Chapter - 6

Linear Inequalities

In equation:- A statement involving variables and the sign of inequality viz $>, <, \geq$ or $\leq$ is called an inequation or an inequality.

i.g $4x+5\geq 0, 3x-2<0, 2x+3\leq 0, 5x-7>0, x^2+3x+2<0, x^2+5x+1>0$ etc

Types of Inequalities:-

(a) Numerical inequality: An inequality that does not involve any variable is called a numerical inequality i.g $4>3, 7<15$.

(b) Literal inequality:- An inequality which has variables is called literal inequality e.g $x<8, y\geq 12, x-y\leq 4$

(c) Strict inequality:- An inequality which has only < or > is called strict inequality.

(d) Slack inequality:- An inequality that has only $\leq$ or $\geq$ is called slack inequality.

Linear inequality:- An inequality is said to be linear if the variables occur in first degree only and there is no term involving the product of the variables. E.g $ax+by+c\leq 0, ax+b>0$

Linear inequality in one variable:- A linear inequality that has only one variable is called linear inequality in one variable. i.e $ax+b<0, a\neq 0$

Note:- An inequality is one variable in which the degree of the variable is 2, is called quadratic inequality in one variable. E.g $ax^2+bx+c\geq 0, a\neq 0$

Linear Inequality in two variables:-

A linear inequality which has only two variables is called linear inequality in two variables.

e.g $3x+4y\leq 0, 4x+5y>0$

Properties:-

(a) **Addition or subtraction:-** Some numbers may be added core subtracted to form both sides of an inequality i.e if a > b, then for any number C, $a+c>b+c$ or $a-c>b-c$.

(b) Multiplication or division:- If both sides of an inequality are multiplied (or divided) by the same positive number, then the sign of inequality remains the same. But when both sides are multiplied (divided) by the same negative number, then the sign of inequality is reversed. Let a, b, c be three real numbers such that $a>b$ and $c\neq 0$

(i) If c>0, then $\frac{a}{c}>\frac{b}{c}$ and $ac>bc$ (ii) If c < 0, then $\frac{a}{c}<\frac{b}{c}$ and $ac<bc$

Solution Set:- The set of all solutions of an inequality is called the solution set of the inequality.

The algebraic solution of linear inequalities in one variable:-

Any solution of a linear inequality in one variable is a value of the variable which makes it a true statement. E.g $x=1$ is the solution of the linear inequality $2x+5>0$

Method to solve a linear inequality in one variable:-

Step – I

Collect all terms involving the variable (x) on one side and constant terms on the other side.

Step – II

Divide this inequality by the coefficient of the variable (x). This gives the solution set of the given inequality.

Step – III

Write the solution set

Example – 1

Solve $24x<100$ when (a) x is a natural number (b) x is an integer

Solution:-

(a) $24x<100$

$\Rightarrow x<\frac{100}{24}$ Values of x are 1, 2, 3, 4

(b) $24x<100 \quad \Rightarrow x<\frac{100}{24}$ The solution set of inequality is $\{.....,-4,-3,-2,-1,0,1,2,3,4\}$

Example – 2 Solve the linear inequality $-5x+25>0$

Solution:- We have $-5x+25>0$

$\Rightarrow 5x<25 \quad \Rightarrow x<5$ Hence, the required solution set is $(-\infty,5)$

Example – 3 Solve the inequality $\frac{5-3x}{3}\le\frac{x}{6}-5$

Solution:- We have $\frac{5-3x}{3}\le\frac{x}{6}-5$

$\Rightarrow \frac{5-3x}{3}\le\frac{x-30}{6}$

$\Rightarrow 2(5-3x)\le x-30$

$\Rightarrow x\ge\frac{40}{7}$ i.e $x\in\left[\frac{40}{7},\infty\right)$

Hence, the required solution set is $\left[\frac{40}{7},\infty\right)$

Representation of solution of a linear inequality in one variable on the number line:-

To represent the solution of a linear inequality in one variable on a number line uses the following rules.

(a) To represent x < a (or x > a) on a number line put a circle (o) on the number a and dark the line to the left (or right) of the number.

(b) To represent $x \le a(\text{or}\, x \ge a)$ on a number line put a dark circle (.) on the number a and dark the line to the left (or right) of the number a.

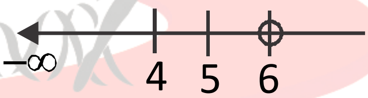
Example:- 1

Solve the following inequality and show the graph of the solution in each case on the number line

(a) $x+\frac{x}{2}+\frac{x}{3}<11$ (b) $\frac{1}{2}\left(\frac{3}{5}x+4\right)\ge\frac{1}{3}(x-6)$

Solution:-

(a) $x+\frac{x}{2}+\frac{x}{3}<11$



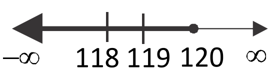
$\Rightarrow \frac{11x}{6}<11$

$\Rightarrow x<6$

$\Rightarrow x\in(-\infty,6)$

(b) $\frac{1}{2}\left(\frac{3}{5}x+4\right)\ge\frac{1}{3}(x-6)$

$\Rightarrow \frac{1}{2}\left(\frac{3x+20}{5}\right)\ge\frac{1}{3}(x-6)$



$\Rightarrow 3(3x+20)\ge 10(x-6)$

$\Rightarrow 9x+60\ge 10x-60$

$\Rightarrow x\le 120$

$\Rightarrow x\in(-\infty,120]$

System of inequalities in one variable and their solutions:-

Two or more inequalities taken together comprise a system of inequalities and solution of the system of inequalities are the solutions common to all the inequalities comprising the system.

Type - I:-

When two separate linear inequalities are given. If the given system of inequalities comprises two separate linear inequalities, then to solve these we use the following working steps.

Step – I

Solve each inequality separately and obtain their solution sets.

Step – II

Find the intersection of the solution sets obtained in step-I.

Example:-2

Solve the following system of inequalities $2x-3<7 \text{and} 2x>-4$. Also, represent the solution graphically on the number line.

Solution:-

We have $2x-3<7$ .... (1)

$2x>-4$ .... (ii)

From inequality (1)

$2x-3<7$

$\Rightarrow x<5$

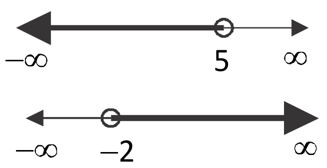
The solution set is $(-\infty,5)$

From inequality (2) we have

$2x>-4$ $\qquad$ $2x>-2$

$\therefore$ The solution set is $(-2,\infty)$

Now let us draw the graphs of the solutions of both inequalities on the number line



Common to both are lying in the interval (-2, 5)

Type – II

When inequalities of the form $a \le \frac{cx+d}{e} \le b$, when $a,b,c,d, \in R$. This type of inequalities will be formed by combining the inequalities $a \le \frac{cx+d}{e}$ and $\frac{cx+d}{e} \le b$

To solve such type of inequalities, make the middle term free from constant (i.e $f \le x \le g$)

Example:- 3

Solve the inequalities $-15 < \frac{3(x-2)}{5} \le 0$

Solution:- $-15 < \frac{3(x-2)}{5} \le 0$

$\Rightarrow \frac{-75}{3} < x-2 \le 0$

$\Rightarrow -25 < x-2 \le 0$

$\Rightarrow -23 < x \le 2$

$\Rightarrow x \in (-23, 2]$

Type – III

When the inequality of the form $\frac{ax+b}{cx+d} \begin{smallmatrix} \ge \\ < \end{smallmatrix} K$, where K is a constant

Working Rule:-

Step – I

Collect all terms in the L.H.S of the given inequality to make R.H.S zero and then reduce it any one of the forms

$\frac{px+q}{cx+d} > 0$ or $\frac{px+q}{cx+d} \ge 0$ or $\frac{px+q}{cx+d} < 0$ or $\frac{Px+q}{cx+d} \le 0$

Step – II

Multiply the above inequality by the square of the denominator to reduce it in the form of the product of two expressions i.e $(px+q)(cx+d) \ge 0$

Step – III

Now solve both expressions separately with suitable inequality sign and get solution sets. Use the result

(a) If the product of two terms < 0 then both terms have opposite sign

(b) If the product of two terms > 0 then both terms have the same sign

Step – IV

Take the union of the above solution sets, which gives the solution o given inequality.

Example – 4

Solve the inequality $\frac{x-2}{x+5} > 2$

Solution:- $\frac{x-2}{x+5} > 2$

$\Rightarrow \frac{x-2}{x+5} - 2 > 0$

$\Rightarrow \frac{x-2-2x-10}{x+5} > 0$

$\Rightarrow \frac{-(x+12)}{x+5} > 0$

$\Rightarrow \frac{x+12}{x+5} < 0$

$\Rightarrow \frac{(x+12)(x+5)^2}{x+5} < 0 \times (x+5)^2$

$\Rightarrow (x+12)(x+5) < 0$

$\Rightarrow x \in (-12, -5)$

Linear inequalities in two variables and their graphical solutions:-

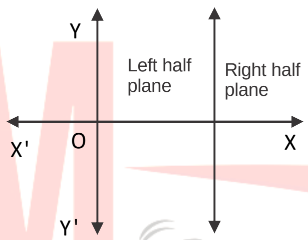
An inequality of the form $ax+by+c>0$ or $ax+by+c\geq 0$ or $ax+by+c\leq 0$ or $ax+by+c<0$ where $a\neq 0, b\neq 0$ is called a linear inequality in two variables x and y. The region containing all the solutions of an inequality is called the solution region.

Concept of Half planes:-

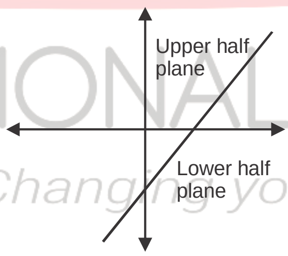
The graph $ax+by+c=0$ is a straight line that divides the XY-plane into two parts. Each part is known as half-plane.

Types of half-planes:-

(a) Left and right half-planes:- A vertical line will divide the XY-plane into two parts, left half-plane and right half-plane.



(b) Lower and upper half-planes:- A non-vertical line will divide the XY-plane into two parts, lower half-plane, and upper half-plane.



(c) Closed half-plane:- A half-plane in XY-plane is called a closed half-plane if the line separating the plane is also included in the half-plane. (The inequality involving sign $\leq$ or $\geq$)

(d) Open half-plane:- A half-plane in XY-plane is called and open half-plane if the line separating the plane is not included in the half-plane. (The inequality involving sign < or >)

Graph of a linear inequality in two variables:-

Working rule:-

Step – I

Consider the equation $ax+by+c=0$ in the plane of the given inequality.

$(ax+by+c>0, ax+by+c\geq 0, ax+by+c<0 \text{ or } ax+by+c\leq 0)$

Step – II

Find the intercepts on the co-ordinate axes. (To find x-int put y = 0 and for y-int put x = 0)

If the line passing through origin find another point other than the origin.

Step – III

Draw a line joining the points obtained in. Step – II

If the inequality is of the form of < or > then draw a dotted line otherwise draw a thick or dark line.

Step – IV

Take any point (preferable origin) not lying on the line and check whether this satisfies the given linear inequality or not

Step – V

If the inequality is satisfied by this point then shade that portion of the plane which contains the chosen point. The shaded region obtained in step – v represents the graph of a linear inequality in two variables.

Example – 1,

Solve the inequality $2x+y\geq 3$ graphically

Solution:-

We have $2x+y\geq 3$

In equation from $2x+y-3=0$

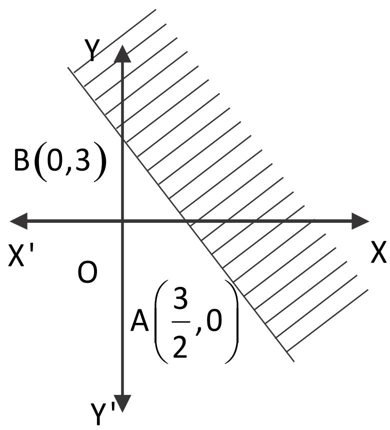
x	$3/2$	0
y	0	3

Points are $A\left(\frac{3}{2},0\right)$ and $(0,3)$

Now take a point not lying on the line say (0, 0)

$2\times 0+0\times 3\geq 3$

$\Rightarrow 0\geq 3$, which is not correct. So we shade the portion which does not contain (0, 0)



Example -2 Solve the inequality $5x+2y<10$ graphically.

Solution:- Given equality is $5x+2y<10$ corresponding equation line is $5x+2y=10$

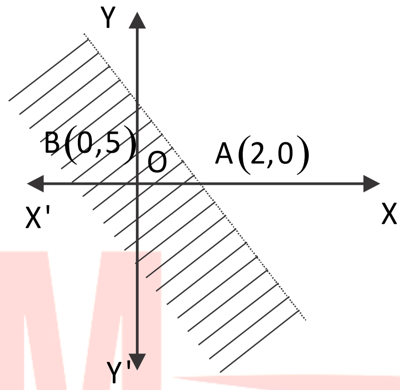
x	2	0
y	0	5

The points are (2, 0) and (0, 5)

Join the points with a dotted line on putting (0,0) is given inequality, we get

$5\times 0+2\times 0<10$

$\Rightarrow 0<10$, which is true



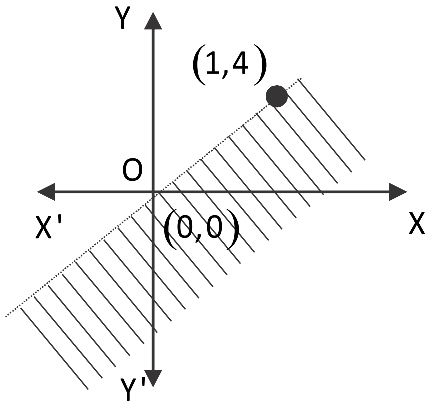
Hence the shaded region represents the solution region of the given inequality.

Example – 3 Solve $4x-y>0$ graphically

Solution:- We have $4x-y>0$ its equation form is $4x-y=0$

x	0	1
y	0	4

The line passes through the points (0, 0) and (1, 4) since the inequality is if the form <. So we join the points (0,0) and (1, 4) by a dotted line.



Now take the point, say (1, 2) not lying on the line, to check whether it satisfies the given linear inequality or not.

At (1, 2) $4(1)-2>0$

$\Rightarrow 2>0$, which is true

Thus the shaded region gives the solution region.

Graphical solution of a system of linear inequations in two variables:-

A system of linear inequalities in two variables can be solved by graphical method.

Working Rules:-

Step – I

Draw the graph of all the given inequalities

Step – II

Find the common shaded region, which satisfies all the given linear inequalities

Step – III

This common region is the required solution region of the system of given inequalities. If there is no common region, then the system of inequalities has no solution.

Example:-

Solve the system of inequalities graphically $3x+4y\le 60, x+3y\le 30, x\ge 0, y\ge 0$

Solution:-

We have the following inequalities

$3x+4y\le 60$ .... (i)

$x+3y\le 30$ .... (ii)

$x\ge 0$ ....(iii)

$y\ge 0$ .... (iv)

Take inequality (i) $3x+4y<60$

In equation form, it can be written as $3x+4y=60$ let us construct the following table

x	20	0
y	0	15

Thus the line intersects at A (20, 0) and B (0, 15) to the co-ordinate axes.

On putting (0, 0) in the given inequality we get $0\le 60$, which is true.

So for $3x+4y<60$ shade the half-plane which contain (0, 0)

Take inequality (ii) $x+3y\le 30$

In the equation form, it can be written as $x+3y=30$

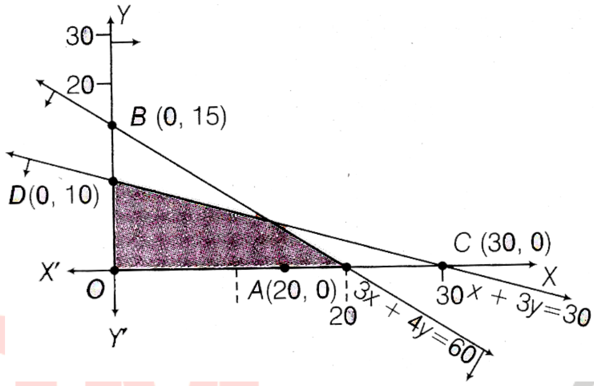
Now, let us construct the following table.

x	30	0
y	0	10

Thus, the line intersects the x-axis at C(30,0) and y-axis at D(0, 10) on putting (0, 0) in the given inequality we get $0 \le 30$, which is true

So for $x+3y \le 30$ shade the half-plane which contain (0, 0)

Take inequalities $x \ge 0, y \ge 0$ So choose the region which lines 1st quadrant only.



Here common shaded region represents the solution region for the system of inequalities.

Question:-

Solve the system of inequalities graphically

$x+y \le 5$

$4x+y \ge 4$

$x+5y \ge 5$

$x \le 4$

$y \le 3$

Solution:-

We have

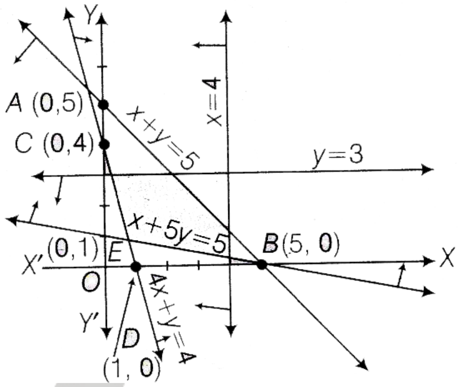
$x+y\le 5$ .... (i)

$4x+y\ge 4$ .... (ii)

$x+5y\ge 5$ .... (iii)

$x\le 4$ .... (iv)

$y\le 3$ ....(v)



Take inequality $x+y\le 5$

In equation form, it can be written as $x+y=5$

x	5	0
y	0	5

The line passes through the points A (0, 5) and B (5, 0).

Now on joining points A and B by a dark line, putting $x=0, y=0$

We get $0\le 5$, which is true

Take the inequality (ii) $4x+y\ge 4$

In equation form, it can be written as $4x+y=4$

x	0	1
y	4	0

Passes through the points C (0, 4) and D(1, 0). Now joining the points C and D by a dark line we get the line CD

On putting x = 0 and y = 0

$0\ge 4$, which is false

Take the inequality (iii) $x+5y\ge 5$

In equation form, it can be written as $x+5y=5$

x	0	5
y	1	0

The points E(0, 1) and B (5,0)

On putting x = 0 and y = 0 we get $0\ge 5$, which is false

Take the inequality (iv) $x\le 4$

In equation form, it can be written as x = 4

Which is parallel to the y-axis

The half-plane contains the origin. Take inequality (v) $y \le 3$

In equation form, it can be written as y = 3

This is a line parallel to the x-axis

The half-plane contains the origin

The common shaded region represents the solution region for the system of inequalities

Some Applications of linear inequations in one variable.:-

In this section, we shall utilize the knowledge of solving linear inequations in one variable in solving different problems from various fields such as science, engineering, economics, etc. Following examples, we illustrate the same.

Example – 1

Find all pairs of consecutive odd positive integers both of which are smaller than 18, such that their sum is more than 20.

Solution:- Let x be the smaller of the two consecutive odd positive integers. Then, the other odd integer is x + 2. It is given that both the integers are smaller than 18 and their sum is more than 20.

Therefore,

$x + 2 < 18 \text{and}, x + (x + 2) > 20$

$\Rightarrow x < 16 \text{and} 2x + 2 > 20$

$\Rightarrow x < 16 \text{ and } 2x > 18$

$\Rightarrow x < 16 \text{ and } x > 9$

$\Rightarrow 9 < x < 16$

$\Rightarrow x = 11, 13, 15$

Hence, the required pairs of odd integers are (11, 13), (13, 15), and (15, 17).

Example – 2 Find all pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25.

Solution:- Let x be the smaller of the two consecutive even positive integers. Then, the other even integer is $x + 2$. It is given that both the integers are larger than 8 and their sum is less than 25. Therefore, $x > 8$ and $x + x + 2 < 25$

$\Rightarrow x > 8 \text{and} 2x + 2 < 25$

$\Rightarrow x > 8 \text{and} 2x < 23$

$\Rightarrow x > 8 \text{ and } x < \frac{23}{2}$

$\Rightarrow 8 < x < \frac{23}{2}$

$\Rightarrow x = 10$ (Choose only even) Hence, the required pair of even integers is (10, 12)

Example – 3

The cost and revenue functions of a product are given by $C(x)=2x+400$ and $R(x)=6x+20$ respectively, where x is the number of items produced by the manufacture. How many items the manufacturer must sell to realize some profit?

Solution:-

We have, Profit = Revenue – Cost

Therefore, to earn some profit, we must have

Revenue > Cost

$\Rightarrow 6x+20>2x+400$

$\Rightarrow 6x-2x>400-20$

$\Rightarrow 4x>380$

$\Rightarrow x>\frac{380}{4}=95$

Hence, the manufacture must sell more than 95 items to realize some profit.

Example – 4

IQ of a person is given by the formula $IQ=\frac{MA}{CA}\times 100$

Where MA is mental age and CA is chronological age. If $80\le IQ\le 140$ for a group of 12-year children, find the range of their mental age.

Solution:-

We have, CA = 12 years

$\therefore IQ=\frac{MA}{CA}\times 100 \Rightarrow IQ=\frac{MA}{12}\times 100=\frac{25}{3}MA$

Now, $80\le IQ\le 140$

$\Rightarrow 80\le \frac{25}{3}MA\le 140$

$\Rightarrow 240\le 25MA\le 420$

$\Rightarrow \frac{240}{25}\le MA\le \frac{420}{25}$

$\Rightarrow 9.6\le MA\le 16.8$

Example – 5

In the first four papers each of 100 marks, Rishi got 95,72, 73, 83 marks. If he wants an average of greater than or equal to 75 marks and less than 80 marks, find the range of marks he should score in the fifth paper.

Solution:-

Suppose scores X marks in the fifth paper. Then,

$$75 \le \frac{95+72+73+83+x}{5} < 80$$

$$\Rightarrow 75 \le \frac{323+x}{5} < 80$$

$$\Rightarrow 375 < 323 + x < 400$$

$$\Rightarrow 52 < x < 77$$

Hence, Rishi must score between 52 and 77 marks.