Introduction:

An algebraic expression consisting of two terms, connected by + or – sign is called a binomial expression.

The terms like $a+b, x-3y^2, \frac{2}{x}-\frac{1}{x^2}$ etc are binomial expressions.

Binomial Theorem:

The binomial theorem refers to the expansion of integral power of such a binomial $i.e.$, of the form $(x+y)^n, (a+b)^n, (3x+4y)^n$ etc.

The binomial expansion for the case $n=2$ was used by the Greek mathematician Euclid. However, Omar Khayyam, the Arab mathematician is credited with the binomial expansion for higher natural numbers. Later Sir Isaac Newton generalized the binomial theorem for negative integral and fractional indices.

In earlier classes, we have already studied that

$(i)\ (a+b)^0 = 1$

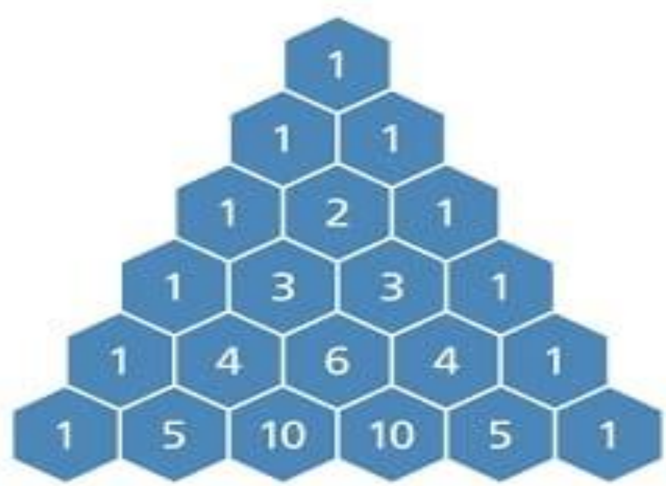
$(ii)\ (a+b)^1 = a+b$

$(iii)(a+b)^2 = a^2+2ab+b^2$

$(iv)(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(v)(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

The coefficients in the above expansions follow a particular pattern as given below:



We also observe that each row is bounded by 1 on both sides. Any entry, except the first and last, in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right. The above pattern or structure of numbers is known as Pascal's triangle. (Blaise Pascal (1623 – 1662))

The above coefficients can be written in a combinatorial form as

$$^0C_0$$
$$^1C_0 \quad ^1C_1$$
$$^2C_0 \quad ^2C_1 \quad ^2C_2$$
$$^3C_0 \quad ^3C_1 \quad ^3C_2 \quad ^3C_3$$
$$^4C_4 \quad ^4C_1 \quad ^4C_2 \quad ^4C_3 \quad ^4C_4$$
$$^5C_0 \quad ^5C_1 \quad ^5C_2 \quad ^5C_3 \quad ^5C_4 \quad ^5C_5$$
$$^6C_0 \quad ^6C_1 \quad ^6C_2 \quad ^6C_3 \quad ^6C_4 \quad ^6C_5 \quad ^6C_6$$
$$^7C_0 \quad ^7C_1 \quad ^7C_2 \quad ^7C_3 \quad ^7C_4 \quad ^7C_5 \quad ^7C_6 \quad ^7C_7$$
$$^8C_0 \quad ^8C_1 \quad ^8C_2 \quad ^8C_3 \quad ^8C_4 \quad ^8C_5 \quad ^8C_6 \quad ^8C_7 \quad ^8C_8$$

(Pascal triangle)

Binomial Theorem for any Positive Integral Index:

If a and b are any two real numbers, then for any positive integer n, we have

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_n\, a^0 b^n$$

$$\Rightarrow\ (a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_n\, b^n = \sum_{r=0}^{n} n_{C_r}\, a^{n-r} b^r$$

where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, … , ${}^nC_n$ are called binomial coefficients. These binomial coefficients can also be written as $C_0, C_1, C_2, \ldots, C_n$.

Some Important Observations:

1. The total number of terms in the expansion of $(a+b)^n$ is $n+1$.
2. In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a+b$ $i.e., n$.
3. In the successive terms of the expansion, powers of the first quantity a go on decreasing by 1 whereas the powers of the second quantity b increase by 1.

4. The binomial coefficients of terms equidistant from the beginning and end are equal.

Particular Cases of Binomial expansion:

1. Expansion of $(a-b)^n$:

We have

$$(a-b)^n = n_{C_0}a^n - n_{C_1}a^{n-1}b + n_{C_2}a^{n-2}b^2 - \cdots + (-1)^r\, n_{C_r}\, a^{n-r}b^r + \cdots + (-1)^n n_{C_n}b^n$$

2. Expansion of $(1+x)^n$:

Replacing a by 1 and b by x , we have

$$(1+x)^n = n_{C_0} + n_{C_1}x + n_{C_2}x^2 + \cdots + n_{C_r}x^r + \cdots + n_{C_n}x^n$$

3. Expansion of $(1-x)^n$:

Replacing x by $-x$, we get

$$(1-x)^n = n_{C_0} - n_{C_1}x + n_{C_2}x^2 + \cdots + (-1)^r\, n_{C_r}x^r + \cdots + (-1)^n\, n_{C_n}x^n$$

4. Expansion of $(a+b)^n + (a-b)^n$:

$$(a+b)^n + (a-b)^n = 2\left[n_{C_0}a^n + n_{C_2}a^{n-2}b^2 + n_{C_4}\, a^{n-4}b^4 + \cdots\right]$$

5. Expansion of $(a+b)^n - (a-b)^n$:

$$(a+b)^n - (a-b)^n = 2\left[n_{C_1}a^{n-1}b + n_{C_3}a^{n-3}b^3 + \cdots\right]$$

Points to Remember:

1. If n is odd, then $(a+b)^n + (a-b)^n$ and $(a+b)^n - (a-b)^n$, both have $\left(\frac{n+1}{2}\right)$ terms.
2. If n is even, then $(a+b)^n + (a-b)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(a+b)^n - (a-b)^n$ has $\frac{n}{2}$ terms.

Example: Find the number of terms in the expansion of the following:

(i) $(2x-3y)^9$

Sol: The number of terms is $9+1=10$.

(ii) $(1+z)^4$

Sol: The number of terms is $4+1=5$.

(iii) $(1+5\sqrt{2}x)^9 + (1-5\sqrt{2}x)^9$

Sol: The number of terms is $\frac{9+1}{2} = 5$.

$(iv) (3x + y)^8 - (3x - y)^8$

Sol: The number of terms is $\frac{8}{2} = 4$.

$(v)\ (1 + 2x + x^2)^{20}$

Sol: The number of terms in the expansion of $(1 + 2x + x^2)^{20} = (1 + x)^{40}$ is 41.

Example: Expand $(x^2 + 2y)^5$ by binomial theorem.

Sol: Using the binomial theorem,

$$(x^2 + 2y)^5 = 5_{C_0}(x^2)^5 + 5_{C_1}(x^2)^4(2y) + 5_{C_2}(x^2)^3(2y)^2 + 5_{C_3}(x^2)^2(2y)^3 + 5_{C_4}(x^2)(2y)^4 + 5_{C_5}(2y)^5$$

$$= x^{10} + 5x^8(2y) + 10x^6(4y^2) + 10x^4(8y^3) + 5x^2(16y^4) + 32y^5$$

$$= x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 80x^2y^4 + 32y^5$$

Example: Using the binomial theorem, expand $\left(\frac{2x}{3} - \frac{3}{2x}\right)^4$.

Sol: We have,

$$\left(\frac{2x}{3} - \frac{3}{2x}\right)^4 = 4_{C_0}\left(\frac{2x}{3}\right)^4 - 4_{C_1}\left(\frac{2x}{3}\right)^3\left(\frac{3}{2x}\right) + 4_{C_2}\left(\frac{2x}{3}\right)^2\left(\frac{3}{2x}\right)^2 - 4_{C_3}\left(\frac{2x}{3}\right)\left(\frac{3}{2x}\right)^3 + 4_{C_4}\left(\frac{3}{2x}\right)^4$$

$$= \frac{16x^4}{81} - 4 \times \frac{8x^3}{27} \times \frac{3}{2x} + 6 \times \frac{4x^2}{9} \times \frac{9}{4x^2} - 4 \times \frac{2x}{3} \times \frac{27}{8x^3} + \frac{81}{16x^4}$$

$$= \frac{16}{81}x^4 - \frac{16}{9}x^2 + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$$

Example: Using the binomial theorem, expand $\{(x + y)^5 + (x - y)^5\}$ and hence find the value of $\left\{(\sqrt{2} + 1)^5 + \left(\sqrt{2} - 1\right)^5\right\}$.

Sol: We have, $(x + y)^5 + (x - y)^5 = 2\{5_{C_0}x^5 + 5_{C_2}x^3y^2 + 5_{C_4}x^1y^4\}$

$$= 2(x^5 + 10x^3y^2 + 5xy^4)$$

Putting $x = \sqrt{2}$ and $y = 1$, we get

$$\left(\sqrt{2} + 1\right)^5 + \left(\sqrt{2} - 1\right)^5 = 2\left\{\left(\sqrt{2}\right)^5 + 10\left(\sqrt{2}\right)^3 + 5\sqrt{2}\right\} = 2\left(4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}\right) = 58\sqrt{2}$$

General Term in a Binomial Expansion:

The $(r+1)th$ term is called the general term of the expansion $(a+b)^n$ and it is denoted by T_{r+1}.

Thus $\boldsymbol{T_{r+1} = n_{C_r}\, a^{n-r} b^r}$

Example: Find the 9th term in the expansion of $\left(\frac{x}{a}+\frac{3a}{x^2}\right)^{12}$.

Sol: The 9th term $= T_9 = 12_{C_8}\left(\frac{x}{a}\right)^{12-8}\left(\frac{3a}{x^2}\right)^8$

$= 495\left(\frac{x^4}{a^4}\right)\frac{3^8a^8}{x^{16}} = 495 \times 3^8 \times \frac{a^4}{x^{12}}$

Example: Find the 6th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.

Sol: The 6th term $= T_6 = 9_{C_5}\left(\frac{4x}{5}\right)^{9-5}\left(-\frac{5}{2x}\right)^5$

$= 9_{C_5}\frac{4^4x^4}{5^4}(-1)^5\frac{5^5}{2^5x^5} = -\frac{9\times8\times7\times6\times5}{5\times4\times3\times2\times1}\frac{4^4x^4}{5^4}\frac{5^5}{2^5x^5} = -\frac{5040}{x}$

Example: Find the coefficient of 5th term in the expansion of $\left(\frac{x}{3}-3y\right)^7$.

Sol: The 5th term $= T_5 = 7_{C_4}\left(\frac{x}{3}\right)^3(-3y)^4$

$= \frac{7\times6\times5\times4}{4\times3\times2\times1}\times\frac{x^3}{27}\times 81y^4 = 105\,x^3y^4$

Thus, the coefficient of the 5th term is 105.

Example: Find a, if 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal.

Sol: We have, $T_{17} = 50_{C_{16}}2^{50-16}\,a^{16} = 50_{C_{16}} \times 2^{34}\times a^{16}$

and, $T_{18} = 50_{C_{17}}2^{50-17}\,a^{17} = 50_{C_{17}} \times 2^{33}\times a^{17}$

It is given that the 17th and 18th terms are equal.

$i.e.\, T_{17} = T_{18}$

$\Rightarrow\ 50_{C_{16}}\times 2^{34}\times a^{16} = 50_{C_{17}}\times 2^{33}\times a^{17}$

$\Rightarrow \frac{50!}{16!34!}\times 2 = \frac{50!}{17!33!}\times a$

$\Rightarrow a = \frac{17\times2}{34} = 1.$

Middle term(s) in a Binomial Expansion:

We know that, the binomial expansion of $(a+b)^n$ has $n+1$ terms. So, for middle terms, there are two cases.

Case – 1: If n is even, then the total number of terms $(i.e., n+1)$ is odd. In this case, there is only one middle term which is the $\left(\frac{n}{2}+1\right)th$ term.

Case – 2: If n is odd, then the total number of terms $(i.e., n+1)$ is even. In this case, there are two middle terms, which are $\left(\frac{n+1}{2}\right)th$ term and $\left(\frac{n+3}{2}\right)th$ term.

Example: Find the middle term in the expansion of $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^{20}$.

Sol: Here $n=20$, which is even. So, $\left(\frac{20}{2}+1\right)th$ term $i.e., 11th$ term is the middle term.

Hence middle term $= T_{11} = 20_{C_{10}}\left(\frac{2x^2}{3}\right)^{20-10}\left(-\frac{3}{2x}\right)^{10} = 20_{C_{10}}x^{10}$

Example: Find the middle terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^7$.

Sol: Here $n=7$, which is odd.

So, $\frac{7+1}{2}\,th$ and $\frac{7+3}{2}\,th$ $i.e., 4th$ and $5th$ terms are two middle terms.

Now first middle term $= T_4 = 7_{C_3}(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3 = -\frac{105}{8}x^{13}$

and the second middle term $= T_5 = 7_{C_4}(3x)^{7-4}\left(-\frac{x^3}{6}\right)^4 = \frac{35}{48}x^{15}$.

Example: If the middle term of $\left(\frac{1}{x}+x\,sinx\right)^{10}$ is equal to $7\,\frac{7}{8}$, then find the value of x.

Sol: Here $n=10$, which is an even number.

So, the middle term is given by $\left(\frac{10}{2}+1\right)th$ term $i.e.$, $6th$ term.

Now, $T_6 = 10_{C_5}\left(\frac{1}{x}\right)^{10-5}(x\,sinx)^5 = 10_{C_5}\frac{1}{x^5}x^5sin^5x = 252\,sin^5x$

Given that middle term $= 7\frac{7}{8} = \frac{63}{8}$.

$\therefore 252\,sin^5x = \frac{63}{8}$

$\Rightarrow \ sin^5x = \frac{63}{8\ .\ 252} \Rightarrow \ sin^5x = \frac{1}{32}$

$\Rightarrow sinx = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow x = n\pi + \ (-1)^n\frac{\pi}{6}, n\ \in Z$

Example: Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1\ .3\ .5\ .....(2n-1)}{n!}\ 2^n x^n$, where n is a positive integer.

Sol: As $2n$ is even, the middle term of the expansion $(1+x)^{2n}$ is $\left(\frac{2n}{2}+1\right) th\ i.e., (n+1)th$ term.

Now, $T_{n+1} = \ 2n_{C_n}\ (1)^{2n-n}\ x^n = \ 2n_{C_n}x^n$

$= \frac{(2n)!}{n!n!}\ x^n = \frac{1.2.3.4.5......(2n-2)(2n-1)(2n)}{n!n!}\ x^n$

$= \frac{\{1.3.5.....(2n-1)\}\{2.4.6.....(2n-2)(2n)\}}{n!n!}\ x^n$

$= \frac{\{1.3.5....(2n-1)\}\ \{1.2.3....(n-1)(n)\}\ 2^n}{n!n!}\ x^n$

$= \frac{\{1\ .3.5....(2n-1)\}\ n!\ 2^n}{n!n!}\ x^n = \frac{1\ .3\ .5\ .....(2n-1)}{n!}\ 2^n x^n$

Example: Prove that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of the middle terms in the expansion of $(1+x)^{2n-1}$.

Sol: As $2n$ is even, so the expansion of $(1+x)^{2n}$ has only one middle term which is $\left(\frac{2n}{2}+1\right) th\ i.e., (n+1)th$ term.

The $(n+1)th$ term is $2n_{C_n}x^n$.

So, the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is $2n_{C_n}$.

Similarly, $(2n-1)$ being odd, the other expansion has two middle terms.

Middle terms are $\left(\frac{2n-1+1}{2}\right)\ th$ and $\left(\frac{2n-1+3}{2}\right) th\ i.e., nth$ and $(n+1)th$ terms.

Now, $T_n = \ 2n-1_{C_{n-1}}\ x^{n-1}$ and $T_{n+1} = \ 2n-1_{C_n}\ x^n$

So, the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$ are $2n-1_{C_{n-1}}$ and $2n-1_{C_n}$.

$\therefore$ The sum of these coefficients $= \ 2n-1_{C_{n-1}} + \ 2n-1_{C_n} = \ 2n-1+1_{C_n}$

$= \ 2n_{C_n} =$ Coefficient of the middle term in the expansion of $(1+x)^{2n}$.

Example: Find the value of k for which the coefficients of the middle terms in the expansion of $(1+kx)^4$ and $(1-kx)^6$ are equal.

Sol: In the expansion of $(1+kx)^4$, middle term = $4_{C_2}(kx)^2 = 6k^2x^2$

In the expansion of $(1-kx)^6$, middle term = $6_{C_3}(-kx)^3 = -20\,k^3x^3$

It is given that: coefficient of the middle term in $(1+kx)^4$ = coefficient of the middle term in $(1-kx)^6$.

$\Rightarrow 6k^2 = -20\,k^3 \Rightarrow k = 0, k = -\frac{3}{10}$

Equidistant Terms:

In the binomial expansion of $(a+b)^n$, the $(r+1)th$ term from the end is $\{(n+1)-r\} = (n-r+1)th$ term from the beginning.

The $(r+1)th$ term from the end in the expansion of $(a+b)^n$ is same as the $(r+1)th$ term from the beginning in the expansion of $(b+a)^n$.

Example: Find the 11th term from the end in the expansion of $\left(2x-\frac{1}{x^2}\right)^{25}$.

Sol: The given expansion contains 26 terms.

So, the 11th term from the end = $(26-11+1)th$ term from the beginning $i.e.$, 16th term from the beginning.

∴ Required term = $T_{16} = 25_{C_{15}}(2x)^{25-15}\left(-\frac{1}{x^2}\right)^{15}$

$= 25_{C_{15}} \times 2^{10} \times x^{10} \times \frac{(-1)^{15}}{x^{30}} = -25_{C_{15}} \times \frac{2^{10}}{x^{20}}$

Example: Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.

Sol: Clearly, the fifth term from the beginning = $T_5 = n_{C_4}\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = n_{C_4} \times 2^{\frac{n-4}{4}} \times \frac{1}{3}$

The fifth term from the end = $(n+1-5+1)th$ term from the beginning

$= (n-3)th$ term from the beginning

So, $T_{n-3} = n_{C_{n-4}}\left(\sqrt[4]{2}\right)^{n-(n-4)}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = n_{C_{n-4}} \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}$

It is given that $\frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$

$\Rightarrow \frac{n_{C_4} \times 2^{\frac{n-4}{4}} \times \frac{1}{3}}{n_{C_{n-4}} \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}} = \sqrt{6} \Rightarrow 2^{\frac{n-4}{4}-1} \times 3^{\frac{n-4}{4}-1} = 6^{\frac{1}{2}}$

$\Rightarrow (2 \times 3)^{\frac{n-4}{4}-1} = (2 \times 3)^{\frac{1}{2}}$

$\Rightarrow \frac{n-4}{4} - 1 = \frac{1}{2} \Rightarrow \frac{n-8}{4} = \frac{1}{2} \Rightarrow n - 8 = 2 \Rightarrow n = 10.$

Example: If O be the sum of odd terms and E that of even terms in the expansion of $(x + a)^n$, prove that

$(i)\ O^2 - E^2 = (x^2 - a^2)^n$

$(ii)\ 4OE = (x + a)^{2n} - (x - a)^{2n}$

$(iii) 2(O^2 + E^2) = (x + a)^{2n} + (x - a)^{2n}$

Sol: We have, $(x + a)^n = n_{C_0} x^n a^0 + n_{C_1} x^{n-1} a^1 + n_{C_2} x^{n-2} a^2 + \cdots + n_{C_{n-1}} x^1 a^{n-1} + n_{C_n} x^0 a^n$

$= \left(n_{C_0} x^n a^0 + n_{C_2} x^{n-2} a^2 + \cdots\right) + \left(n_{C_1} x^{n-1} a^1 + n_{C_3} x^{n-3} a^3 + \cdots\right) = O + E$ ... (1)

and

$(x - a)^n = n_{C_0} x^n a^0 - n_{C_1} x^{n-1} a^1 + n_{C_2} x^{n-2} a^2 - \cdots + n_{C_{n-1}} x^1 (-1)^{n-1} a^{n-1} + n_{C_n} x^0 (-1)^n a^n$

$= \left(n_{C_0} x^n a^0 + n_{C_2} x^{n-2} a^2 + \cdots\right) - \left(n_{C_1} x^{n-1} a^1 + n_{C_3} x^{n-3} a^3 + \cdots\right) = O - E$ ... (2)

(i) Multiplying (1) and (2), we get

$(x + a)^n (x - a)^n = (O + E)(O - E)$

$\Rightarrow (x^2 - a^2)^n = O^2 - E^2$

(ii) We have, $4OE = (O + E)^2 - (O - E)^2$

$\Rightarrow 4OE = \{(x + a)^n\}^2 - \{(x - a)^n\}^2$

$\Rightarrow 4OE = (x + a)^{2n} - (x - a)^{2n}$

(iii) Squaring (1) and $(2)$and then adding, we get

$\{(x + a)^n\}^2 + \{(x - a)^n\}^2 = (O + E)^2 + (O - E)^2$

$\Rightarrow (x + a)^{2n} + (x - a)^{2n} = 2(O^2 + E^2)$

Some More Problems Related to Binomial Theorem:

(Finding the Coefficient for a given index of the variable)

Example: Find the coefficient of x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, when $x \neq 0$.

Sol: Suppose $(r+1)th$ term contains x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$

Now, $T_{r+1} = 11_{C_r}(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = (-1)^r 11_{C_r} 2^{(11-r)}.3^r.x^{22-3r}$ ... (i)

If T_{r+1} contains x^{10}, then $22 - 3r = 10 \Rightarrow r = 4$.

So, $(4+1)th$ $i.e.$ $5th$ term contains x^{10}

Putting $r = 4$ in (i), we get

$T_5 = (-1)^4 11_{C_4} 2^{(11-4)}.3^4.x^{10} = 11_{C_4} \times 2^7 \times 3^4 \times x^{10}$

$\therefore$ Coefficient of $x^{10} = 11_{C_4} \times 2^7 \times 3^4$

Example: Find the coefficient of x^6y^3 in the expansion of $(x+2y)^9$.

Sol: Suppose x^6y^3 occurs in $(r+1)^{th}$ term of the expansion of $(x+2y)^9$

Now, $T_{r+1} = 9_{C_r} \times (x)^{9-r} \times (2y)^r = 9_{C_r} \times 2^r \times x^{9-r} \times y^r$

This will contain x^6y^3, if

$9 - r = 6$ and $r = 3 \Rightarrow r = 3$

$\therefore$ Coefficient of $x^6y^3 = 9_{C_3} \times 2^3 = \frac{9!}{3!\,6!} \times 2^3 = \frac{9\times8\times7\times6!}{3!\times6!} \times 8 = 672$

Example: Find the coefficient of x^{40} in the expansion off $(1+2x+x^2)^{27}$

Sol: We have $(1+2x+x^2)^{27} = \{(1+x)^2\}^{27} = (1+x)^{54}$

Suppose x^{40} occurs in $(r+1)th$ term in the expansion of $(1+x)^{54}$.

Now, $T_{r+1} = 54_{C_r}x^r$

For this term to contain x^{40}, we must have $r = 40$

So, the coefficient of $x^{40} = 54_{C_{40}}$.

Example: Prove that there is no term involving x^6 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, where $x \neq 0$.

Sol: Suppose x^6 occurs in $(r+1)th$ term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$.

Now, $T_{r+1} = 11_{C_r}\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = 11_{C_r}\,(-1)^r\,2^{11-r}\,3^r\,x^{22-3r}$

For this term to contain x^6, we must have $22 - 3r = 6 \Rightarrow r = \frac{16}{3}$, which is a fraction.

But, r is a natural number. Hence, there is no term containing x^6.

(Finding the term independent of the variable)

Example: Find the term independent of x in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$.

Sol: Let $(r+1)th$ term is independent of x in the given expression.

Now, $T_{r+1} = 10_{C_r}(3x^2)^{10-r}\left(-\frac{1}{2x^3}\right)^r = 10_{C_r}\,3^{10-r}\left(-\frac{1}{2}\right)^r x^{20-5r} \ldots (i)$

This term will be independent of x, if $20 - 5r = 0 \Rightarrow r = 4$.

So, $(4+1)th\ i.e.\ 5th$ term is independent of x.

Putting $r = 4$ in (i), we get

$T_5 = 10_{C_4}3^6\left(-\frac{1}{2}\right)^4 = \frac{10\times9\times8\times7}{4\times3\times2\times1}\times\frac{729}{16} = \frac{76545}{8}$

Hence, the required term $= \frac{76545}{8}$.

Example: Find the value of a so that the term independent of x in $\left(\sqrt{x} + \frac{a}{x^2}\right)^{10}$ is 405.

Sol: Let $(r+1)th$ term in the expansion of $\left(\sqrt{x} + \frac{a}{x^2}\right)^{10}$ be independent of x.

Now, $T_{r+1} = 10_{C_r}\left(\sqrt{x}\right)^{10-r}\left(\frac{a}{x^2}\right)^r = 10_{C_r}\,x^{5-\frac{r}{2}-2r}\,a^r \ldots (i)$

This will be independent of x, if $5 - \frac{r}{2} - 2r = 0 \Rightarrow 5 - \frac{5r}{2} = 0 \Rightarrow r = 2$

Putting $r = 2$ in (i), we get: $T_3 = 10_{C_2}a^2$

It is given that the term independent of x is equal to 405.

$\therefore\ 10_{C_2} a^2 = 405 \Rightarrow 45\, a^2 = 405 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3.$

(Problems based on Consecutive terms or Consecutive Coefficients)

Example: The coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio $1 : 7 : 42$. Find n.

Sol: Let the three consecutive terms be rth, $(r+1)th$, and $(r+2)th$ terms. Their coefficients in the expansion of $(1+x)^n$ are $n_{C_{r-1}}, n_{C_r}$, and $n_{C_{r+1}}$ respectively.

It is given that $n_{C_{r-1}} : n_{C_r} : n_{C_{r+1}} = 1 : 7 : 42$

Now, $\frac{n_{C_{r-1}}}{n_{C_r}} = \frac{1}{7} \Rightarrow \frac{r}{n-r+1} = \frac{1}{7} \Rightarrow n - 8r + 1 = 0 \dots (i)$

and, $\frac{n_{C_r}}{n_{C_{r+1}}} = \frac{7}{42} \Rightarrow \frac{r+1}{n-r} = \frac{1}{6} \Rightarrow n - 7r - 6 = 0 \dots (ii)$

Solving (i) and (ii), we get $r = 7$ and $n = 55$.

Example: If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in $A.P.$, prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

Sol: The coefficients of a^{r-1}, a^r and a^{r+1} in the binomial expansion of $(1+a)^n$ are $n_{C_{r-1}}, n_{C_r}$, and $n_{C_{r+1}}$ respectively.

It is given that $n_{C_{r-1}}, n_{C_r}$, and $n_{C_{r+1}}$ are in $A.P.$

So, $2\, n_{C_r} = n_{C_{r-1}} + n_{C_{r+1}}$

$\Rightarrow 2 = \frac{n_{C_{r-1}}}{n_{C_r}} + \frac{n_{C_{r+1}}}{n_{C_r}} \Rightarrow 2 = \frac{r}{n-r+1} + \frac{n-r}{r+1}$

$\Rightarrow 2 = \frac{r(r+1)+(n-r)(n-r+1)}{(n-r+1)(r+1)} \Rightarrow 2(n-r+1)(r+1) = r(r+1) + (n-r)(n-r+1)$

$\Rightarrow 2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + r^2 + n - r$

$\Rightarrow\ n^2 - 4nr - n + 4r^2 - 2 = 0 \Rightarrow\ n^2 - n(4r+1) + 4r^2 - 2 = 0.$

Example: Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Sol: We have $(0.99)^5 = (1 - 0.01)^5$

$= 5_{C_0} - 5_{C_1}(0.01) + 5_{C_2}(0.01)^2 - 5_{C_3}(0.01)^3 + 5_{C_4}(0.01)^4 - 5_{C_5}(0.01)^5$

$= 1 - 5 \times (0.01) + 10 \times (0.0001) - 10 \times (0.000001) - (0.0000000001)$

$= 1 - 0.05 + 0.001 = 0.951$

Example: Which is larger $(1.01)^{1000000}$ or, $10,000$?

Sol: We have $(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$= 1000000_{C_0} + 1000000_{C_1}(0.01) + 1000000_{C_2}(0.01)^2 + \cdots + 1000000_{C_{1000000}}(0.01)^{1000000}$

$= 1 + 1000000 \times (0.01) +$ other positive terms

$= 1 + 10000 +$ other positive terms

$= 10001 > 10000.$

$\therefore (1.01)^{1000000} > 10000.$

Example: Using, binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

Sol: We have, $6^n - 5n = (1 + 5)^n - 5n$

$= n_{C_0} + n_{C_1}(5) + n_{C_2}(5)^2 + n_{C_3}(5)^3 + \cdots + n_{C_n}(5)^n - 5n$

$= 1 + 5n + n_{C_2}(5)^2 + n_{C_3}(5)^3 + \cdots + n_{C_n}(5)^n - 5n$

$\Rightarrow 6^n - 5n - 1 = n_{C_2}(5)^2 + n_{C_3}(5)^3 + \cdots + n_{C_n}(5)^n$

$\Rightarrow 6^n - 5n - 1 = 5^2(n_{C_2} + n_{C_3} \times 5 + n_{C_4} \times 5^2 + \cdots + n_{C_n} \times 5^{n-2})$

$\Rightarrow 6^n - 5n - 1 = 25 \times$ an integer

$\Rightarrow 6^n - 5n = 25 \times$ an integer $+1$

$\Rightarrow 6^n - 5n$ leaves the remainder 1 when divided by 25.