Chapter- 05

Complex Numbers and Quadratic Equations

Introduction to Imaginary Numbers

Introduction:

The equations of the form $x^2 + 1 = 0,\ x^2 + 4 = 0$ etc. are not solvable in $R\ i.e.$ there is no real number whose square is a negative real number.

Need for Complex Numbers

From about 250 A.D. onwards, mathematicians have been coming across quadratic equations of the form $ax^2 + bx + c = 0$, where $a, b, c \in R,\ a \neq 0$ and $b^2 - 4ac < 0$ whose solution is not possible in the set of real numbers. It was in the 16th century that the Italian Mathematicians **Cardano** and **Bombelli** started a serious discussion on extending the number system to include square roots of negative numbers.

Consider the equation $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \sqrt{-1} = i$

In 1977, the Swiss Mathematician **Euler** was the first mathematician to introduce the symbol $i(iota)$ for the square root of $-1\ i.e.$ a solution of $x^2 + 1 = 0$ with the property $i^2 = -1$.

He also called this symbol as the **imaginary unit.**

Integral powers of IOTA ($\boldsymbol{i}$)

We have $i = \sqrt{-1}$

$i^2 = -1,\ i^3 = i^2.i = (-1)i = -i,\ \ i^4 = (i^2)^2 = (-1)^2 = 1$

To compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then, $n = 4m + r$ where $0 \leq r < 4$

$\Rightarrow i^n = i^{4m+r} = (i^4)^m i^r = i^r$

Now, $i^{4n} = (i^4)^n = 1^n = 1$

$i^{4n+1} = i^{4n}.\ i = 1.i = i$

$i^{4n+2} = i^{4n}.i^2 = 1 \times (-1) = -1$

$i^{4n+3} = i^{4n}.i^3 = 1 \times (-i) = -i$ where $n \in N$.

Note: $i^0 = 1$.

Example: Evaluate the following:

$(i)\ i^{135} = i^{4\times33+3} = i^3 = -i$

$(ii)\ i^{457} = i^{4\times114+1} = i^1 = i$

$(iii)\ i^{-998} = \dfrac{1}{i^{998}} = \dfrac{1}{i^{4\times249+2}} = \dfrac{1}{i^2} = \dfrac{1}{-1} = -1$

Example: Show that

$(i)\ \left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = -4$

Sol: $\left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = \left\{i^{19} + \frac{1}{i^{25}}\right\}^2 = \left\{i^3 + \frac{1}{i}\right\}^2 = \left\{-i + \frac{i}{i^2}\right\}^2 = \left\{-i + \frac{i}{-1}\right\}^2$

$= (-i - i)^2 = (-2i)^2 = 4i^2 = 4(-1) = -4$

$(ii)\ i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, for all $n \in N$.

Sol: $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n + i^n \times i + i^n \times i^2 + i^n \times i^3$

$= i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i)$

$= i^n \times 0 = 0.$

Example: Evaluate the following.

$(i)\ 1 + i^{10} + i^{100} + i^{1000}$

Sol: $1 + i^{10} + i^{100} + i^{1000}$

$= 1 + i^{4\times2+2} + i^{4\times25} + i^{4\times250}$

$= 1 - 1 + 1 + 1 = 2$

$(ii)\ i.\ i^2.i^3.i^4. \dots .\ i^{1000}$

Sol: $i.\ i^2.i^3.i^4. \dots .\ i^{1000} = i^{1+2+3+ \dots +1000}$

$= i^{\frac{1000\times 1001}{2}} = i^{500500} = 1$

$(iii)\ \dfrac{i^{582}+i^{584}+i^{586}+i^{588}+i^{590}}{i^{592}+i^{594}+i^{596}+i^{598}+i^{600}}$

Sol: $\frac{i^{582}+i^{584}+i^{586}+i^{588}+i^{590}}{i^{592}+i^{594}+i^{596}+i^{598}+i^{600}}$

$= \dfrac{-1+1-1+1-1}{1-1+1-1+1} = -\dfrac{1}{1} = -1$

$(iv)\ \sum_{n=1}^{13}(i^n + i^{n+1})$

Sol: $\sum_{n=1}^{13}(i^n + i^{n+1}) = (i^1+i^2)+(i^2+i^3)+(i^3+i^4)+ \ \ldots\ldots\ldots + (i^{13}+i^{14})$

$= (i-1)+(-1-i)+(-i+1)+(1+i)+ \ \ldots. \ + (i-1)$

$= -1 + i$

Example: If n is an odd positive integer, then prove that $i^n + i^{2n} + i^{3n} + i^{4n} = 0$.

Sol: $i^n + i^{2n} + i^{3n} + i^{4n} = \ i^n + (i^2)^n + (i^3)^n + (i^4)^n$

$= \ i^n + (-1)^n + (-i)^n + 1^n$

$= \ i^n - 1 - i^n + 1 = 0$

Imaginary Quantities

If $x^2 + 4 = 0 \Rightarrow x = \sqrt{-4} = \sqrt{4\times(-1)} = \sqrt{4}\sqrt{-1} = \pm 2i$

The product of a real number and an imaginary unit is called an imaginary number.

Ex: $2i,\ \ -3i,\ \ \frac{7}{4}i,\ \ \sqrt{2}i$ are imaginary numbers.

The square of a real number is always non-negative, but the square of an imaginary number is always negative.

Ex: $(2i)^2 = 4i^2 = -4$

$(-\sqrt{7}i)^2 = 7i^2 = -7$

Note:

- For any positive real number a, we have $\sqrt{-a} = \ \sqrt{(-1)\times a} = i\sqrt{a}$

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is not true when both a and b negative real numbers.

Standard Form of Complex Numbers and Equality of Complex Numbers

Introduction

Consider an equation: $x^2 - 4x + 13 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i = 2 + 3i,\ 2 + (-3)i$

Definition:

Any number which can be expressed as in the form $x + iy$, where $x,\ y \in R$ is called a complex number.

Or, the sum of a real number and an imaginary number is called a complex number.

The set of all complex numbers is denoted by $\mathcal{C}$.

$i.e.\ C = \{z = x + iy : x,\ y \in R\}$

Let $z = x + iy \in C$

Here ‘ ’ is called the real part and ‘y’is called the imaginary part of z

$i.e.\ Re\, z = x$ and $Im\, z = y$

If $y = 0$, then $z = x$, which is purely real.

If $x = 0$, then $z = iy$, which is purely imaginary.

Since a real number ‘ ’ can be written as $x + i0$, so every real number is a complex number. Hence $R \subset C$. Also, every imaginary number is a complex number.

Equality of Complex Numbers:

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$

$i.e.\ Re\,(z_1) = Re\,(z_2)$ and $Im(z_1) = Im(z_2)$

Example: If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y.

Sol: We have $z_1 = z_2 \Rightarrow 2 - iy = x + 3i \Rightarrow x = 2,\ \ y = -3$

Example: Find x and y if $(x + y) + 3i = -7 + (x - y)i$

Sol: We have, $x + y = -7$ and $x - y = 3$

$\Rightarrow \quad x = -2$ and $y = -5$

Note:

- Complex numbers are neither positive nor negative.
- Complex numbers cannot be compared.

Example: Find the values of x and y if

$(i)\ (x + iy)\ (2 - 3i) = 4 + i$

Sol: $x + iy = \frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2-3i)(2+3i)}$

$$= \frac{8 + 12i + 2i - 3}{4 + 9} = \frac{5 + 14i}{13}$$

$$= \frac{5}{13} + \frac{14}{13}i$$

$\Rightarrow x = \frac{5}{13},\ y = \frac{14}{13}$

$$(ii)\ \frac{x-1}{3+i} + \frac{y-1}{3-i} = i$$

Sol: $\frac{(x-1)(3-i)+(y-1)(3+i)}{9+1} = i \Rightarrow \frac{3x-ix-3+i+3y+iy-3-i}{10} = i$

$\Rightarrow\ 3x + 3y - 6 + i(y - x) = 10i$

Equating real and imaginary parts,

$3x + 3y - 6 = 0$ and $y - x = 10$

$\Rightarrow x + y = 2$ and $y - x = 10$

Solving we get $x = -4$ and $y = 6$.

Example: If $a + ib = \frac{c+i}{c-i}$, where c is real, prove that $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

Sol: We have, $a + ib = \frac{c+i}{c-i}$

$$\Rightarrow a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)} = \frac{(c+i)^2}{c^2-i^2} = \frac{c^2-1+2ic}{c^2+1} = \frac{c^2-1}{c^2+1} + i\,\frac{2c}{c^2+1}$$

Equating real and imaginary parts,

$a = \frac{c^2-1}{c^2+1}$ and $b = \frac{2c}{c^2+1}$

$\Rightarrow \quad a^2 + b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2 + \left(\frac{2c}{c^2+1}\right)^2$ and $\frac{b}{a} = \left(\frac{2c}{c^2+1}\right) \div \left(\frac{c^2-1}{c^2+1}\right)$

$\Rightarrow \quad a^2 + b^2 = \frac{(c^2-1)^2+4c^2}{(c^2+1)^2}$ and $\frac{b}{a} = \frac{2c}{c^2-1}$

$a^2 + b^2 = \frac{(c^2+1)^2}{(c^2+1)^2} = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$

Example: If $(x+iy)^{\frac{1}{3}} = a + ib,\ x,\ y, a,\ b \in R$, then show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$

Sol: We have $(x+iy)^{\frac{1}{3}} = a + ib \Rightarrow x + iy = (a+ib)^3$

$\Rightarrow \quad x + iy = a^3 + 3a^2 ib + 3ai^2b^2 + i^3b^3$

$\Rightarrow \quad x + iy = a^3 + i\,3a^2b - 3ab^2 - ib^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$

Equating real and imaginary parts, $x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4a^2 - 4b^2 = 4(a^2 - b^2)$

Example: Find the smallest positive integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

Sol: Let $z = \frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n}{(1-i)^n(1-i)^{-2}} = \frac{(1+i)^n}{(1-i)^n}(1-i)^2$

$$= \left\{\frac{(1+i)(1+i)}{(1-i)(1+i)}\right\}^n (1-i)^2 = \left\{\frac{(1+i)^2}{1+1}\right\}^n (1-i)^2 = \left(\frac{1-1+2i}{2}\right)^n (1-i)^2 = i^n(1-i)^2$$
$$= i^n(1-1-2i) = -2i^{n+1}$$

Since z is real so, $n + 1 = 2, 4,\ 6,\ \ldots\ldots \Rightarrow n = 1, 3, 5,\ \ldots..$

Hence, the least value of n is 1.

Example: What is the smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$?

Sol: We have $(1+i)^{2n} = (1-i)^{2n}$

$\Rightarrow \quad \{(1+i)^2\}^n = \{(1-i)^2\}^n \Rightarrow (1-1+2i)^n = (1-1-2i)^n$

$\Rightarrow \quad (2i)^n = (-2i)^n$

$\Rightarrow \ 2^n i^n = (-1)^n 2^n i^n \Rightarrow (-1)^n = 1$

$\Rightarrow$ n is a multiple of 2 $\Rightarrow n = 2, 4, 6, \ldots..$

So, the smallest positive value of n is 2.

Example: Find the real value of 'a' for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real.

Sol: Let $z = 3i^3 - 2ai^2 + (1-a)i + 5 = -3i + 2a + (1-a)i + 5$

$= (2a+5) + i(1-a-3)$

Since z is real so its imaginary part is zero.

$i.e.\ 1 - a - 3 = 0$

$\Rightarrow \ a = -2$

Algebra of Complex Numbers

1. Closure Law: Let $z_1, z_2 \in C$ such that $z_1 = a + ib$ and $z_2 = c + id$

Addition: $z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d) \in C$

Subtraction: $z_1 - z_2 = (a+ib) - (c+id) = (a-c) + i(b-d) \in C$

Multiplication: $z_1.z_2 = (a+ib).(c+id) = ac + iad + ibc - bd$

$= (ac - bd) + i(ad + bc) \in C$

Division: $\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac-iad+ibc+bd}{c^2+d^2}$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \in C$$

Since the sum, difference, product, and quotient of any two complex numbers is a complex number so the set of complex numbers is closed under addition, subtraction, multiplication, and division.

2. Commutative Laws:

If $z_1, z_2 \in C$ then $z_1 + z_2 = z_2 + z_1$ and $z_1.z_2 = z_2.z_1$

3. Associative Laws:

If $z_1, z_2, z_3 \in C$ then $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1.(z_2.z_3) = (z_1.z_2).z_3$

4. Distributive Laws:

If $z_1, z_2, z_3 \in C$ then $z_1(z_2 + z_3) = (z_1 z_2 + z_1 z_3)$

5. Existence of Additive Identity:

If $z \in C$, then there exists $0 = 0 + i0 \in C$ such that $z + 0 = z = 0 + z$

So $0 + i0$ is called the additive inverse.

6. Existence of Additive Inverse:

If $z \in C$, then there exists $-z \in C$ such that $z + (-z) = 0 = (-z) + z$

So, $-z$ is called the additive inverse of z.

7. Existence of Multiplicative Identity:

If $z \in C$, then there exists $1 = 1 + i0 \in C$ such that $z\,.\,1 = z = 1.z$.

So, $1 + i0$ is called the multiplicative identity.

8. Existence of Multiplicative Inverse:

If $z \neq 0 \in C$, then there exists $\frac{1}{z} \in C$ such that $z.\frac{1}{z} = 1 = \frac{1}{z}.z$. So $\frac{1}{z} = z^{-1}$ is called the multiplicative inverse or reciprocal of z.

Example: Express the following complex numbers in the form $x + iy$.

$(i)(-3i)\left(\frac{1}{9}i + 2\right)$

Sol: $(-3i)\left(\frac{1}{9}i + 2\right) = \frac{(-3i)i}{9} + 2(-3i) = \frac{1}{3} - 6i$

$(ii)(1 + i)^4$

Sol: $(1 + i)^4 = \{(1 + i)^2\}^2 = (1 - 1 + 2i)^2 = (2i)^2 = 4i^2 = -4$

Example: Find the additive and multiplicative inverse of the complex number $z = (2 + \sqrt{3}i)^2$.

Sol: The additive inverse of z is $-z = -\left(2 + \sqrt{3}i\right)^2 = -\left(4 - 3 + 4\sqrt{3}i\right)$

$= -\left(1 + 4\sqrt{3}i\right) = -1 - 4\sqrt{3}i$

The multiplicative inverse of z is $\frac{1}{z} = \frac{1}{(2+\sqrt{3}i)^2} = \frac{1}{1+4\sqrt{3}i}$

$$= \frac{(1-4\sqrt{3}i)}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)} = \frac{(1-4\sqrt{3}i)}{1+48} = \frac{1}{49} - \frac{4\sqrt{3}i}{49}$$

Example: If $x = -5 + 2\sqrt{-4}$, then find the value of $x^4 - 9x^3 + 35x^2 - x + 4$.

Sol: Given that $x = -5 + 2\sqrt{-4} = -5 + 4\mathrm{i}$

$\Rightarrow x + 5 = 4i \Rightarrow (x+5)^2 = (4i)^2 \Rightarrow x^2 + 10x + 25 = -16$

$\Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 - 9x^3 + 35x^2 - x + 4 = x^2(x^2 + 10x + 41) - x^3 - 6x^2 - x + 4$

$= -x^3 - 6x^2 - x + 4 = -x(x^2 + 10x + 41) + 4x^2 + 40x + 4$

$= 4x^2 + 40x + 4 = 4(x^2 + 10x + 41) - 160 = -160$

Example: Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

Sol: Consider $z = \left(\frac{1+i}{1-i}\right)^n = \left\{\frac{(1+i)(1+i)}{(1-i)(1+i)}\right\}^n$

$$= \left\{\frac{(1+i)^2}{1+1}\right\}^n$$

$$= \left(\frac{1-1+2i}{2}\right)^n = i^n$$

Since i^n is real so the least positive value of n is 2.

{when $n = 1. i^n = i$ and when $n = 2, i^n = i^2 = -1$}

Example: Show that $(3+i)^{-2} + (3-i)^{-2} = \frac{4}{25}$

Sol: L.H.S. $= (3+i)^{-2} + (3-i)^{-2}$

$$= \frac{1}{(3+i)^2} + \frac{1}{(3-i)^2}$$

$$= \frac{1}{9-1+6i} + \frac{1}{9-1-6i}$$

$= \dfrac{1}{8+6i} + \dfrac{1}{8-6i}$

$= \dfrac{16}{(8+6i)(8-6i)}$

$= \frac{16}{64+36} = \frac{16}{100} = \frac{4}{25}$ =R.H.S.

Conjugate and Modulus of Complex Numbers

Conjugate of a Complex Number:

If $z = x + iy$, then the conjugate of z denoted by $\bar{z}$ and is defined by $\bar{z} = x - iy$.

It follows from the definition that the conjugate of a complex number is obtained by replacing i by $-i$.

For example, if $x = 3 + 4i$, then $\bar{z} = 3 - 4i$.

Example: Find the conjugate of the following complex numbers.

$(i) z = -3 + 4i \Rightarrow \bar{z} = -3 - 4i$

$(ii)\ z = -\sqrt{7} - \sqrt{3}i \Rightarrow \bar{z} = -\sqrt{7} + \sqrt{3}i$

$(iii)\ z = -\frac{1}{2}i + 5 \Rightarrow \bar{z} = 5 + \frac{1}{2}i$

Properties:

(i) The conjugate of a real number is the number itself.

Let $z = x = x + i0 \Rightarrow \bar{z} = x - i0 = x$. So $z = \bar{z}$.

(ii) The double conjugate of a complex number is the number itself.

Let $z = x + iy \Rightarrow \bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy$. So, $\bar{\bar{z}} = z$.

$(iii)\ z + \bar{z} = 2\ Re(z)$

$(iv)\ z - \bar{z} = 2\ i\ Im(z)$

(v) If $z + \bar{z} = 0$, then z is purely imaginary.

(vi) If $z - \bar{z} = 0$, then z is purely real.

$(vii)\ z\ .\ \bar{z} = (Rez)^2 + (Imz)^2$

$(viii)$ If z_1, z_2 are two complex numbers, then

$(a)\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$(b)\ \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

$(c)\ \overline{z_1 . z_2} = \overline{z_1}\ .\ \overline{z_2}$

$(d)\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}},\ z_2 \neq 0.$

Example: Express the following complex numbers in the form $x + iy$.

$(i)\ \dfrac{1}{3-4i}$

Sol: $z = \frac{1}{3-4i} = \frac{(3+4i)}{(3-4i)(3+4i)} = \frac{3+4i}{9+16} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}\ i$

$(ii)\ \dfrac{1+2i}{2+i}$

Sol: $z = \frac{1+2i}{2+i} = \frac{(1+2i)(2-i)}{(2+i)(2-i)} = \frac{2-i+4i+2}{4+1} = \frac{4+3i}{5} = \frac{4}{5} + \frac{3}{5}i$

$(iii)\ \dfrac{(3-2i)(2+3i)}{(2+5i)(5-2i)}$

Sol: $z = \frac{(3-2i)(2+3i)}{(2+5i)(5-2i)} = \frac{6+9i-4i+6}{10-4i+25i+10} = \frac{12+5i}{20+21i} = \frac{(12+5i)(20-21i)}{(20+21i)(20-21i)}$

$$= \frac{240 - 252i + 100i + 105}{400 + 441} = \frac{345 - 152i}{841} = \frac{345}{841} - \frac{152}{841}i$$

$(iv)\ \left(\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right)$

Sol: $z = \left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

$$= \left(\frac{1+i+3-6i}{1+i-2i+2}\right)\left(\frac{3+4i}{2-4i}\right) = \left(\frac{4-5i}{3-i}\right)\left(\frac{3+4i}{2-4i}\right)$$

$$= \frac{12+16i-15i+20}{6-12i-2i-4}$$

$$= \frac{32+i}{2-14i} = \frac{(32+i)(2+14i)}{(2-14i)(2+14i)}$$

$= \dfrac{64 + 448i + 2i - 14}{4 + 196} = \dfrac{50 + 450i}{200}$

$= \dfrac{50}{200} + \dfrac{450}{200} i = \dfrac{1}{4} + \dfrac{9}{4}\ i$

Example: Express the following complex numbers in the standard form. Also find their conjugate.

$(i)\ \ z = \dfrac{1 - \text{i}}{1 + \text{i}} = \dfrac{(1 - \text{i})^2}{(1 + i)(1 - i)} = \dfrac{1 - 1 - 2i}{1 + 1} = -\dfrac{2i}{2} = -i$

$\Rightarrow\ \ \bar{z} = i$

$(ii)\ z = \frac{(2+3i)^2}{2-i} = \frac{4-9+12i}{2-i} = \frac{-5+12i}{2-i} = \frac{(-5+12i)(2+i)}{(2-i)(2+i)} = \frac{-10-5i+24i-12}{4+1}$

$= \dfrac{-22 + 19i}{5} = -\dfrac{22}{5} + \dfrac{19}{5} i$

$\Rightarrow\ \ \bar{z} = -\frac{22}{5}\ - \frac{19}{5} i$

$(iii)\ z = (\sqrt{7} + 5i)^2 = 7 - 25 + 10\sqrt{7}i = -18 + 10\sqrt{7}i$

$\Rightarrow \bar{z} = -18 - 10\sqrt{7}i = (\sqrt{7} - 5i)^2$

Example: Find the values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugates of each other.

Sol: Since the given complex numbers are conjugate of each other, so

$-3 + ix^2y = \overline{(x^2 + y) + 4i} \ \ \Rightarrow -3 + ix^2y = x^2 + y - 4i$

Equating real and imaginary parts

$x^2 + y = -3$ ------ (1)

and $x^2y = -4$ ------ (2)

From (2), we get $y = -\frac{4}{x^2}$

Then from (1), we get $x^2 - \frac{4}{x^2} = -3 \Rightarrow x^4 - 4 = -3x^2 \Rightarrow x^4 + 3x^2 - 4 = 0$

$\Rightarrow (x^2 - 1)(x^2 + 4) = 0 \Rightarrow x^2 - 1 = 0$ or $x^2 + 4 = 0.$

Since x is real so $x^2 + 4 \neq 0.$ Thus $x^2 - 1 = 0 \Rightarrow x = \pm 1$

When $x = \pm 1,\ y = -\frac{4}{1} = -4$

Example: If $z_1 = 2 + 3i,\quad z_2 = 3 - 4i$, then prove the following

$(i)\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Sol: L.H.S. $= \overline{z_1 + z_2} = \overline{(2+3i) + (3-4i)} = \overline{5 - i} = 5 + i$

R.H.S.$= \overline{z_1} + \overline{z_2} = \overline{2+3i} + \overline{3-4i} = 2 - 3i + 3 + 4i = 5 + i$

$\therefore$ L.H.S. = R.H.S.

$(ii)\ \overline{z_1 . z_2} = \overline{z_1}\ .\ \overline{z_2}$

Sol: L.H.S. $= \overline{z_1 . z_2} = \overline{(2+3i)(3-4i)} = \overline{6 - 8i + 9i + 12}$

$= \overline{18 + i} = 18 - i$

R.H.S. $= \overline{z_1}\ .\ \overline{z_2} = \left(\overline{2+3i}\right)\left(\overline{3-4i}\right) = (2-3i)(3+4i)$

$= 6 + 8i - 9i + 12 = 18 - i$

$\therefore$ L.H.S. = R.H.S.

$(iii)\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Sol: L.H.S. $= \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{2+3i}{3-4i}\right)} = \overline{\left(\frac{(2+3i)(3+4i)}{9+16}\right)} = \overline{\left(\frac{6+8i+9i-12}{25}\right)} = \overline{\left(\frac{-6+17i}{25}\right)}$

$= \overline{\left(-\frac{6}{25} + \frac{17}{25}i\right)} = -\frac{6}{25}\ - \frac{17}{25}i$

R.H.S. $= \frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{2+3i}}{\overline{3-4i}} = \frac{2-3i}{3+4i} = \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i-9i-12}{9+16} = \frac{-6-17i}{25} = -\frac{6}{25}\ - \frac{17}{25}i$

$\therefore$ L.H.S. = R.H.S.

Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is denoted by $|z|$ and is defined as

$|z| = \sqrt{z} = \sqrt{x^2 + y^2} = \sqrt{(Re\ z)^2 + \ (Im\ z)^2}$

For example, if $z = 3\ - 4i$, then $|z| = \sqrt{3^2 + (-4)^2} = 5$

Properties of Modulus

Let $z,\ z_1, z_2$ are complex numbers. Then

$(i)\ |z| = 0 \Rightarrow Rez = 0,\ \ Im\, z = 0.$

$(ii)\ |z| = |\overline{z}| = |-z|$

(iii) z. $\overline{z} = |z|^2$

$(iv) |z_1 . z_2| = |z_1||z_2|$

$(v)\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0.$

$(vi) |z_1 + z_2| \leq\ |z_1| + |z_2|$ (Triangle Inequality)

$(vii) |z_1 + z_2|^2 =\ |z_1|^2 + |z_2|^2 + 2\, Re(z_1 .\ \overline{z_2})$

$(viii)\ |z_1 - z_2|^2 =\ |z_1|^2 + |z_2|^2 - 2\, Re\left(z_1 .\ \overline{z_2}\right)$

$(ix)\ \ |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(\ |z_1|^2 + |z_2|^2)$

$(x) |az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(\ |z_1|^2 + |z_2|^2)$

Note:

$(i)\ |z_1 - z_2| \leq\ |z_1| + |z_2|$

$(ii)\ |z_1 - z_2|\ \geq\ \big||z_1| - |z_2|\big|$

$(iii)\ |z_1 - z_2| \geq |z_1| - |z_2|$

Example: If $z = x + iy$, show that $|x| + |y| \leq \sqrt{2}|z|$.

Sol: We have $z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

Now $(|x| - |y|)^2 \geq 0$

$\Rightarrow \qquad |x|^2 + |y|^2 - 2|x||y| \geq 0$

$\Rightarrow \qquad |x|^2 + |y|^2 \geq 2|x||y|$

$\Rightarrow \quad x^2 + y^2 \geq 2|x||y|$

$\Rightarrow \quad 2(x^2 + y^2) \geq x^2 + y^2 + 2|x||y|$

$\Rightarrow \quad 2|z|^2 \geq (|x| + |y|)^2$

$\Rightarrow \quad |x| + |y| \leq \sqrt{2}|z|.$

Example: If $z = x + iy$ and $|2z - 1| = |z + 1|$, show that $x^2 + y^2 = 2x$

Sol: We have $z + 1 = (x + iy) + 1 = (x + 1) + iy$

Also $2z - 1 = 2(x + iy) - 1 = (2x - 1) + i(2y)$

$\therefore \; |2z - 1| = |z + 1|$

$\Rightarrow \quad \sqrt{(2x-1)^2 + (2y)^2} = \sqrt{(x+1)^2 + y^2}$

$\Rightarrow \quad (2x - 1)^2 + 4y^2 = (x + 1)^2 + y^2$

$\Rightarrow \quad (2x - 1)^2 - (x + 1)^2 = y^2 - 4y^2$

$\Rightarrow \quad 3x^2 - 6x + 3y^2 = 0 \Rightarrow x^2 - 2x + y^2 = 0 \Rightarrow x^2 + y^2 = 2x$

Example: Find x if $(1 - i)^x = 2^x$.

Sol: We have $(1 - i)^x = 2^x$

Taking modulus of both sides we get $|(1 - i)^x| = |2^x|$

$\Rightarrow \; |1 - i|^x = |2|^x$

$\Rightarrow \; \left(\sqrt{1^2 + (-1)^2}\right)^x = 2^x \Rightarrow \left(\sqrt{2}\right)^x = 2^x$

$\Rightarrow \; \Rightarrow 2^{\frac{x}{2}} - 2^x = 0 \Rightarrow 2^{\frac{x}{2}}\left(1 - 2^{\frac{x}{2}}\right) = 0 \Rightarrow 1 - 2^{\frac{x}{2}} = 0 \; [\, since \; 2^{\frac{x}{2}} \neq 0]$

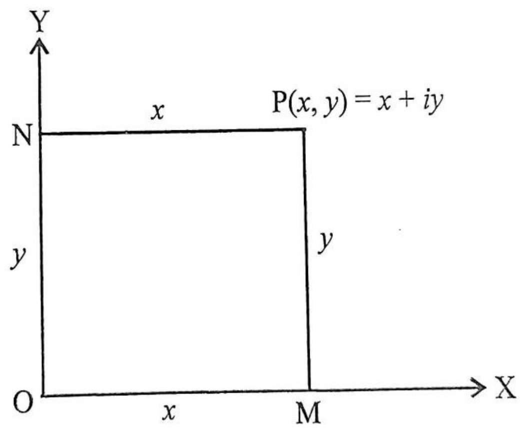
$\Rightarrow \; 2^{\frac{x}{2}} = 1 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$

Geometrical and Polar Representation of Complex Numbers

Geometrical Representation of a Complex Number

We know that a real number can be represented geometrically on the number line.

A complex number $z = x + iy$ can be represented by a point $(x,\ y)$ on the plane which is known as the Argand plane. To represent $z = x + iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively real and imaginary parts of z. This point $P(x,\ y)$ represents the complex number $z = x + iy$.



If a complex number is purely real, then its Imaginary part is zero. Therefore, a purely real number is represented by a point on $x - axis$. A purely imaginary number is represented by a point on $y - axis$.

So $x - axis$ is known as the real axis and $y - axis$ is known as the imaginary axis.

Conversely, if $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the affix of the point P.

Thus, there exists a one-one correspondence between the points of the plane and the members(elements) of the set C of all complex numbers, $i.e.$, for every complex number $z = x + iy$ there exists uniquely a point $(x,\ y)$ on the plane and for every point $(x,\ y)$ of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or **Gaussian plane**. The point P, plotted on the Argand plane is called the Argand diagram.

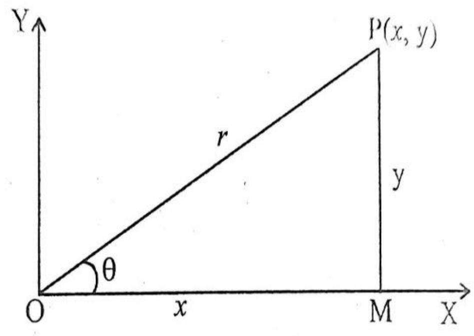
The length of the line segment OP is called the **modulus** of z and is denoted by $|z|$.

Here, $OM = x,\ MP = y$

In the right-angled triangle $OMP,\ |OP| = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2} = r(say)$

Thus, if $= x + iy$, then $|z| = r = \sqrt{(Rez)^2 + (Imz)^2}$

Geometrically, $|z|$ is the distance of z from the origin.



The angle θ which OP makes with the positive direction of $x-axis$ in an anticlockwise sense is called the **argument** or **amplitude** of z and is denoted by $\arg(z)$ or $amp(z)$. In the right-angled triangle OMP, $tan\theta = \frac{y}{x}$.

So if $z = x + iy$, then $\arg z = \theta$.

This angle θ has infinitely many values differing by multiples of 2π. $i.e.$ $-\infty < \arg z < \infty$

The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the amplitude or principal argument. The argument of z depends upon the quadrant in which the point P lies.

Techniques to determine the principal argument

Let $z = x + iy$ such that $Parg\ z = \theta$

Step I: Find the acute angle α given by $\tan\alpha = \left|\frac{y}{x}\right|$

Step II: If

(i) $x > 0,\ y = 0$, then $\theta = 0$.

(ii) $x > 0,\ y > 0$, then $\theta = \alpha$

(iii) $x = 0,\ y > 0$, then $\theta = \frac{\pi}{2}$

(iv) $x < 0,\ y > 0$, then $\theta = \pi - \alpha$

(v) $x < 0,\ y = 0$, then $\theta = \pi$

(vi) $x < 0,\ y < 0$, then $\theta = -(\pi - \alpha)$

$(vii)\ x = 0,\ y < 0,$ then $\theta = -\frac{\pi}{2}$

$(viii) x > 0,\ y < 0,$ then $\theta = -\alpha$

Example: Find the modulus and argument of each of the following complex numbers.

$(i) 1 + i\sqrt{3}$

Sol: Let $z = 1 + i\sqrt{3}$

Here $x = 1,\ y = \sqrt{3}$

So, $|z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Consider $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{\sqrt{3}}{1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

Since $x > 0$ and $y > 0$, so $\arg(z) = \alpha = \frac{\pi}{3}$

$(ii) \frac{1+i}{1-i}$

Sol: Let $z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{(1+i)^2}{1-i^2} = \frac{1-1+2i}{1+1} = \frac{2i}{2} = i = 0 + i$

Here $x = 0,\ y = 1 > 0$

$|z| = \sqrt{0+1} = 1$

Since $x = 0$ and $y > 0$, so $\arg(z) = \frac{\pi}{2}$

Polar Form of a Complex Number

Let $z = x + iy$ be a complex number represented by a point $P(x, y)$ in the Argand plane.

In $\Delta\, POM$, we have $os\theta = \frac{OM}{OP} = \frac{x}{r}, sin\theta = \frac{PM}{OP} = \frac{y}{r}$

$\therefore z = x + iy \Rightarrow z = r\, cos\theta + i\, r\, sin\theta$

$= r\,(cos\theta + i\, sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z.

Note: $z = r\, e^{i\theta}$ is called exponential form, where $e^{i\theta} = \cos\theta + i\, sin\theta$

Multiplication of a Complex Number by IOTA

Let $z = x + iy = r(cos\theta + i\,sin\theta)$

Then $r = |z|$ and $\arg(z) = \theta$

Now $iz = i\,r\big(cos\theta + i\,sin\theta\big) = r(-sin\theta + i\,cos\theta)$

$$= r\left\{\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right\}$$

Thus iz is a complex number such that

$|iz| = r = |z|$ and $\arg(iz) = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \arg(z)$

So, the multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

Example: Write the following complex numbers in the polar form:

(i) $\boldsymbol{\sqrt{3} + i}$

Sol: Let $z = \sqrt{3} + i$. Here $x = \sqrt{3} > 0,\ y = 1 > 0$

Then $r = |z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = 2$

Now $tan\alpha = \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$. So $\theta = \alpha = \frac{\pi}{6}$

Hence the polar form is $z = r(\cos\theta + i\,sin\theta) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

(ii) $\boldsymbol{1 - i}$

Sol: Let $z = 1 - i$. Here $x = 1 > 0,\ y = -1 < 0$

Then $r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$

Now $tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$. So $\theta = -\alpha = -\frac{\pi}{4}$

Hence the polar form is $z = r\big(\,cos\theta + i\,sin\theta\big) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\}$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

(iii) $\boldsymbol{-2i}$

Sol: Let $z = -2i = 0 + (-2)i$

Here $x = 0,\ y = -2 < 0$

Then $r = |z| = \sqrt{0 + (-2)^2} = 2$

Here $\theta = -\frac{\pi}{2}$

So the polar form is $z = r(cos\theta + i\ sin\theta)$

$= 2\left\{\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right\} = 2(\cos\frac{\pi}{2}\ - i\sin\frac{\pi}{2})$

Properties of Argument

$(i)\ \arg(z_1.z_2) = \arg z_1 + \arg z_2$

$(ii)\ \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\ ,\ \ z_2 \neq 0$

$(iii)\ \arg z^n = n\arg z$

$(iv)\ \arg\left(\frac{z}{\bar{z}}\right) = 2\arg z$

$(v)\ \arg\bar{z} = -\arg z$

$(vi)\ \arg(-z) = \arg\big((-1)z\big) = \arg(-1) + argz = \pi + argz$

$(vii)\ \arg(iz) = \arg(i) + argz = \frac{\pi}{2} + argz$

Example: If z_1, z_2 are two complex numbers, then prove the following.

$|\boldsymbol{z_1}.\boldsymbol{z_2}| = |\boldsymbol{z_1}||\boldsymbol{z_2}|$

Sol: Let $z_1 = r_1(cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(cos\theta_2 + i\sin\theta_2)$

So, $|z_1| = r_1,\ \ |z_2| = r_2,\ \ argz_1 = \theta_1,\ argz_2 = \theta_2$

Now $z_1.z_2 = r_1(cos\theta_1 + i\sin\theta_1).r_2(cos\theta_2 + i\sin\theta_2)$

$= r_1r_2(cos\theta_1cos\theta_2 + i\ cos\theta_1\sin\theta_2 + i\ sin\theta_1.cos\theta_2 - sin\theta_1sin\theta_2)$

$= r_1r_2\{(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2) + i\ (cos\theta_1\sin\theta_2 + sin\theta_1.cos\theta_2)\}$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

So, $|z_1.z_2| = r_1r_2 = |z_1||z_2|$ and $\arg(z_1.z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

Example: If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

Sol: We have $iz^3 + z^2 - z + i = 0$

Dividing both sides by i, we get $z^3 - iz^2 + iz + 1 = 0$

$\Rightarrow \quad z^2(z-i) + i(z-i) = 0 \Rightarrow (z-i)(z^2+i) = 0$

$\Rightarrow \quad z = i$ or, $z^2 = -i$

Now, $z = i \Rightarrow |z| = |i| = 1$

and $z^2 = -i \Rightarrow |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

Hence, in either case, we have $|z| = 1$.

Square roots of a Complex Number

Let us compute $\sqrt{a+ib}$

Let $\sqrt{a+ib} = x + iy$

Squaring both sides we have $a + ib = (x+iy)^2 = x^2 - y^2 + i\,2xy$

Equating real and imaginary parts

$x^2 - y^2 = a$ …… (1)

and $2xy = b$ …… (2)

Now $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = a^2 + b^2$

$\Rightarrow x^2 + y^2 = \sqrt{a^2+b^2}$ …… (3)

Adding (1) and (3), $2x^2 = \sqrt{a^2+b^2} + a \qquad \Rightarrow x = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}$

Subtracting (1) from (3), we get

$2y^2 = \sqrt{a^2+b^2} - a \Rightarrow y = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}$

From (2), $xy = \frac{b}{2}$

If b is positive then x and y are of the same sign.

So the required square root = $\pm\frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{a^2+b^2}+a}+i\sqrt{\sqrt{a^2+b^2}-a}\;\right]$

If b is negative then x and y are of different signs.

So the required square roots = $\pm\frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{a^2+b^2}+a}-i\sqrt{\sqrt{a^2+b^2}-a}\;\right]$

Note: If $z=a+ib$, then $\sqrt{z}=\begin{cases}\pm\left[\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right], & b>0\\ \pm\left[\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right], & b<0\end{cases}$

Example: Find the square roots of $-4-3i$

Sol: *Method-1:* Let $\sqrt{-4-3i}=x+iy$

$\Rightarrow -4-3i=(x+iy)^2=x^2-y^2+i\,2xy$

Equating real and imaginary parts, we have

$x^2-y^2=-4$ .... (1) and $2xy=-3$ .... (2)

Now $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=(-4)^2+(-3)^2=25$

$\Rightarrow x^2+y^2=5$ .... (3)

Adding (1) and (3), we get $2x^2=1\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

Subtracting (1) from (3), we get $2y^2=9\Rightarrow y=\pm\frac{3}{\sqrt{2}}$

Since $xy<0$, so x and y are of opposite signs

We have $x+iy=\frac{1}{\sqrt{2}}-i\frac{3}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}+i\frac{3}{\sqrt{2}}$

$\therefore\ \sqrt{-4-3i}=\pm\frac{1}{\sqrt{2}}(1-3i)$

Method – 2: $-4-3i=\frac{1}{2}(-8-6i)=\frac{1}{2}(1-9-2.3i)$

$=\frac{1}{2}\{1^2+(3i)^2-2.1.3i\}=\frac{1}{2}(1-3i)^2$

$\Rightarrow\sqrt{-4-3i}=\pm\frac{1}{\sqrt{2}}(1-3i)$

Method -3: Here $z = a + ib = -4 - 3i,\ |z| = \sqrt{16 + 9} = 5$

Since $b = -3 < 0$, so $\sqrt{z} = \pm\left[\sqrt{\frac{5+(-4)}{2}} - i\sqrt{\frac{5-(-4)}{2}}\right]$

$= \pm\left[\sqrt{\frac{1}{2}} - i\sqrt{\frac{9}{2}}\right]$

$= \pm\left[\frac{1}{\sqrt{2}} - i\,\frac{3}{\sqrt{2}}\right]$

$= \pm\frac{1}{\sqrt{2}}(1 - 3i)$

Example: Find the square roots of the following complex numbers.

(i) $\boldsymbol{7 - 24i}$

Sol: $\sqrt{7 - 24i} = \pm\sqrt{16 - 9 - 24i} = \pm\sqrt{4^2 + (3i)^2 - 2.\ 4.3i}$

$= \pm\sqrt{(4 - 3i)^2} = \pm(4 - 3i)$

(ii) $\boldsymbol{i}$

Sol: $\sqrt{i} = \sqrt{\frac{1}{2}.\ 2i} = \pm\frac{1}{\sqrt{2}}\sqrt{2i} = \pm\frac{1}{\sqrt{2}}\sqrt{1 - 1 + 2i} = \pm\frac{1}{\sqrt{2}}\sqrt{1^2 + i^2 + 2.1.i}$

$= \pm\frac{1}{\sqrt{2}}\sqrt{(1 + i)^2} == \pm\frac{1}{\sqrt{2}}(1 + i)$

(iii) $\boldsymbol{\frac{a^2}{b^2} - \frac{b^2}{a^2} + 2i}$

Sol: $\sqrt{\frac{a^2}{b^2} - \frac{b^2}{a^2} + 2i} = \pm\sqrt{\left(\frac{a}{b}\right)^2 + \left(i\frac{b}{a}\right)^2 + 2\frac{a}{b}\,.\frac{b}{a}i} = \pm\sqrt{\left(\frac{a}{b} + i\frac{b}{a}\right)^2} = \pm\left(\frac{a}{b} + i\frac{b}{a}\right)$

(iv) $\boldsymbol{4ab + 2i(a^2 - b^2)}$

Sol: $\sqrt{4ab + 2i(a^2 - b^2)}$

$= \pm\sqrt{(a + b)^2 - (a - b)^2 + 2i(a + b)(a - b)}$

$= \pm\sqrt{(a + b)^2 + \{i(a - b)\}^2 + 2(a + b)(a - b)i}$

$= \pm\sqrt{\{(a+b)+i(a-b)\}^2}$

$= \pm\{(a+b)+i(a-b)\}$

(v) $\boldsymbol{x^2+\frac{1}{x^2}\ -\frac{4}{i}\ \left(x-\frac{1}{x}\right)-6}$

Sol: $\sqrt{x^2+\frac{1}{x^2}\ -\frac{4}{i}\ \left(x-\frac{1}{x}\right)-6} = \ \pm\sqrt{x^2+\frac{1}{x^2}-2-\frac{4i}{i^2}\left(x-\frac{1}{x}\right)-4}$

$= \ \pm\sqrt{\left(x-\frac{1}{x}\right)^2+4i\left(x-\frac{1}{x}\right)-4} = \pm\sqrt{\left(x-\frac{1}{x}\right)^2+2.\left(x-\frac{1}{x}\right).2i+(2i)^2}$

$= \ \pm\sqrt{\left(x-\frac{1}{x}+2i\right)^2} = \pm\left(x-\frac{1}{x}+2i\right)$

Note: The square roots a complex number are the additive inverse of each other.

Solutions of Quadratic Equation in the set of Complex Numbers

The equation $ax^2+bx+c=0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x. A quadratic equation cannot have more than two roots. If $b^2-4ac<0$, then the solution is given in the set of complex numbers.

Complex roots of an equation with real coefficients always occur in conjugate pairs. However, this may not be true in the case of equations with complex coefficients.

Fundamental Theorem of Algebra

We know that every polynomial equation $f(x)=0$ has at least one root, real or imaginary(complex). The theorem states that " A polynomial equation of degree n has n roots.

If $b^2-4ac<0$, then the roots of the quadratic equation with real coefficients are imaginary and complex conjugates of each other.

Complex roots of an equation with real coefficients always occur in conjugate pairs like $2+3i$ and $2-3i$.

However, this may not be true in the case of equations with complex coefficients.

For example, $x^2\ -2ix-1=0$ has both roots equal to i .

Example: Solve each of the following equations.

(i) $\mathbf{4x^2 + 9 = 0}$

Sol: We have $4x^2 + 9 = 0 \Rightarrow 4x^2 = -9 \Rightarrow x^2 = -\frac{9}{4}$

$\Rightarrow \quad x = \ \pm\frac{3}{2}i$

Hence, the roots of the given equation are $\frac{3}{2}i$ and $-\frac{3}{2}i$.

(ii) $\mathbf{x^2 - 4x + 13 = 0}$

Sol: We have, $x^2 - 4x + 13 = 0$

$\Rightarrow \quad x = \frac{4\pm\sqrt{16-52}}{2} = \frac{4\pm6i}{2} = 2 \pm 3i$

Hence, the roots of the given equation are $2 + 3i,\ 2 - 3i$.

(iii) $\mathbf{\sqrt{5}\,x^2 + x + \sqrt{5} = 0}$

Sol: Here, the discriminant is $1^2 - 4 \times \sqrt{5} \times \sqrt{5} = -19$

Therefore, the solutions are $\frac{-1 \pm\sqrt{-19}}{2\sqrt{5}} = \frac{-1\pm \sqrt{19}\,i}{2\sqrt{5}}$

Quadratic Equations with Complex Coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$ … (1)

where $a,\ b,\ c$ are complex numbers and $a \neq 0$.

So the roots are complex numbers.

Since the order relation is not defined in case of complex numbers, therefore, we cannot assign positive or negative sign to the discriminant $D = \ b^2 - 4ac$.

However, equation (1) has complex roots which are equal, if $D = b^2 - 4ac = 0$ and unequal roots if $D = b^2 - 4ac \neq 0$.

Example: Solve each of the following equations.

(i) $\mathbf{x^2 - 5ix - 6 = 0}$

Sol: We have, $x^2 - 5ix - 6 = 0 \Rightarrow x = \ \frac{5i\pm\sqrt{-25+24}}{2} = \frac{5i\pm i}{2} = 3i\ ,\ 2i$

Hence, the roots of the given equation are $3i$ and $2i$.

(ii) $\boldsymbol{ix^2 - 4x - 4i = 0}$

Sol: We have, $ix^2 - 4x - 4i = 0 \Rightarrow i(x^2 + 4ix - 4) = 0$

$\Rightarrow \quad x^2 + 4ix - 4 = 0 \Rightarrow (x + 2i)^2 = 0 \Rightarrow x = -2i$

Hence the root of the given equation is $-2i$.

(iii) $\boldsymbol{x^2 - (7 - i)x + (18 - i) = 0}$

Sol: Here, discriminant is $\{-(7 - i)^2\} - 4 \times 1 \times (18 - i)$

$= \; 49 - 1 - 14i - 72 + 4i$

$= -24 - 10i$

$= 1 - 25 - 10i$

$= \; 1^2 + (-5i)^2 - 2 \times 1 \times 5i = \; (1 - 5i)^2$

Therefore, the solutions are $\frac{-\{-(7-i)\} \pm \sqrt{(1-5i)^2}}{2\times1}$

$= \dfrac{(7 - i) \pm (1 - 5i)}{2}$

$= \dfrac{7 - i + 1 - 5i}{2}, \dfrac{7 - i \; - 1 + 5i}{2}$

$= 4 \; - 3i$ and $3 + 2i$