and Sub multiple Angles

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Trigonometric Ratios of Multiple and Submultiple Angles

If A is any angle, then angles $2A, 3A, 4A, \ldots$ etc are called multiple angles and the angles $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \ldots$ etc are called sub - multiple angles of A.

Trigonometric Ratios of angle $2A$ in terms of that of angle A

$(i)\ \sin 2A = \sin(A + A) = sinA\ cosA + cosA\ sinA = 2\ sinA\ cosA$

(ii) We have $sin2A = 2\ sinA\ cosA = \frac{2\ sinA\ cos^2A}{cosA} = \frac{2\ tanA}{sec^2A} = \frac{2\ tanA}{1 + tan^2A}$

$(iii)\ \cos 2A = \cos(A + A) = cosA\ cosA - sinA\ sinA = cos^2A - sin^2A$

$(iv)\ \cos 2A = cos^2A - sin^2A = cos^2A - (1 - cos^2A) = 2cos^2A - 1$

$(v)\ \cos 2A = cos^2A - sin^2A = 1 - sin^2A - sin^2A = 1 - 2sin^2A$

$(vi)\ cos2A = cos^2A - sin^2A = \frac{cos^2A - sin^2A}{cos^2A + sin^2A} = \frac{1 - tan^2A}{1 + tan^2A}$

(vii) We have $\cos 2A = 1 - 2sin^2A \Rightarrow \ sin^2A = \frac{1-cos2A}{2}$

$(viii)$ We have $cos2A = 2cos^2A - 1 \Rightarrow cos^2A = \frac{1+\cos 2A}{2}$

(ix) $\tan 2A = \tan(A + A) = \dfrac{\tan A + tanA}{1 - tanA\ tanA} = \dfrac{2\ tanA}{1 - \ tan^2A}$

(x) $tan^2\ A = \frac{1 - \cos 2A}{1+\cos 2A}$

Trigonometric Ratios of the Angle A in terms of Angle $\frac{A}{2}$

Replacing A by $\frac{A}{2}$ in the relations of multiple angles of $2A$, we obtain

$(i)\ sinA = 2\sin\frac{A}{2}\cos\frac{A}{2}$

$(ii)\ sinA = \dfrac{2\tan\frac{A}{2}}{1 + tan^2\frac{A}{2}}$

$(iii)\ cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2}$

$(iv)\ cosA = 2\ cos^2\frac{A}{2} - 1$

$(v)\ cosA = 1 - 2\ sin^2\frac{A}{2}$

$$(vi)\ cosA = \frac{1 - \ tan^2 \frac{A}{2}}{1 + \ tan^2 \frac{A}{2}}$$

$$(vii)\ sin^2 \frac{A}{2} = \frac{1 - cosA}{2}$$

$$(viii)\ cos^2 \frac{A}{2} = \frac{1 + cosA}{2}$$

$$(ix)\ tanA = \frac{2 \tan \frac{A}{2}}{1 - tan^2 \frac{A}{2}}$$

$$(x)\ tan^2 \frac{A}{2} = \frac{1 \ - \cos A}{1 + \cos A}$$

Trigonometric Ratios of Angle $3A$ in terms of Angle A

(i) $\sin 3A = 3\ sinA - 4sin^3A$

Sol: $\sin 3A = \sin(A + 2A) = sinA\ cos2A + sin2A\ cosA$

$= sinA\ (1 - 2sin^2A) + (2\ sinA\ cosA)cosA$

$= sinA - 2sin^3A + 2\ sinA\ (1 - sin^2A)$

$= sinA\ - 2sin^3A + 2\ sinA - 2\ sin^3A = 3\ sinA\ - 4sin^3A$

$(ii)\ cos3A = 4cos^3A\ - 3\ cosA$

$$(iii)\ tan3A = \frac{3\ tanA - \ tan^3A}{1 - 3tan^2A}$$

Trigonometric Ratios of Angle A in terms of Angle $\frac{A}{3}$

Replacing A by $\frac{A}{3}$ in the above formulae, we obtain

$$(i)\ sinA = 3\sin\left(\frac{A}{3}\right) - 4\ sin^3\left(\frac{A}{3}\right)$$

$$(ii)\ cosA = 4\ cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$$

$$(iii) tanA = \frac{3\tan\left(\frac{A}{3}\right) - tan^3\left(\frac{A}{3}\right)}{1 - 3\ tan^2\left(\frac{A}{3}\right)}$$

Value of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $sinA$

We have $(\cos\frac{A}{2} + \sin\frac{A}{2})^2 = cos^2\frac{A}{2} + sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2} = 1 + sinA$

$\Rightarrow \quad \cos\frac{A}{2} + \sin\frac{A}{2} = \pm\sqrt{1 + sinA}$ .... (1)

Similarly, $\cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{1 - sinA}$ .... (2)

By adding (1) and (2) and subtracting , we have

$2\cos\frac{A}{2} = \pm\sqrt{1 + sinA} \pm \sqrt{1 - sinA}$ and

$2\sin\frac{A}{2} = \pm\sqrt{1 + sinA} \mp \sqrt{1 - sinA}$

There are 4 values of $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$, when A is not known but $sinA$ is given. If A is known, then definite sign of $\cos\frac{A}{2} + \sin\frac{A}{2}$ and $\cos\frac{A}{2} - \sin\frac{A}{2}$ may be obtained in the following manner.

$\cos\frac{A}{2} + \sin\frac{A}{2} = \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{A}{2}\right)$, $\quad \cos\frac{A}{2} - \sin\frac{A}{2} = \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{A}{2}\right)$

$\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{A}{2}\right) = \pm\sqrt{1 + sinA}$ and $\sqrt{2}\cos\left(\frac{\pi}{4} + \frac{A}{2}\right) = \pm\sqrt{1 - sinA}$

Note:

1. $\sin\frac{A}{2} = \begin{cases} \sqrt{\frac{1-\cos A}{2}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ I \ or \ II \\ -\sqrt{\frac{1-\cos A}{2}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ III \ or \ IV \end{cases}$

2. $\cos\frac{A}{2} = \begin{cases} \sqrt{\frac{1+cosA}{2}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ \text{I or IV} \\ -\sqrt{\frac{1+cosA}{2}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ \text{II or III} \end{cases}$

3. $\tan\frac{A}{2} = \begin{cases} \sqrt{\frac{1-cosA}{1+cosA}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ \text{I or III} \\ -\sqrt{\frac{1-cosA}{1+cosA}} \ if \ \frac{A}{2} \ lies \ in \ quadrant \ \text{II or IV} \end{cases}$

Example: If $sinA = \frac{3}{5}$, where $0° < A < 90°$, find the values of $sin2A, cos2A, tan2A$.

Sol: We have $sinA = \frac{3}{5}$, where $0° < A < 90°$

Since $cos^2A = 1 - sin^2A$

So, $cosA = \sqrt{1 - sin^2A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore tanA = \dfrac{sinA}{cosA} = \dfrac{3}{4}$

Now, $sin2A = 2\ sinA\ cosA = 2.\frac{3}{5}.\frac{4}{5} = \frac{24}{25}$

$cos2A = 1 - 2\ sin^2A = 1 - 2\left(\dfrac{3}{5}\right)^2 = \dfrac{7}{25}$

$\tan 2A = \dfrac{2\ tanA}{1 - tan^2A} = \dfrac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \dfrac{24}{7}$

Example: Prove that $\frac{1+sin2\theta+cos2\theta}{1+sin2\theta-cos2\theta}=cot\theta$

Sol: $LHS=\frac{1+sin2\theta+cos2\theta}{1+sin2\theta-cos2\theta}=\frac{(1+cos2\theta)+sin2\theta}{(1-cos2\theta)+sin2\theta}=\frac{2cos^2\theta+2\,sin\theta\,cos\theta}{2\,sin^2\theta+2\,sin\theta\,cos\theta}$

$=\frac{2\,cos\theta\,(\,cos\theta+sin\theta)}{2\,sin\theta\,(\,cos\theta+sin\theta)}=cot\theta=RHS$

Example: Prove that $\frac{\sin 5x-2\,sin3x+sinx}{\cos 5x-cosx}=tanx$

Sol: $LHS=\frac{\sin 5x-2\,sin3x+sinx}{\cos 5x-cosx}=\frac{\sin 5x+sinx-2\sin 3x}{cos5x-cosx}$

$=\frac{2\,sin3x\,cos2x\;-2\,sin3x}{-2\,sin3x\,sin2x}=\frac{sin3x(\,2\,cos2x-1)}{-2sin3x\,sin2x}$

$=\frac{1-cos2x}{sin2x}=\frac{2\,sin^2x}{2\,sinx\,cosx}=tanx=RHS$

Example: Prove that $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

Sol: We have $\cos\frac{7\pi}{8}=\cos\left(\pi-\frac{\pi}{8}\right)=-\cos\frac{\pi}{8}$

Also $\cos\frac{5\pi}{8}=\cos\left(\pi-\frac{3\pi}{8}\right)=-\cos\frac{3\pi}{8}$

$LHS=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$=\left(1-cos^2\frac{\pi}{8}\right)\left(1-cos^2\frac{3\pi}{8}\right)=sin^2\frac{\pi}{8}\ sin^2\frac{3\pi}{8}$

$=\frac{1-\cos\frac{\pi}{4}}{2}\times\frac{1-\cos\frac{3\pi}{4}}{2}=\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)=\frac{1}{4}\left(1-\frac{1}{2}\right)=\frac{1}{8}=RHS$

Example: Prove that $cos^2x + \ cos^2\left(x+\frac{\pi}{3}\right) + \ cos^2\ \left(x-\frac{\pi}{3}\right) = \frac{3}{2}$

Sol: $LHS = cos^2x + \ cos^2\left(x+\frac{\pi}{3}\right) + \ cos^2\ \left(x-\frac{\pi}{3}\right)$

$$= \ \frac{1+cos2x}{2} + \frac{1+\cos 2\left(x+\frac{\pi}{3}\right)}{2} + \frac{1+\cos 2\left(x-\frac{\pi}{3}\right)}{2}$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\ \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\ [3+\cos 2x\ -\cos 2x] = \frac{3}{2} = RHS$$

Example: Find the value of $\tan\frac{\pi}{8}$

Sol: Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$

Now, $\tan 2x = \frac{2\,tanx}{1- tan^2 x} \Rightarrow \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1- tan^2\frac{\pi}{8}} \Rightarrow 1 = \frac{2\tan\frac{\pi}{8}}{1- tan^2\frac{\pi}{8}}$

$\Rightarrow \quad 1 - tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$

$\Rightarrow \quad tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

$\Rightarrow \quad \tan\frac{\pi}{8} = \frac{-2 \pm\sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $\frac{\pi}{8}$ lies in the first quadrant so $\tan\frac{\pi}{8}$ is positive.

Hence $\tan\frac{\pi}{8} = -1 + \sqrt{2} = \sqrt{2} - 1$

Example: If $tanx = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, find the values of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Sol: Since $\pi < x < \frac{3\pi}{2}$, so $\cos x$ is negative.

Also, $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$, so $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $sec^2x = 1 + tan^2x = 1 + \frac{9}{16} = \frac{25}{16}$

$\Rightarrow \quad cos^2x = \frac{16}{25} \Rightarrow cosx = -\frac{4}{5}$

Now $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$

$$\cos\frac{x}{2} = -\sqrt{\frac{1 + cosx}{2}} = -\sqrt{\frac{1 - \frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = -3$$

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Example: Show that $2\ sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$

Sol: $LHS = 2\ sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

$= \ 2\ sin^2\beta + 2\cos(\alpha+\beta)\ (2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$

$= 2sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$

$= 2sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\ cos^2(\alpha+\beta) + 2\ cos^2\ (\alpha+\beta) - 1$

$= 2sin^2\beta + 2(\ cos^2\alpha - \ sin^2\beta) - 1$

$= 2sin^2\beta + 2\ cos^2\alpha\ - 2sin^2\beta\ - 1$

$= 2cos^2\alpha - 1 = \cos 2\alpha = RHS$

Example: Prove that $\sqrt{3}\ cosec\ 20° - \sec 20° = 4$

Sol: $LHS = \sqrt{3}\ cosec\ 20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° \ - \sin 20°}{\sin 20° \cos 20°}$$

$$= \frac{2(\frac{\sqrt{3}}{2}\cos 20° \ - \frac{1}{2}\sin 20°\ )}{\frac{1}{2}\ 2\sin 20° \cos 20°}$$

$$= \frac{4\ (\sin 60° \cos 20° \ - \cos 60° \sin 20°)}{\sin 40°} \ = \frac{4\sin 40°}{\sin 40°} = 4 = RHS$$

Example: Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

Sol: $LHS = \cos 20° \cos 40° \cos 60° \cos 80°$

$= \frac{1}{2} \cos 20° \cos 40° \cos 80°$

$= \frac{1}{2} \cos 20° \cos(60° - 20°) \cos(60° + 20°)$

$= \frac{1}{2} \cos 20° \, (cos^2 20° - sin^2 \, 60°)$

$= \frac{1}{2} \cos 20° \left(cos^2 20° - \frac{3}{4} \right) = \frac{1}{8} \cos 20° \, (4 \, cos^2 \, 20° - 3)$

$= \frac{1}{8} \, (4 \, cos^3 \, 20° - 3 \cos 20°)$

$= \frac{1}{8} \cos 60° = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16} = RHS$

Example: If $\sin(\theta+\alpha)=a$ and $\sin(\theta+\beta)=b$, prove that $\cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$
$=1-2a^2-2b^2$

Sol: Now $\cos(\theta+\alpha)=\sqrt{1-sin^2(\theta+\alpha)}=\sqrt{1-a^2}$

Also $\cos(\theta+\beta)=\sqrt{1-sin^2(\theta+\beta)}=\sqrt{1-b^2}$

We have $\cos(\alpha-\beta)=\cos\{(\theta+\alpha)-(\theta+\beta)\}$

$=\cos(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\alpha)\sin(\theta-\beta)$

$=\sqrt{1-a^2}\sqrt{1-b^2}+ab=ab+\sqrt{1-a^2-b^2+a^2b^2}$

$\therefore\ \cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)=2\,cos^2(\alpha-\beta)-4ab\cos(\alpha-\beta)$

$=\ 2\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}^2-1\ -4ab\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}$

$=\ 2\left\{ab+2ab\sqrt{1-a^2-b^2+a^2b^2}\ +1-a^2-b^2+a^2b^2\right\}-1-4a^2b^2$

$-4ab\sqrt{1-a^2-b^2+a^2b^2}$

$=2-2a^2-2b^2-1=1-2a^2-2b^2$

Example: If $cos\theta = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$, prove that $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$

Sol: We have $cos\theta = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$

$$\Rightarrow \frac{1 - tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta}{1 - cos\alpha\cos\beta}$$

$$\Rightarrow \frac{\left(1 - tan^2\frac{\theta}{2}\right) + \left(1 + tan^2\frac{\theta}{2}\right)}{\left(1 - tan^2\frac{\theta}{2}\right) - \left(1 + tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha - \cos\beta) + (1 - cos\alpha\cos\beta)}{(\cos\alpha - \cos\beta) - (1 - cos\alpha\cos\beta)}$$

$$\Rightarrow \frac{2}{-2\,tan^2\frac{\theta}{2}} = \frac{1 + cos\alpha - cos\beta - cos\alpha\,cos\beta}{-(1 + cos\alpha - cos\beta - cos\alpha\,cos\beta)}$$

$$\Rightarrow \frac{1}{tan^2\frac{\theta}{2}} = \frac{(1 + cos\alpha)(1 - cos\beta)}{(1 - cos\alpha)(1 + cos\beta)}$$

$$\Rightarrow tan^2\frac{\theta}{2} = tan^2\frac{\alpha}{2}\;cot^2\frac{\beta}{2}$$

$$\Rightarrow \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

Example: Prove that $sin3x\, sin^3x + \cos 3x\, cos^3x = cos^3 2x$

Sol: $LHS = sin3x\, sin^3x + \cos 3x\, cos^3x$

$= \frac{1}{4}\ (sin3x\ 4sin^3x + \cos 3x\ 4cos^3x)$

$= \frac{1}{4}\ \{\sin 3x\ (3\ sinx\ - \sin 3x) + \cos 3x\ (\cos 3x + 3\ cosx)\}$

$= \frac{1}{4}\ \{3\ sin3x\ sinx\ -\ sin^2\ 3x +\ cos^2 3x + 3 \cos 3x\ cosx\}$

$= \frac{1}{4}\{3(\cos 3x\ cosx + sin3x\ sinx) + (cos^2 3x - sin^2 3x)\}$

$= \frac{1}{4}\ \{3 \cos(3x - x) + \cos 2(3x)\}$

$= \frac{1}{4}\ \left(\ 3 \cos 2x + \cos 3(2\mathrm{x})\right)$

$= \frac{1}{4}\ 4\ cos^3 2x =\ cos^3\ 2x = RHS$

Example: Prove that $\frac{\tan 5x+\tan 3x}{\tan 5x -\tan 3x} = 4\cos 2x \cos 4x$

Sol: $LHS = \frac{\tan 5x+\tan 3x}{\tan 5x -\tan 3x}$

$$= \frac{\frac{sin5x}{\cos 5x}+\frac{sin3x}{cos3x}}{\frac{sin5x}{\cos 5x}+\frac{sin3x}{cos3x}} = \frac{sin5x\cos 3x+\cos 5x\, sin3x}{\sin 5x\cos 3x-\cos 5x\, sin3x}$$

$$= \frac{\sin(5x+3x)}{\sin(5x-3x)} = \frac{sin8x}{\sin 2x}$$

$$= \frac{2\sin 4x\cos 4x}{\sin 2x}$$

$$= \frac{2\,(2\sin 2x\cos 2x)\cos 4x}{\sin 2x}$$

$$= 2\cos 2x\cos 4x$$

$$= RHS$$

Example: Prove that $\frac{\cos 5x+\cos 4x}{1-2\cos 3x} = -(\cos 2x + cosx)$

Sol: $LHS = \frac{\cos 5x+\cos 4x}{1-2\cos 3x} = \frac{sin3x\,(\cos 5x+\cos 4x)}{sin3x\,(1-2\cos 3x)}$

$$= \frac{(2\sin\frac{3x}{2}\cos\frac{3x}{2})\,(2\cos\frac{9x}{2}\cos\frac{x}{2})}{\sin 3x - 2\sin 3x\cos 3x}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin(\frac{3x-6x}{2})\cos\left(\frac{3x+6x}{2}\right)} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(-\frac{3x}{2}\right)\cos\frac{9x}{2}}$$

$$= \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\sin\left(\frac{3x}{2}\right)\cos\frac{9x}{2}} = -2\cos\frac{3x}{2}\cos\frac{x}{2} = -(cos2x + cosx) = RHS$$

Example: If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, prove that $\cos\alpha = \frac{\cos\theta - e}{1 - e\cos\theta}$

Sol: We have $\cos\alpha = \frac{1-tan^2\frac{\alpha}{2}}{1+tan^2\frac{\alpha}{2}}$

$$\Rightarrow \cos\alpha = \frac{1-\frac{1+e}{1-e}tan^2\frac{\theta}{2}}{1+\frac{1+e}{1-e}tan^2\frac{\theta}{2}}$$

$$\Rightarrow \cos\alpha = \frac{(1-e)-(1+e)tan^2\frac{\theta}{2}}{(1-e)+(1+e)\frac{tan^2\theta}{2}} \Rightarrow \cos\alpha = \frac{\left(1-tan^2\frac{\theta}{2}\right)-e\left(1+tan^2\frac{\theta}{2}\right)}{\left(1+tan^2\frac{\theta}{2}\right)-e\left(1-tan^2\frac{\theta}{2}\right)}$$

$$\Rightarrow \cos\alpha = \frac{\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}-e}{1-e\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}}$$ [Dividing numerator and denominator by $1+\ tan^2\frac{\theta}{2}$]

$$\Rightarrow \cos\alpha = \frac{\cos\theta - e}{1-e\cos\theta}$$ [since $\cos\theta = \frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$]

THANKING YOU

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