Introduction:

The system of real numbers possesses the property of order $i.e.$, they can be arranged in such a way that each number is greater than or less than any other number. Such an arrangement is called the inequalities in real number system.

Let x be any real number. Then we say that x is either positive or negative or zero.

When x is positive, we write $x > 0$. When x is negative, we write $x < 0$.

Further, for any two real numbers x and y, if $x \neq y$, then either $x > y$ or $x < y$.

We have $x > y$ when $x - y > 0$ and $x < y$ when $x - y < 0$.

If $x \nless y$, then either $x > y$ or $x = y$. The two relations $x > y$ and $x = y$ can be written together as $x \geq y$.

Similarly, if $x \ngtr y$, then we write $x \leq y$.

The symbols '$>$', '$<$', '$\geq$', '$\leq$' are called the signs of inequalities.

Inequalities:

Two real numbers or two algebraic expressions related by the symbol '$>$', '$<$', '$\geq$' or '$\leq$' form an equality.

Hence, a statement involving the above symbols is called an inequality.

For example: $4 > 2$, $x \leq 5$, $2x + 3y > 6$..

Different Forms of Inequalities:

1. Numerical inequalities: The inequalities which do not involve variables, are called numerical inequalities.

For example: $6 < 2$, $13 \leq 4$.

2. Literal inequalities: The inequalities involving variables are called literal inequalities.

For example: $x > 2$, $y \geq 6$, $2x - 3y \leq 0$.

3. Linear inequalities in one variable: Linear inequalities in one variable x are of the forms $ax + b > 0$, $ax + b < 0$, $ax + b \leq 0$, $ax + b \geq 0$, $ax + b \leq c$, $ax + b \geq c$, where a, b, c are real numbers and $a \neq 0$.

4. Linear inequalities in two variables: Linear inequalities in two variables x and y are of the forms $ax + by > c$, $ax + by < c$, $ax + by \leq c$, $ax + by \geq c$, where a, b, c are real numbers and $a \neq 0$, $b \neq 0$.

5. Strict inequalities: The inequalities involving the symbol '$<$' or '$>$' are called strict inequalities. The inequalities $ax+b>c\ (a\neq 0)$, $ax+b<c(a\neq 0)$, $ax+by>c\ \left(a\neq 0,\ b\neq 0\right)$, $ax+by<c(a\neq 0,\ \ b\neq 0)$ are strict inequalities.

6. Slack inequalities: The inequalities involving the symbol '$\geq$' or '$\leq$' are called slack inequalities.

For example: $x\geq 5,\ 2x+3y\leq 6$.

7. Unconditional inequalities: An unconditional inequality is one that holds for all values of the variables.

For example: $x^2+5>2x+1$ is true for all real values of x.

8. Conditional inequalities: A conditional inequality is true only for certain values of the variable.

For example: $3x+2>14$ is true only for those values of x which are greater than 4.

Rules of Inequalities:

1. If $a>b$ and c is any number, then $a+c>b+c$ and $a-c>b-c$.
2. If $a>b$ and $c>0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$.
3. If $a>b$ and $c<0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$
4. $a-c>b\Rightarrow a>b+c$ or $-c>b-a$
5. If $a>b$, then $-a<-b$.
6. If $a>b$ then $a^n>b^n$ and $a^{-n}<b^{-n}$, $n\in N$.
7. If $a,\ b\in R$, then $a^2+b^2\geq 2ab$.
8. $\frac{a+b}{2}\geq \sqrt{ab}\geq \frac{2ab}{a+b}$, if a and b are positive real numbers. $i.e.,\ A.M.\ \geq G.M.\geq H.M.$

Example: Find the minimum value of $4^x+4^{1-x},\ x\in R$.

Sol: We have, $A.M.\ \geq G.M.$

$\Rightarrow \frac{4^x+4^{1-x}}{2}\geq \sqrt{4^x.4^{1-x}}$

$\Rightarrow \frac{4^x+4^{1-x}}{2}\geq 2$

$\Rightarrow 4^x+4^{1-x}\geq 4$

So, the minimum value of $4^x + 4^{1-x}$ is 4.

Solutions of an inequation:

The value(s) of the variable(s) which makes the inequality a true statement is called its solution.

Solving an Inequation:

It is the process of obtaining all possible solutions of an inequation.

Solution Set:

The set of all solutions of an inequality is called the solution set.

Solving Linear Inequation in One Variable:

Solving inequations of the form $ax > b,\ ax \geq b,\ ax < b,\ ax \leq b\ (a \neq 0)$

Example: Solve $20x < 105$ when (i) x is a natural number (ii) x is an integer.

Solution: We have $20x < 105 \Rightarrow x < \frac{105}{20} = \frac{21}{4}$

(i) When x is a natural number, the values of x which makes the statement true are 1, 2, 3, 4, 5.

$\therefore$ The solution set of the inequality is $\{1,\ 2,\ 3,\ 4,\ 5\}$.

(ii) When x is an integer, the solution of the given inequality are $\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5$.

$\therefore$ The solution set of the inequality is $\{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5\}$.

Example: Solve $30x < 200$

(i) x is a natural number, (ii) x is an integer.

Solution: We are given $30x < 200$

$\Rightarrow \frac{3ox}{30} < \frac{200}{30}$

$\Rightarrow x < \frac{20}{3}$

(i) When x is a natural number, in this case the following values of x make the statement true.

$1,\ 2,\ 3,\ 4,\ 5,\ 6$

The solution set of the inequality is $\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

(ii) When x is an integer, the solutions of the given inequality are

$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

The solution set of the inequality is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

Example: Solve the following linear inequations:

(i) $2x - 4 \leq 0.$

Solution: We have, $2x - 4 \leq 0 \Rightarrow 2x \leq 4 \Rightarrow x \leq 2.$

The solution set of the given inequation is $(-\infty, 2]$.

(ii) $7x + 9 > 30.$

Solution: We have, $7x + 9 > 30 \Rightarrow 7x > 21 \Rightarrow x > 3$

The solution set of the given inequation is $(3, \infty)$.

(iii) $-3x + 12 \geq 0.$

Solution: We have, $-3x + 12 \geq 0 \Rightarrow -3x \geq -12 \Rightarrow x \leq 4.$

The solution set of the given inequation is $(-\infty, 4]$.

Solving inequations of the form

$\boldsymbol{ax + b > cx + d \text{ or } ax + b < cx + d \text{ or } ax + b \geq cx + d \text{ or } ax + b \leq cx + d.}$

Example: Solve: $5x - 3 < 3x + 1$ when (i) x is a real number (ii) x is an integer (iii) x is a natural number.

Solution: We have, $5x - 3 < 3x + 1 \Rightarrow 5x - 3x < 1 + 3 \Rightarrow 2x < 4 \Rightarrow x < 2$

(i) If $x \in R \Rightarrow x \in (-\infty, 2)$

So, the solution set is $(-\infty, 2)$.

(ii) If $x \in Z$ then $x = 1, 0, -1, -2, -3, \ldots.$

So, the solution set is $\{\ldots., -3, -2, -1, 0, 1\}$

(iii) If $x \in N$ then $x = 1$. So, the solution set is $\{1\}$.

Example: Solve the following inequations:

(i) $3x + 17 \leq 2(1 - x)$

Solution: We have, $3x + 17 \leq 2(1 - x) \Rightarrow 3x + 17 \leq 2 - 2x$

$\Rightarrow 3x + 2x \leq 2 - 17 \Rightarrow 5x \leq -15 \Rightarrow x \leq -3.$

Hence, the solution set of the given inequation is $(-\infty,\ -3]$.

$(ii)\ 2(2x+3) - 10 \leq 6(x-2)$

Solution: We have, $2(2x+3) - 10 \leq 6(x-2)$

$\Rightarrow 4x + 6 - 10 \leq 6x - 12 \Rightarrow 4x - 6x \leq -12 + 4 \Rightarrow -2x \leq -8$

$\Rightarrow x \geq 4$

Hence, the solution set of the given inequation is $[4,\ \infty)$.

Example: Solve the following inequalities:

$(i)\ \frac{2x-3}{4} + 9 \geq 3 + \frac{4x}{3}$

Solution: We have, $\frac{2x-3}{4} + 9 \geq 3 + \frac{4x}{3} \Rightarrow \frac{2x-3}{4} - \frac{4x}{3} \geq 3 - 9$

$\Rightarrow \frac{3(2x-3)-16x}{12} \geq -6 \Rightarrow \frac{6x-9-16x}{12} \geq -6 \Rightarrow -9 - 10x \geq -72$

$\Rightarrow -10x \geq -63 \Rightarrow x \leq \frac{63}{10} = 6.3$

Hence, the solution set of the given inequation is $(-\infty,\ 6.3]$

$(ii)\ \frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$

Solution: We have, $\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$

$\Rightarrow \frac{5(5x-2)-3(7x-3)}{15} > \frac{x}{4} \Rightarrow \frac{25x-10-21x+9}{15} > \frac{x}{4} \Rightarrow \frac{4x-1}{15} > \frac{x}{4}$

$\Rightarrow 4(4x-1) > 15x \Rightarrow 16x - 4 > 15x \Rightarrow 16x - 15x > 4 \Rightarrow x > 4$

Hence, the solution set of the given inequation is $(4,\ \infty)$.

$(iii)\ \frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x-6)$

Solution: We have, $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x-6)$

$\Rightarrow \frac{1}{2}\left(\frac{3x+20}{5}\right) \geq \frac{1}{3}\ (x-6) \Rightarrow \geq 9x + 60 \geq 10x - 60$

$\Rightarrow 9x - 10x \geq -60 - 60 \Rightarrow -x \geq -120 \Rightarrow x \leq 120$

Hence, the solution set of the given inequation is $(-\infty,\ 120]$.

$(iv)\ \frac{1}{x-2} < 0$

Solution: We have, $\frac{1}{x-2} < 0 \Rightarrow x - 2 < 0 \Rightarrow x < 2$

So, the solution set of the given inequation is $(-\infty,\ 2)$.

$(v)\ \frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$

Solution: We have $\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3} \Rightarrow 9(x-2) \geq 25(2-x)$

$\Rightarrow 9x - 18 \geq 50 - 25x \Rightarrow 9x + 25x \geq 50 + 18$

$\Rightarrow 34x \geq 68 \Rightarrow x \geq 2$

So, the solution set of the given inequation is $[2,\ \infty)$.

Solving Inequation of the form $\frac{ax+b}{cx+d} > k,\ or\ \frac{ax+b}{cx+d} \geq k.\ or\ \frac{ax+b}{cx+d} < k,\ or\ \frac{ax+b}{cx+d} \leq k$

Example: Solve the following inequations:

$(i)\ \frac{2x+4}{x-1} \geq 5$

Solution: We have, $\frac{2x+4}{x-1} \geq 5 \Rightarrow \frac{2x+4}{x-1} - 5 \geq 0 \Rightarrow \frac{2x+4-5(x-1)}{x-1} \geq 0$

$\Rightarrow \frac{2x+4-5x+5}{x-1} \geq 0 \Rightarrow \frac{-3x+9}{x-1} \geq 0$

$\Rightarrow \frac{3x-9}{x-1} \leq 0 \Rightarrow \frac{x-3}{x-1} \leq 0$

$\Rightarrow 1 < x \leq 3$

Hence, the solution set of the given inequation is $(1,\ 3]$.

$(ii)\ \frac{x+3}{x-2} \leq 2$

Solution: We have, $\frac{x+3}{x-2} \leq 2$

$\Rightarrow \frac{x+3}{x-2} - 2 \leq 0$

$\Rightarrow \frac{x+3-2x+4}{x-2} \leq 0$

$\Rightarrow \frac{-x+7}{x-2} \leq 0$

$\Rightarrow \frac{x-7}{x-2} \geq 0$

$\Rightarrow x \in (-\infty,\ 2) \cup [7,\ \infty)$

Hence, the solution set of the given inequation is $(-\infty,\ 2) \cup [7,\ \infty)$.

Solving inequations with a modulus sign

For this, we use the inequalities:

- For $a > 0,\ |x| \leq a \iff -a \leq x \leq a$
- $|x| \geq a \iff x \geq a\ or\ x \leq -a$

Example: Solve $|5x + 3| < 4$

Solution: We have, $|5x + 3| < 4$

$\Rightarrow -4 < 5x + 3 < 4 \Rightarrow -4 < 5x + 3$ and $5x + 3 < 4$

$\Rightarrow -7 < 5x$ and $5x < 1 \Rightarrow x > \frac{7}{5}$ and $x < \frac{1}{5}$

$\Rightarrow -\frac{7}{5} < x < \frac{1}{5}$

Hence, the solution set is $\left\{x: -\frac{7}{5} < x < \frac{1}{5}\right\}$.

Example: Express the following subset of R in the interval form: $\{x:\ |5x - 3| > 12\}$

Solution: We have, $\{x:\ |5x - 3| > 12\} = \{x : 5x - 3 < -12 \text{ or } 5x - 3 > 12\}$

$= \{x: 5x < -9\ \text{ or }\ 5x > 15\}\quad = \{x: x < -\frac{9}{5} \text{ or } x > 3\}$

Required interval is $\left(-\infty,\ -\frac{9}{5}\right) \cup (3,\ \infty)$.

Example: Solve the following inequality and represent the solution set on the number line.

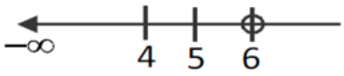
$x + \frac{x}{2} + \frac{x}{3} < 11$

Solution: Given that $x + \frac{x}{2} + \frac{x}{3} < 11$

$\Rightarrow \frac{11x}{6} < 11\ \Rightarrow x < 6$

Hence, the solution set is $(-\infty, 6)$.

The solution set can be represented on the number line as follows:



Graphical Solution of Linear Inequations in one variable:

Consider graphs of the inequalities $x > 2$ and $x \geq 2$:

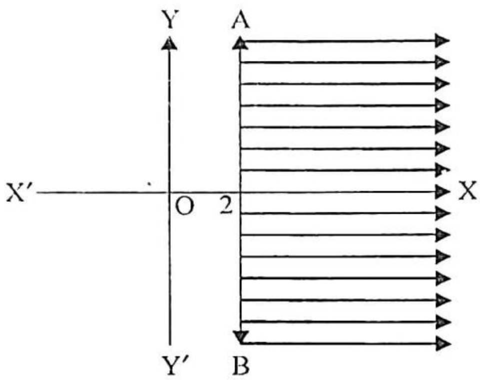
We know that the graph of $x = 2$ is a straight line say $\overleftrightarrow{AB}$, parallel to $y - axis$ at a distance 2 from it, $i.e.$, the $x -$ coordinate of any point on this graph is 2.

Now to draw the graph of the inequality $x > 2$, we see that the line $x = 2$, $i.e.$, $\overleftrightarrow{AB}$ divides the plane in two parts, one lying on the right side of $\overleftrightarrow{AB}$ and the other on the left side of it. The x −coordinate of any point lying in the region on the right side of $\overleftrightarrow{AB}$ is greater than 2 and no point of this region lie on $\overleftrightarrow{AB}$.

Since for all points on the plane region lying on the right side of $\overleftrightarrow{AB}$, we have $x > 2$. This plane region (shown in fig) extended to the right of $\overleftrightarrow{AB}$ (no point of this region lying on $\overleftrightarrow{AB}$) is the graph of the inequality $x > 2$. The graph is called the solution set of the inequality $x > 2$.

To draw the graph of the inequality $x \geq 2$, we see that it consists of $x > 2$ and $x = 2$. The equality $x = 2$ is true for all points lying on the line $\overleftrightarrow{AB}$. Thus all points lying on the line $\overleftrightarrow{AB}$ and to the right of $\overleftrightarrow{AB}$ satisfy the inequality $x \geq 2$.

Hence the plane region (shown in fig) extended from $\overleftrightarrow{AB}$ to the right of it is the graph of the inequality $x \geq 2$ and this region is called the solution set of $x \geq 2$.



Graphical solution of Linear Inequations in two variables:

If a, b and c are real numbers, then the equation $ax + by + c = 0$ is called a linear equation in two variables x and y whereas the inequalities $ax + by \leq c$, $ax + by \geq c$, $ax + by < c$ and $ax + by > c$ are called linear inequations in two variables x and y.

We know that the graph of the equation $ax + by = c$ is a straight line which divides the $xy - plane$ into two parts which are represented by $ax + by \leq c$ and $ax + by \geq c$. These two parts are known as the closed half – spaces. The regions represented by $ax + by < c$ and $ax + by > c$ are known as the open half – spaces.

The set of points (x, y) satisfying a linear inequation is called the solution set of that inequations and the region containing all the solutions of linear inequation is called the solution region.

Algorithm:

Step I : Convert the given inequation, say $ax + by \leq c$, into equation $ax + by = c$ which represents a straight line in $xy - plane$.

Step II: Put $y = 0$ and $x = 0$ in the equation obtained in step I to get the points where the line meets the coordinate axes.

Step III: Join the points to obtain the graph of the line from the given inequation.

Step IV: Choose a point, if possible $(0, 0)$, not lying on this line: Substitute its coordinates in the inequation. If the inequation is satisfied, then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain the chosen point.

Step V: The shaded region obtained in step IV represents the desired solution set.

Points to Remember: In case of inequalities $ax + by \leq c$ and $ax + by \geq c$, points on the line are also a part of the shaded region while in case of inequalities $ax + by < c$ and $ax + by > c$, points on the line $ax + by = c$ are not in the shaded region.

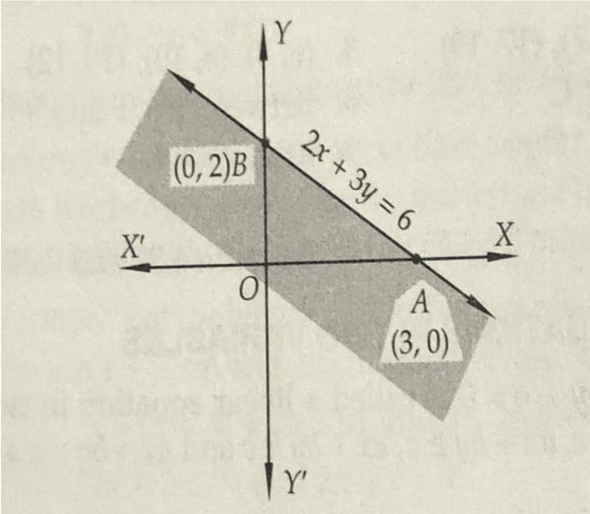
Example: Solve the following inequations graphically:

$2x + 3y \leq 6$.

Solution: Converting the given inequation into equation, we obtain $2x + 3y = 6$.

Putting $y = 0$ and $x = 0$ respectively, we get $x = 3$ and $y = 2$.

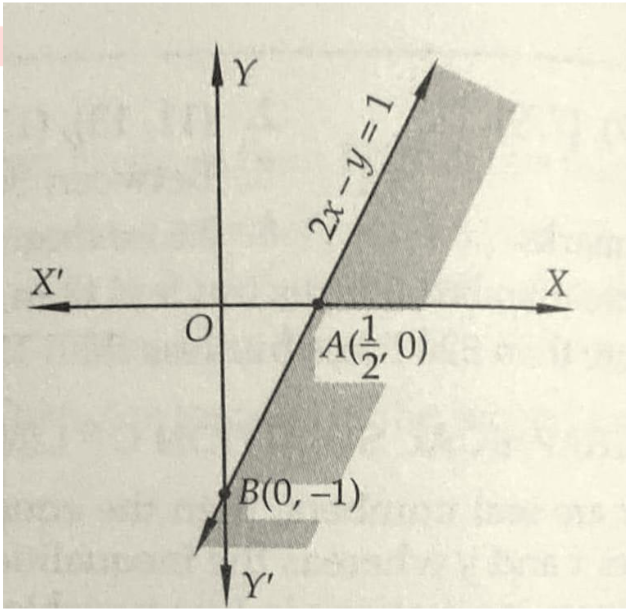
So, this line meets $x - axis$ at $A(3, 0)$ and $y - axis$ at $B(0,2)$. We plot these points and join them by a thick line. This line divides the $xy - plane$ in two parts. Consider the point $(0, 0)$. Clearly, $(0, 0)$ satisfies the inequality. So, the region containing the origin is represented by the given inequation as shown in the figure. This region represents the solution set of the given inequations.



Example: Solve the following inequations graphically:

$2x - y \geq 1$

Solution: Converting the given inequation into equation, we obtain $2x - y = 1$. This line meets x and $y - axes$ at $A\left(\frac{1}{2}\ ,\ 0\right)$ and $B(0,\ -1)$ respectively. Joining these points by a thick line we obtain the line passing through A and B as shown in figure. This line divides the $xy - plane$ into two regions. Consider the point $O(0,\ 0)$. Clearly, $(0,\ 0)$ does not satisfy the inequation $2x - y \geq 1$. So, the region not containing the origin is represented by the given inequation. Clearly, it represents the solution set of the given inequation.



General Solution of System of Linear Inequations in two Variables:

If two or more linear inequations have a common solution, then the inequations are called simultaneous inequations or system of inequations. The solution set of a system of linear inequations in two variables may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight line boundaries.

A system of linear inequations in two variables can be solved by graphical method.

Working Rules:

Step – I : Draw the graph of all the given inequations.

Step – II : Find the common shaded region, which satisfies all the given inequations.

Step – III: This common region is the required solution set of the given system of linear inequations.

Point to Remember:

If there is no common region, then the system of inequations has no solution.

Example: Exhibit graphically the solution set of the linear inequations

$3x + 4y \leq 12,\ 4x + 3y \leq 12,\ x \geq 0,\ y \geq 0$

Solution: Converting the inequations into equations, we get

$3x + 4y = 12,\ 4x + 3y = 12,\ x = 0$ and $y = 0$.

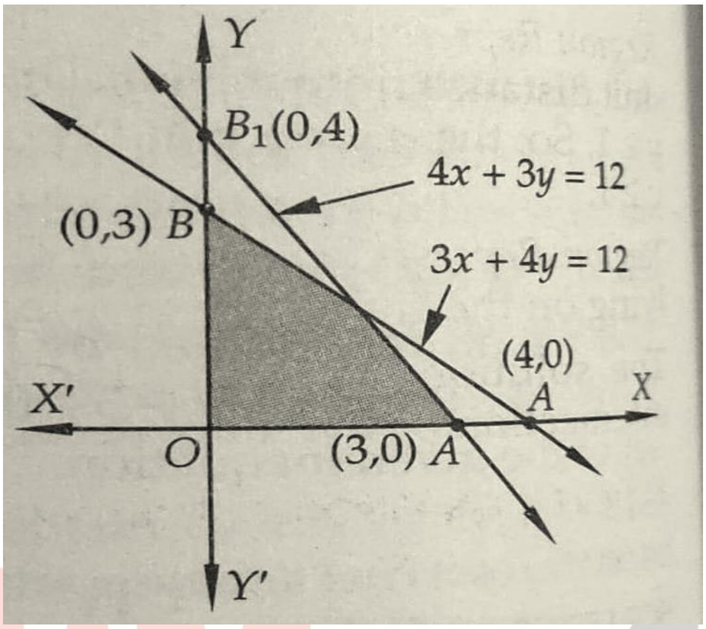
Region represented by $3x + 4y \leq 12$*:* The line $3x + 4y = 12$ meets the coordinate axes at $A(4,0)$ and $B(0,\ 3)$. Draw a thick line joining A and B. We find that $(0,\ 0)$ satisfies inequation $3x + 4y \leq 12$. So, the portion containing origin represents the solution set of the inequation $3x + 4y \leq 12$.

Region represented by $4x + 3y \leq 12$:

The line $4x + 3y = 12$ meets the coordinate axes at $A_1(3,\ 0)$ and $B_1(0,\ 4)$ respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \leq 12$.

Region represented by $x \geq 0$ *and* $y \geq 0$: Clearly, $x \geq 0$ and $y \geq 0$ represent the first quadrant.

Hence the shaded region given in figure represents the solution set of the given linear inequations.



Example: Exhibit graphically the solution set of the linear inequations.

$x + y \leq 5,\ 4x + y \geq 4,\ x + 5y \geq 5,\ x \leq 4,\ y \leq 3.$

Solution: Converting the inequations into equations, we obtain

$x + y = 5,\ 4x + y = 4,\ x + 5y = 5,\ x = 4,\ y = 3$

Region represented by $x + y \leq 5$*:* The line $x + y = 5$ meets the coordinate axes at $A(5,\ 0)$ and $B(0,\ 5)$ respectively. Join these points by a thick line. Clearly, $(0,\ 0)$ satisfies the inequality $x + y \leq 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \leq 5$.

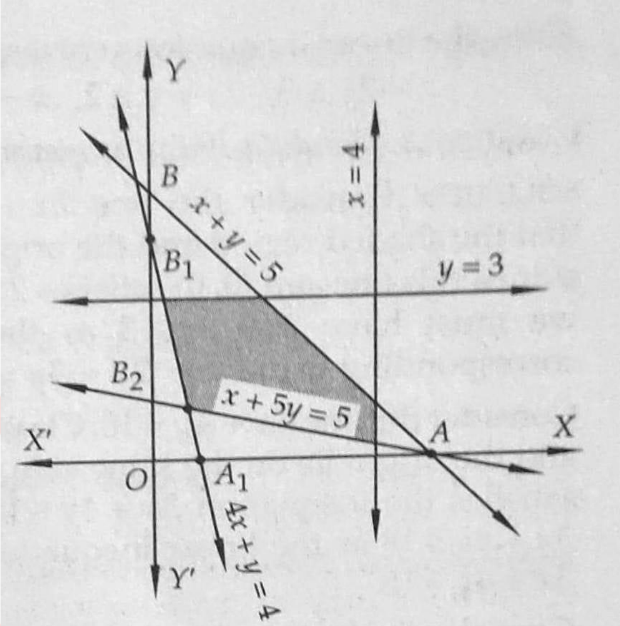
Region represented by $4x + y \geq 4$: The line $4x + y = 4$ meets the coordinate axes at $A_1(1,\ 0)$ and $B_1(0,\ 4)$ respectively. Join these points by a thick line. Clearly, $(0,\ 0)$ does not satisfy the inequation $4x + y \geq 4$. So, the portion not containing the origin is represented by the inequation $4x + y \geq 4$.

Region represented by $x + 5y \geq 5$: The line $x + 5y = 5$ meets the coordinate axes at $A(5,\ 0)$ and $B_2(0,\ 1)$ respectively. Join these two points by a thick line. We find that $(0,\ 0)$ does not satisfy the inequation $x + 5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

Region represented by $x \leq 4$: Clearly, $x = 4$ is a line parallel to $y - axis$ at a distance of 4 units from the origin. Since $(0,\ 0)$ satisfies the inequation $x \leq 4$ so, the portion lying on the left side of $x = 4$ is the region represented by $x \leq 4$.

Region represented by $y \leq 3$: Clearly, $y = 3$ is a line parallel to $x - axis$ at a distance 3 from it. Since $(0,\ 0)$ satisfies $y \leq 3$ so, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear inequations as shown in the figure.



Practical Problems based on Linear Inequalities:

In this section, we shall utilize the knowledge of solving linear inequations in one variable in solving different problems from various fields such as science, engineering, economics etc.

Example: Find all pairs consecutive odd positive integers, both of which are smaller than 18, such that their sum is more than 20.

Solution: Let x be the smaller of the two consecutive odd positive integers. Then, the other odd integer is $x + 2$.

It is given that both the integers are smaller than 18 and their sum is more than 20. Therefore,

$x + 2 < 18$ and, $x + (x + 2) > 20$

$\Rightarrow x < 16$ and $2x + 2 > 20$ $\quad \Rightarrow x < 16$ and $x > 9$

$\Rightarrow 9 < x < 16 \Rightarrow x = 11,\ 13,\ 15$

Hence, the required pairs of odd integers are $(11,\ 13)$, $(13,\ 15)$ and $(15,\ 17)$.

Example: Find all pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25.

Solution: Let x be the smaller of the two consecutive even positive integers. Then, the other even integer is $x+2$.

It is given that both the integers are larger than 8 and their sum is less than 25. Therefore,

$x > 8$ and $x + (x+2) < 25$

$\Rightarrow x > 8$ and $2x + 2 < 25 \Rightarrow x > 8$ and $x < \frac{23}{2}$

$\Rightarrow 8 < x < \frac{23}{2} \Rightarrow x = 10$

Hence, the required pair of even integers is $(10,\ 12)$.

Example: The cost and revenue functions of a product are given by $C(x) = 2x + 400$ and $R(x) = 6x + 20$ respectively, where x is the number of items produced by the manufacturer. How many items the manufacturer must sell to realize some profit?

Solution: We know that Profit = Revenue − Cost. Therefore, to earn some profit, we must have

Revenue > Cost

$\Rightarrow 6x + 20 > 2x + 400$

$\Rightarrow 6x - 2x > 400 - 20$

$\Rightarrow 4x > 380$

$\Rightarrow x > 95$

Hence, the manufacturer must sell more than 95 items to realize some profit.

Example: IQ of a person is given by the formula: $IQ = \frac{MA}{CA} \times 100$, where MA is mental age and CA is chronological age. If $80 \leq IQ \leq 140$ for a group of 12 year children, find the range of their mental age.

Solution: We have: $CA = 12$ years

$\therefore\ IQ = \frac{MA}{CA} \times 100$

$\Rightarrow IQ = \frac{MA}{12} \times 100 = \frac{25}{3}\ MA$

Now, $80 \leq IQ \leq 140$

$\Rightarrow 80 \leq \frac{25}{3}\ MA \leq 140 \Rightarrow 240 \leq 25\ MA \leq 420$

$\Rightarrow \frac{240}{25} \leq MA \leq \frac{420}{25} \Rightarrow 9.6\ \leq MA \leq 16.8$

Example: In the first four papers each of 100 marks, Rishi got 95, 72, 73, 83 marks. If he wants an average of greater than or equal to 75 marks and less than 80 marks, find the range of marks he should score in the fifth paper.

Solution: Suppose scores x marks in the fifth paper. Then

$75 \leq \frac{95+72+73+83+x}{5} < 80$

$\Rightarrow 75 \leq \frac{323+x}{5} < 80$

$\Rightarrow 375 \leq 323 + x < 80$

$\Rightarrow 52 \leq x < 77$

Hence, Rishi must score between 52 and 77 marks.

Example: A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%.

Solution: Let x litres of 30% acid solution be added to 600 litres of 12% solution of acid. Then,

Total quantity of mixture $= (600 + x)$ litres

Total acid content in the $(600 + x)$ litres of mixture $= \frac{30x}{100} + \frac{12}{100} \times 600$

It is given that acid content in the resulting mixture must be more than 15% and less than 18%.

$\therefore$ 15% of $(600 + x) < \left(\frac{30x}{100} + \frac{12}{100} \times 600\right) <$ 18% of $(600 + x)$

$\Rightarrow \frac{15}{100}(600 + x) < \left(\frac{30x}{100} + \frac{12}{100} \times 600\right) < \frac{18}{100}(600 + x)$

$\Rightarrow 15(600 + x) < 30x + 12 \times 600 < 18(600 + x)$

$\Rightarrow 9000 + 15x < 30x + 7200 < 10800 + 18x$

$\Rightarrow 9000 + 15x < 30x + 7200$ and $30x + 7200 < 10800 + 18x$

$\Rightarrow 9000 - 7200 < 30x - 15x$ and $30x - 18x < 10800 - 7200$

$\Rightarrow 1800 < 15x$ and $12x < 3600$

$\Rightarrow 120 < x$ and $x < 300$

$\Rightarrow 120 < x < 300$

Hence, the number of litres of the 30% solution of acid must be more than 120 but less than 300.

Example: A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and third length is to be twice as long as the shortest. What are the possible lengths for the shortest board if third piece is to be at least 5 cm longer than the second?

Solution: Let the length of the shortest piece be x cm.

Then, the lengths of the second and third piece are $(x+3)$ cm and $2x$ cm respectively. Then,

$x+(x+3)+2x \leq 91$ and $2x \geq (x+3)+5$

$\Rightarrow 4x+3 \leq 91$ and $2x \geq x+8$

$\Rightarrow 4x \leq 88$ and $x \geq 8$

$\Rightarrow x \leq 22$ and $x \geq 8 \Rightarrow 8 \leq x \leq 22$

Hence, the shortest piece must be at least 8 cm long but not more than 22 cm long.

Solution of system of Linear inequations in one variable:

The solution set of a system of linear inequations in one variable is the intersection of the solution sets of the linear inequations in the given system.

Example: Solve the following system of linear inequations:

$3x-6 \geq 0,\ 4x-10 \leq 6$

Solution: The given system of inequations is $3x-6 \geq 0 \ldots (i) \quad 4x-10 \leq 6 \ldots (ii)$

Now, $3x-6 \geq 0 \Rightarrow 3x \geq 6 \Rightarrow x \geq 2$

$\therefore$ Solution set of inequation (i) is $[2,\ \infty)$.

and $4x-10 \leq 6 \Rightarrow 4x \leq 16 \Rightarrow x \leq 4$

$\therefore$ Solution set of inequation (ii) is $(-\infty. 4]$.

The solution sets of inequations (i) and (ii) can be represented graphically on real line.

Hence, the solution set of the given system of inequations is $(-\infty,\ 4] \cap [2,\ \infty) = [2,\ 4]$

Example: Solve: $-5 \leq \frac{2-3x}{4} \leq 9$

Solution: We have, $-5 \leq \frac{2-3x}{4} \leq 9 \Rightarrow -20 \leq 2-3x \leq 36 \Rightarrow -22 \leq -3x \leq 34$

$\Rightarrow \frac{-22}{-3} \geq x \geq \frac{34}{-3}$

$\Rightarrow \frac{22}{3} \geq x \geq -\frac{34}{3} \Rightarrow -\frac{34}{3} \leq x \leq \frac{22}{3} \Rightarrow x \in \left[-\frac{34}{3}, \frac{22}{3}\right]$

Hence, the interval $\left[-\frac{34}{3}, \frac{22}{3}\right]$ is the solution set of the given system of inequations.