Chapter- 2

Relations and Functions

Introduction:

We are familiar with the concept of relationships between people. For example, a mother-daughter relationship exists between P and Q if and only if P is the mother of Q or Q is the daughter of P.

In mathematics, we have many relations such as $x < y$, p is divisible by q, $a = b^2$, volume $V = x^3$ of a cube. Hence, we observe that every relationship involves a pair of objects or elements in a particular order.

Ordered Pairs:

There are many common ways of pairing, relating, or associating elements or members of one or more sets with each other. The states with their capitals can be associated through ordered pairs like (Odisha, Bhubaneswar), (Bihar, Patna), (Maharashtra, Mumbai), etc.

Identifying an Ordered Pair:

An ordered pair is a pair of elements whose components are written according to a specific order, separating them by a comma and enclosing the pair in small brackets. Let A and B be two sets. Consider the pair (x, y), where $x \in A$ and $y \in B$. Then (x, y) is called an ordered pair. In an ordered pair, the order in which the two elements occur is important. Thus (x, y) and (y, x) are different ordered pairs.

Equality of Ordered Pairs:

Two ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Note: $(a, b) = (b, a)$ iff $a = b$.

Example: Find x and y if $(x + 3, 5) = (6, 2x + y)$

Sol: $x + 3 = 5 \Rightarrow x = 3$ Also $2x + y = 5 \Rightarrow y = 5 - 2x = 5 - 6 = -1$

Example: If $(a + 2, b - 3) = (5, 7)$, find the values of a and b.

Sol: We have, $a + 2 = 5$ and $b - 3 = 7$

$\Rightarrow a = 3$ and $b = 10$.

Cartesian Product of Sets:

Let A and B be two non – empty sets. The set of all ordered pairs $(a,\ b)$ where $a \in A$ and $b \in B$ is called the Cartesian product of A and B. It is denoted by $A \times B$.

Mathematically, $A \times B = \{(a,\ b): a \in A$ and $b \in B\}$

We have $B \times A = \{(a,\ b): b \in B$ and $a \in A\}$

Since $(a,\ b) \neq (b,\ a)$, so in general $A \times B \neq B \times A$, unless $a = b$.

We write, $A \times A$ as A^2.

The Cartesian product three non – empty sets $A,\ B,\ C$ is defined by

$A \times B \times C = \{(a,\ b,\ c): a \in A,\ b \in B,\ c \in C\}$

Note: The plane is represented in a form of set as $R \times R = R^2$.

Example: Let $A = \{1,\ 2,\ 3\}$ and $B = \{3,\ 4\}$

Then $A \times B = \{(1,\ 3),\ (1,\ 4),\ (2,\ 3),\ (2,\ 4),\ (3,\ 3),\ (3,\ 4)$

Example: Let $A = \{a,\ b\}$ and $B = \{x: x\ is\ a\ prime\ number\ less\ than\ 7\}$. Find $A \times B$ and $B \times A$. Show that $A \times B \neq B \times A$.

Sol: Given sets are $A = \{a,\ b\}$ and $B = \{2,\ 3,\ 5\}$

So, $A \times B = \{(a,\ 2),\ (a,\ 3),\ (a,\ 5),\ (b,\ 2),\ (b,\ 3),\ (b,\ 5)\}$

and $B \times A = \{(2,\ a),\ (2,\ b),\ (3,\ a),\ (3,\ b),\ (5,\ a),\ (5,\ b)\}$

Since, $(a,\ 2) \neq (2,\ a)$ so $A \times B \neq B \times A$.

Note: If $A \times B$ is given, then we can find $B \times A$ from it, just by reversing the position of elements of each ordered pair.

Example: If $A = \{-1,\ 1\}$, then find $A \times A \times A$.

Sol: $A \times A = \{(-1, -1),\ (-1,\ 1),\ (1,\ -1),\ (1,\ 1)\}$

$$A \times A \times A = \{(-1, -1,\ -1),\ (-1,\ 1,\ -1),\ (1,\ -1, -1),\ (1,\ 1,\ -1),\ (-1,\ -1,\ 1), (-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ 1)\}$$

Identifying two sets given their Cartesian Product:

In the Cartesian product $A \times B$, A is the set of first components/elements and B is the set of second components/elements.

Example: If $A \times B = \{(7,\ 2),\ (7,\ 3),\ (7,\ 4),\ (2,\ 2),\ (2,\ 3),\ (2,\ 4)\}$ then determine A and B.

Sol: A = set of first elements of $A \times B = \{\ 7,\ 2\}$

B = set of second elements of $A \times B = \{2,\ 3,\ 4\}$

Number of Elements in Cartesian Product of two Sets:

- If there are p elements in set A and q elements in set B, then there will be pq elements in $A \times\ B.\ i.e.$, if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.
- If A and B are non-empty sets and either A or B is an infinite set, then $A \times B$ will also be an infinite set.
- If either $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

Example: Let A and Bbe two sets such that $A \times B$ contains 6 elements. If three elements of $A \times B$ are $(1,\ 3),\ (2,\ 5)$ and $(3,\ 3)$, then find $A,\ B$, and remaining elements of $A \times B$.

Sol: It is given that $(1,\ 3),\ (2,\ 5)$ and $(3,\ 3) \in A \times B$.

So $A = \{1,\ 2,\ 3\}$ and $B = \{3,\ 5\}$

$\Rightarrow A \times B = \{(1,\ 3),\ (1,\ 5),\ (2,\ 3),\ (2,\ 5),\ (3,\ 3),\ (3,\ 5)\}$

Hence, the remaining elements of $A \times B$ are $(1,\ 5),\ (2,\ 3),\ (3,\ 5)$.

Example: Let $A = \{1,\ 2\}$ and $B = \{3,\ 4\}$. How many subsets will $A \times B$ have?

Sol: Here $n(A) = 2$ and $n(B) = 2$

So, $n(A \times B) = 4$.

Hence, the total number of subsets of $A \times B$ is $2^4 = 16$

Graphical Representation of the Cartesian Product of two Sets:

Example: *Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the following products graphically i.e. by lattices:* (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

SOLUTION (i) We have, $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$.

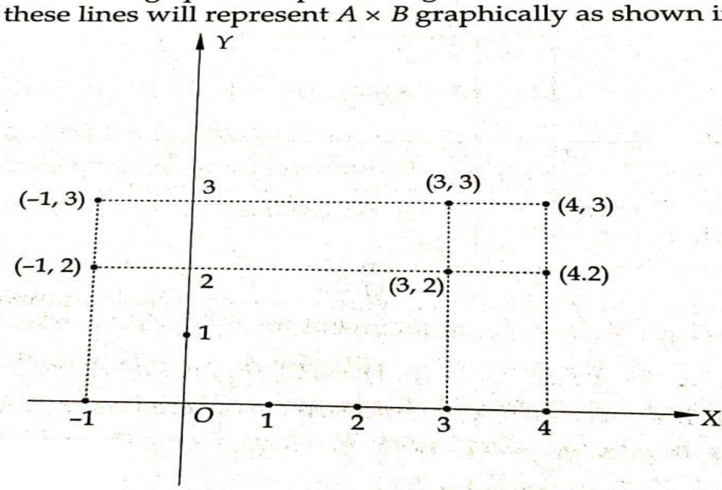
$\therefore \quad A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$

In order to represent $A \times B$ graphically, we follow the following steps:

(a) Draw two mutually perpendicular lines one horizontal and other vertical.

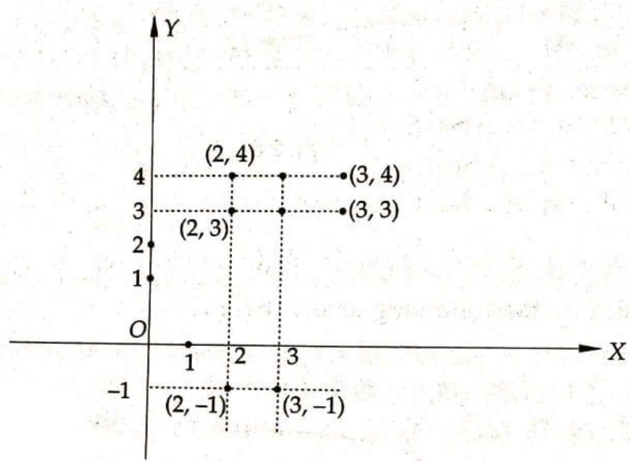
(b) On the horizontal line represent the elements of set A and on the vertical line represent the elements of set B.

(c) Draw vertical dotted lines through points representing elements of A on horizontal line and horizontal lines through points representing elements of B on the vertical line. Points of intersection of these lines will represent $A \times B$ graphically as shown in Fig. 2.3.



(ii) Clearly, $B \times A = \{2, 3\} \times \{-1, 3, 4\} = \{(2, -1), (2, 3), (2, 4), (3, -1), (3, 3), (3, 4)\}$

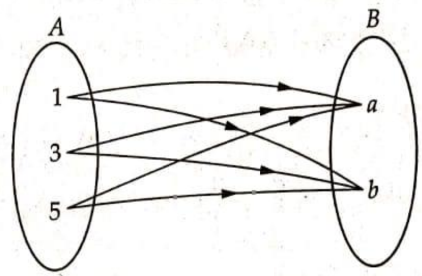
Here, we represent B on the horizontal line and A on vertical line. Graphical representation of $B \times A$ is as shown in Fig. 2.4.



DIAGRAMATIC REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in Fig. 2.2. Now, we draw line segments starting from each element of A and terminating to each element of set B.

If $A = \{1, 3, 5\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.



Some Theorems on Cartesian Product of Sets:

1. For any three non – empty sets A, B and C, we have

   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2. For any three non-empty sets A, B and C, we have $A \times (B - C) = (A \times B) - (A \times C)$

3. If A and B are any two non – empty sets, then $A \times B = B \times A \Leftrightarrow A = B$

4. If A Í B, then $A \times A \subseteq (A \times B) \cap (B \times A)$

5. If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any non – empty set C.

6. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

7. For any non –empty sets A, B, C, and D, we have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

8. For any three sets A, B and C, we have

   (i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

   (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

9. Let A and B be two non –empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

10. Let A be a non –empty set such that $A \times B = A \times C$. Then, $B = C$.

Example: If $A = \{1, 3\}$, $B = \{2, 3\}$ and $C = \{1, 2, 4\}$, then verify that

$(i)\ A \times (B \cap C) = (A \times B) \cap (A \times C)$

Sol: $\cap C = \{2\}$. So, $A \times (B \cap C) = \{(1, 2), (3, 2)\}$

Again, $A \times B = \{(1, 2), (1, 3), (3, 2), (3, 3)\}$

and $A \times C = \{(1, 1), (1, 2), (1, 4), (3, 1), (3, 2), (3, 4)\}$

So, $(A \times B) \cap (A \times C) = \{(1, 2), (3, 2)\}$.

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$(ii)\ A \times (B - C) = (A \times B) - (A \times C)$

Sol: B $-C = \{3\}$. So, $A \times (B - C) = \{(1, 3), (3, 3)\}$

Again, $A \times B = \{(1, 2), (1, 3), (3, 2), (3, 3)\}$

and $A \times C = \{(1, 1), (1, 2), (1, 4), (3, 1), (3, 2), (3, 4)\}$

So, $(A \times B) - (A \times C) = \{(1, 3), (3, 3)\}$

Hence, $A \times (B - C) = (A \times B) - (A \times C)$

$(iii)(A \cup B) \times C = (A \times C) \cup (B \times C)$

Sol: $A \cup B = \{1, 2, 3\}$

So, $(A \cup B) \times C = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3.1), (3, 2), (3, 4)\}$

Again, $A \times C = \{(1, 1), (1, 2), (1, 4), (3, 1), (3, 2), (3, 4)\}$

and $B \times C = \{(2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 4)\}$

So, $(A \times C) \cup (B \times C) = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3.1), (3, 2), (3, 4)\}$

Hence, $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Example:

If A and B are any two non – empty sets, then prove that $A \times B = B \times A \Leftrightarrow A = B$.

Solution: Let $A = B$

So $A \times B = A \times A = B \times A$.

Conversely, Let $A \times B = B \times A$. We have to show that $A = B$.

Let x is an arbitrary element of A $i.e.$ $x \in A$

$\Rightarrow (x,\ y) \in A \times B$, for any $y \in B$ $\Rightarrow (x,\ y) \in B \times A \Rightarrow x \in B$

So, $A \subseteq B$ .... (I)

Again, let y is an arbitrary element of B $i.e.$ $y \in B$.

$\Rightarrow (y,\ x) \in B \times A$, for any $x \in A \Rightarrow (y,\ x) \in A \times B \Rightarrow y \in A$

So, $B \subseteq A$ .... (II)

From (I) and (II), we get $A = B$.

Example: If $A \subseteq B$ and $\subseteq D$, then prove that $A \times C \subseteq B \times D$.

Sol: Let $(x,\ y) \in A \times C \Rightarrow x\ \in A$ and $y \in C$

$\Rightarrow x \in B$ and $y \in D$ (since $A \subseteq B$ and $C \subseteq D$) $\Rightarrow (x,\ y) \in B \times D$

Hence, $A \times C \subseteq B \times D$.

Example: For any sets $A,\ B,\ C$, and D, prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Sol: Let $(x,\ y) \in (A \times B) \cap (C \times D) \Leftrightarrow (x,y) \in (A \times B)$ and $(x,y) \in (C \times D)$

$\Leftrightarrow x \in A,\ y \in B$, and $x \in C,\ y \in D \Leftrightarrow x \in A$ and $x \in C,\ y \in B$ and $y \in D$

$\Leftrightarrow x \in (A \cap C)$ and $y \in (B \cap D) \Leftrightarrow (x,\ y) \in (A \cap C) \times (B \cap D)$

We get,$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

and $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Example: If two non – empty sets A and B have m elements in common, then how many elements do $A \times B$ and $B \times A$ have common?

Sol: We have $n(A \cap B) = m$

Now $n\big((A \times B) \cap (B \times A)\big) = n\big((A \cap B) \times (B \cap A)\big) = n\big((A \cap B) \times (A \cap B)\big) = m.m = m^2$

Example: For any three sets $A,\ B, C$ prove that

$(i)\ A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

Sol: $A \times (B' \cup C')' = A \times ((B')' \cap (C')') = A \times (B \cap C) = (A \times B) \cap (A \times C)$

$(ii)\ A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

Sol: Same as (i).

Relations

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.

Thus R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element.

If R is a relation from a non – empty set A to a non – empty set B and if $(x,\ y) \in R$, then we write $x\ R\ y$ and we say that x is related to y by the relation R.

If $(x,\ y) \notin R$, then we write $x\ R\ \ y$ and we say that x is not related to y by the relation R.

If $A = B$ $i.e.$ $R \subseteq A \times A$, then R is a relation defined on A.

Example: Let $A = \{0,\ 1,\ 2,\ 3\}$ and $B = \{1,\ 3,\ 4\}$

Then $A \times B = \{\ (0,1),\ (0,3),\ (0,\ 4),\ \ldots\ldots (3,\ 4)\}$

$R_1 = Equal\ to = \{\ (1,\ 1),\ (3,\ 3)\}$ $\quad R_2 = greater\ than = \{(2,\ 1),\ (3,\ 1)\}$

Example: If $A = \{1,\ 2,\ 3\}$ and $B = \{a,\ b,\ c\}$, then $R = \{(1,\ a),\ (2,\ c),\ (1,\ a),\ (3,\ a)\}$, being a subset of $A \times B$, is a relation from A to B. Here $(1,\ a) \in R$, so we write $1\ R\ a$.

Example: If $A = \{1,\ 2,\ 3,\ 4\}$ and $B = \{5,\ 6,\ 7,\ 8\}$, then which of the following are relations from A to B ? Answer.

$(i)\ R_1 = \{(1,\ 5),\ (2,\ 7),\ (3,\ 8)\}$ $\qquad (ii)\ R_2 = \{(6,\ 2),\ (3,\ 7),\ (4,\ 7)\}$

Sol: Here $A \times B =$
$\{(1,\ 5), (1,\ 6),\ (1,\ 7),\ (1,\ 8), (2,\ 5),\ (2,6),\ (2,\ 7),\ (2,\ 8),\ (3,\ 5),\ (3,\ 6),\ (3,\ 7),\ (3,\ 8),$

$$(4,\ 5),\ (4,\ 6),\ (4,\ 7),\ (4,\ 8)\}$$

(i) Since, $R_1 \subseteq A \times B$, therefore R_1 is a relation from A to B.

(ii) Since, $(6,\ 2) \in R_2$ but $(6,\ 2) \notin A \times B$, therefore R_2 is not a relation from A to B.

Number of Relations

Let A and B be any two finite sets containing m and n elements respectively. Since each relation from A to B is a subset of $A \times B$, so the number of relations defined from A to B is 2^{mn}.

Example: If $A = \{a,\ b\}$ and $B = \{2,\ 3,\ 4\}$, then find the number of relations from A to B.

Sol: We have $n(A) = 2$ and $n(B) = 3$.

So, the number of relations from A to B is $2^{2\times 3} = 2^6 = 64$.

Types of Relations

1. **Empty / Void Relation:** Since $\emptyset \subseteq A \times B$, so $\emptyset$ is a relation from A to B, which is called an empty relation. It is the smallest relation in $A \times B$.
2. **Universal Relation:** since $A \times B \subseteq A \times B$, so $A \times B$ is a relation from A to B, which is called the universal relation. It is the largest relation in $A \times B$.
3. **Identity Relation:** Let $A = \{x.\ y,\ z\}$. So $I_A = \{(x,\ x),\ (y,\ y),\ (z,\ z)\}$ is called the identity relation on A.

Domain and Range of a Relation:

Let R be a relation from a set A to a set B. Then the set of all first components of ordered pairs belonging to R is called the domain of R, while the set of all second components of the ordered pairs in R is called the range of R.

Thus, $dom - R = \{x: (x,\ y) \in R\}$ and $rng\ - R = \{\ y: (x,\ y) \in R\}$

Example: If $A = \{1,\ 3,\ 5,\ 7\}$, $B = \{2,\ 4,\ 6,\ 8,\ 10\}$ and let $R = \{(1,\ 8),\ (3,\ 6),\ (5,\ 2),\ (1,\ 4)\}$ be a relation from A to B. Then $dom\ - R = \{1,\ 3,\ 5\}$ and $rng\ - R = \{8,\ 6,\ 2,\ 4\}$

The inverse of a Relation:

Let $A,\ B$ be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(y,\ x): (x,\ y) \in R\}$.

From the above example, we get $R^{-1} = \{(8,\ 1),\ (6,\ 3),\ (2,\ 5),\ (4,\ 1)\}$

So, $dom - R^{-1} = \{8,\ 6,\ 2,\ 4\}$ and $rng - R^{-1} = \{1,\ 3,\ 5\}$

Remember:

- If $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$.
- $dom - R = rng - R^{-1}$ and $rng - R = dom - R^{-1}$.
- $domain \subseteq A$ and $range \subseteq B$.
- The set B is called the co-domain of the relation R.
- $range \subseteq co - domain$

Representation of a Relation

A relation from a set A to a set B can be represented in any one of the following forms:

1. **Roster Form:** In this form, a relation is represented by the set of all ordered pairs belonging to R.

Example: A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R.

Solution: Here $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$

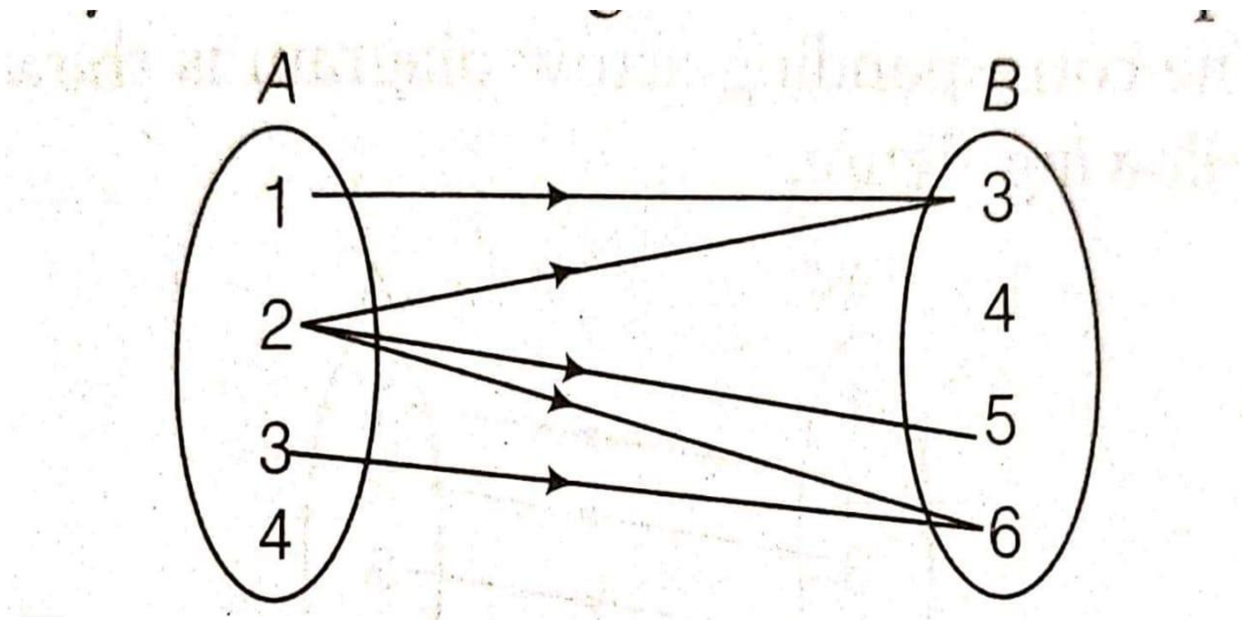
So, $dom - R = \{2, 3, 5\}$ and $rng - R = \{3, 6, 10\}$

2. **Set – builder Form:** In this form, we represent the relation R from set A to set B as $R = \{(x, y): x \in A, y \in B$ and the rule which relates the elements of A and $B\}$

For example, if R is a relation from set $A = \{1, 2, 4, 5\}$ and $B = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\right\}$ such that $R = \left\{(1.1), \left(2, \frac{1}{2}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$, then in set-builder form, R can be written as $R = \{(x, y): x \in A, y \in B$, and $y = \frac{1}{x}\}$

3. **Arrow diagram:** To represent a relation by an arrow diagram, we draw arrows from the first component to the second component of all ordered pairs belonging to relation R.

For example, a relation $R = \{(1,3), (2, 5), (3, 6), (2, 6), (2, 3)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{3, 4, 5, 6\}$ can be represented by the following arrow diagram.



Example: Let R be the relation on the set N of natural numbers defined by

$R = \{(a, b): a + 3b = 12, a \in N, b \in N\}$. Find

(i) R (ii) Domain of R (iii) Range of R (iv) R^{-1}

Sol: We have $a + 3b = 12 \Rightarrow a = 12 - 3b$

Putting, $b = 1, 2, 3$ respectively in the above relation, we get $a = 9, 6, 3$

So $(i) R = \{(9, 1), (6, 2), (3, 3)\}$ $(ii) dom - R = \{9, 6, 3\}$

(iii) $rng - R = \{1, 2, 3\}$ (iv) $R^{-1} = \{(1, 9), (2, 6), (3, 3)\}$

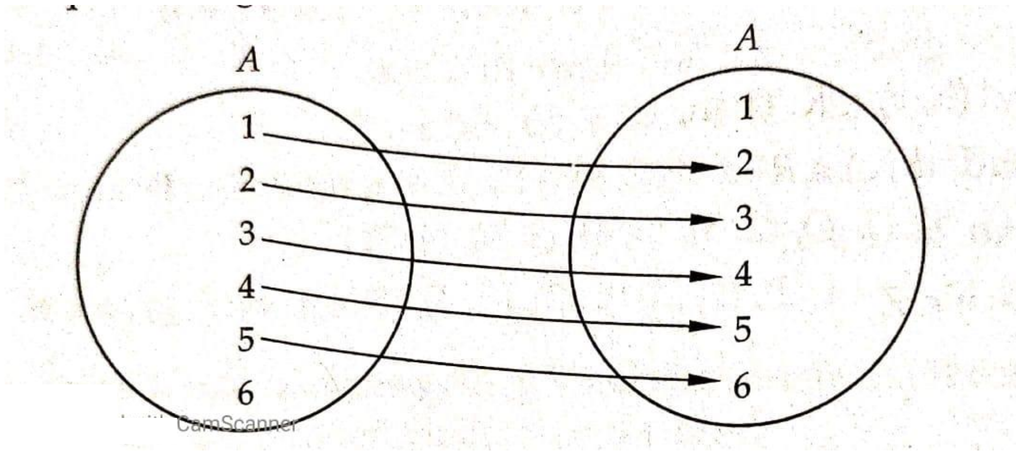
Example: Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R on set A by $R = \{(x, y): y = x + 1\}$

(i) Depict this relation using an arrow diagram

(ii) Write down the domain, co-domain, and range of R.

Sol: Putting $x = 1, 2, 3, 4, 5, 6$ respectively in $y = x + 1$, we get $y = 2, 3, 4, 5, 6, 7$ respectively.

But $7 \notin A$. Hence $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$



Example: The following figure shows a relation R between the sets P and Q. Write the relation R in (i) Roster form (ii) Set builder form. What are its domain and range?

Sol: (i) It is evident from the figure that

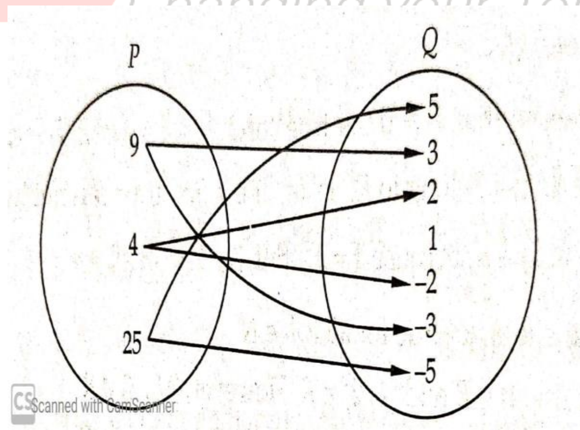
$R = \{(9,3), (9,-3), (4, 2), (4, -2), (25, 5), (25, -5)\}$

(ii) The relation R in set-builder form is

$R = \{(x, y): y^2 = x, x \in P, y \in Q\}$

The domain and range of R are $\{9, 4, 25\}$ and

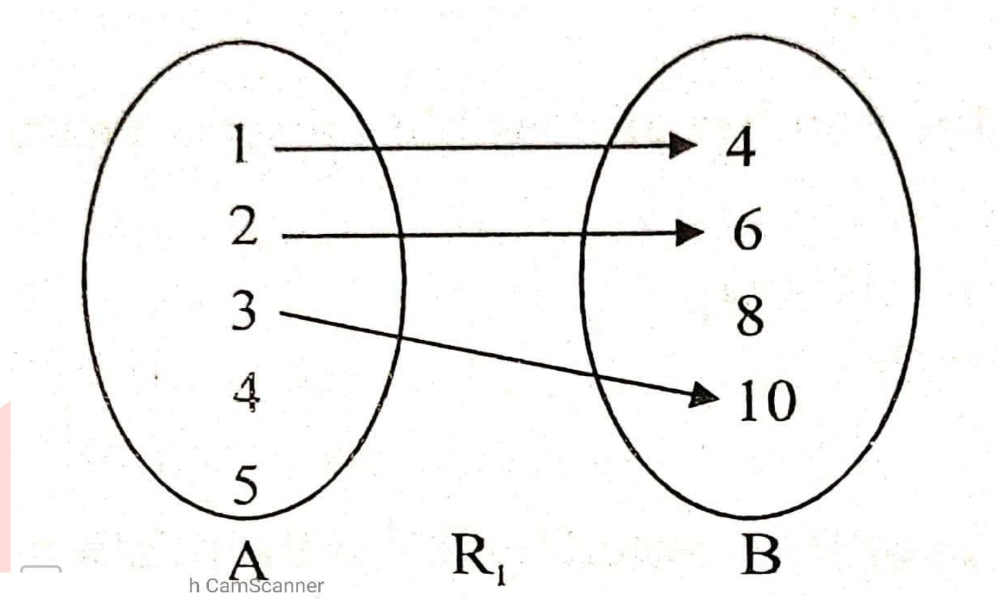
$\{-5, -3, -2, 2, 3, 5\}$ respectively.



Different Kinds of Relations

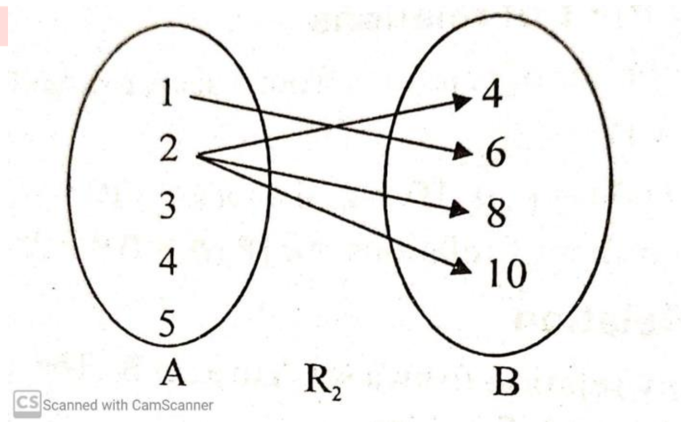
1. **One – One Relation**

A relation $R \subseteq A \times B$ is called one – on if $(x_1,\ y_1) \in R,\ (x_2,\ y_2) \in R$ then $y_1 = y_2 \Rightarrow x_1 = x_2$.



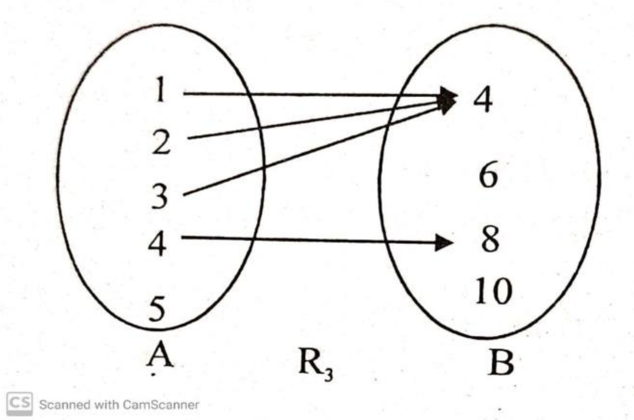
2. **One – Many Relation**

A relation $R \subseteq A \times B$ is called one – many if $(x_1,\ y_1) \in R,\ (x_1, y_2) \in R$ for some $x_1 \in A$ and $y_1, y_2 \in B$ where $y_1 \neq y_2$.



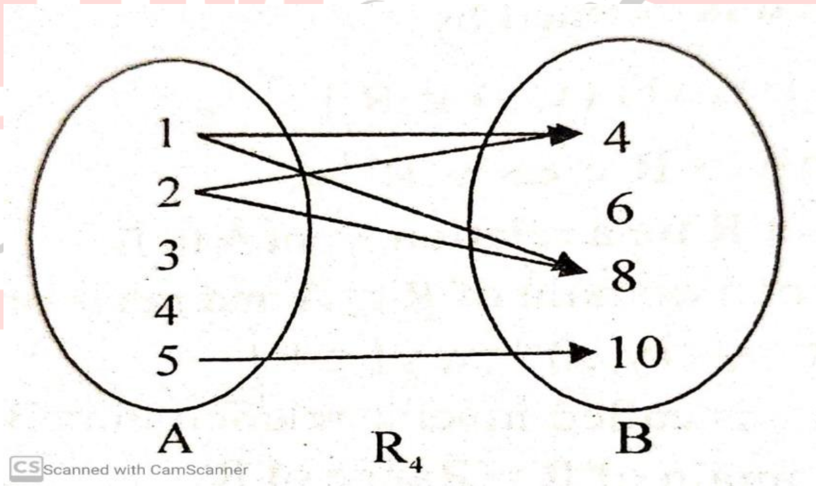
3. **Many – One Relation**

A relation $R \subseteq A \times B$ is called many – one if $(x_1, y_1) \in R$, $(x_2, y_1) \in R$ for some $y_1 \in B$ and $x_1, x_2 \in A$ where $x_1 \neq x_2$.



4. **Many – Many Relation**

A relation $R \subseteq A \times B$ is called many – many if it is both one –many and many – one.



Example: Determine the domain and range of the following relations.

$(i)\ R = \{(4x + 3,\ 1 - x) : x \leq 4,\ x \in N\}$

Sol: Here $R = \{(7,\ 0),\ (11,\ -1),\ (15,\ -2),\ (19,\ -3)\}$

$dom - R = \{7,\ 11,\ 15,\ 19\},\ rng\ - R = \{0,\ -1,\ -2,\ -3\}$

$(ii)\ R = \left\{\left(x + 4,\ \frac{2 + x}{2 - x}\right) : 4 \leq x \leq 6,\ x \in N\right\}$

Sol: Here $R = \{(8, -3), \left(9, -\frac{7}{3}\right), (10, -2)\}$

So, $dom - R = \{8, 9, 10\}$, $\text{rng} - \text{R} = \left\{-3, -\frac{7}{3}, -2\right\}$

Example: Let A and B be the sets of real numbers. Define a relation R from A to B by

$R = \{(x, y) \in A \times B : x^2 + y^2 = 1$ and $|x - y| = 1\}$. Find the domain and range of R.

Sol: Here $R = \{(1,0), (-1, 0), (0, 1), (0, -1)\}$

So, $dom - R = \{-1, 0, 1\}$, $rng - R = \{-1, 0, 1\}$.

Example: Let R be the relation on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n. Prove that

(i) $(x\, x) \in R \,\forall\, x \in Z$

(ii) $(x, y) \in R \Rightarrow (y, x) \in R \,\forall\, x, y \in Z$

(iii) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R \,\forall\, x, y, z \in Z$

Sol: (i) For any $x \in Z$, we have $x - x = 0 = 0 \times n$

$\Rightarrow x - x$ is divisible by n.

$\Rightarrow (x, x) \in R$. Thus $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then $(x, y) \in R$

$\Rightarrow x - y$ is divisible by $n \Rightarrow x - y = kn$ for some $k \in Z$

$\Rightarrow y - x = (-k)n \Rightarrow y - x$ is divisible by $n \Rightarrow (y, x) \in R$

Thus, $(x,y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x,y) \in R$ and $(y,z) \in R$. Then,

$(x,y) \in R \Rightarrow x - y$ is divisible by $n \Rightarrow x - y = an$ for some $a \in Z$

$(y,z) \in R \Rightarrow y - z$ is divisible by $n \Rightarrow y - z = bn$ for some $b \in Z$

$\therefore$ $(x,y) \in R$ and $(y,z) \in R \Rightarrow x - y = an$ and $y - z = bn$

$\Rightarrow x - y + y - z = an + bn \Rightarrow x - z = (a + b)n$

$\Rightarrow x - z$ is divisible by $n. \Rightarrow (x,z) \in R$

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

Functions

The word 'Function' is derived from a Latin word meaning operation and the words correspondence, mapping, and map are synonymous to it.

Function as a special kind of Relation

Let A and B be two non-empty sets. A relation f from A to B $i, e.$, a subset of $A \times B$ is called a function from A to B if

(i) for each $x \in A$ there exist y $\in B$ such that $(x, y) \in f$ $i.e.$ $dom - f = A$.

$(ii)(x,\ y) \in f$ and $(x,\ z) \in f \Rightarrow$ y $= z$.

In other words, a function f from a set A to a set B associates each element of set A to a unique element of B.

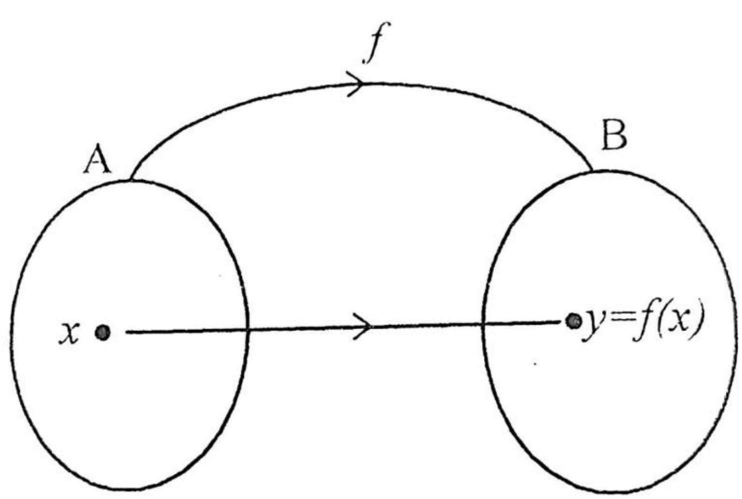
Thus a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first component.

The function f from A to B is denoted by $f : A \rightarrow B$.

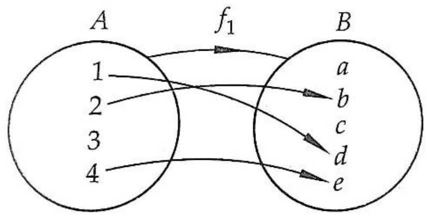
If $(x,\ y) \in f$, then we write $y = f(x)$, where y is called the image of x under f and x is called the pre-image of y. Also, y is called the value of f at x.

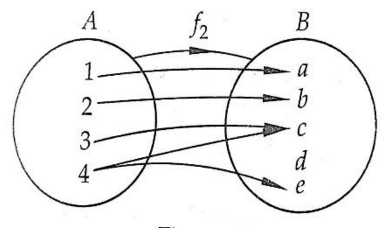
The function $f: A \rightarrow B$ can also be written as $f = \{(x,\ f(x): x \in A\}$.

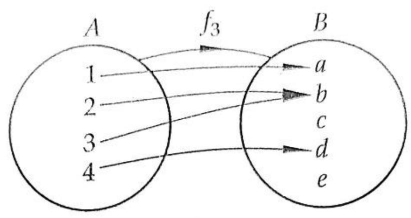
If $y = f(x)$, then x is called an independent variable and y is a dependent variable.

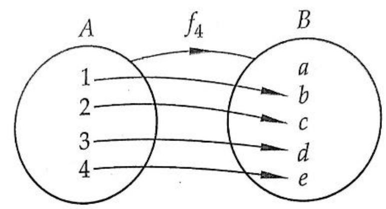


Let $A = \{1,\ 2,\ 3,\ 4\}$ and $B = \{a,\ b,\ c,\ d,\ e\}$ be two sets and let $f_1,\ f_2,\ f_3$ and f_4 be rules associating elements of A to elements of B as shown in the following figures:









We get f_1 is not a function from set A to set B, because there is an element $3 \in A$ which is not associated with any element of B.

Also, f_2 is not a function from A to B because an element $4 \in A$ is associated with two elements c and e in B. But f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated with a unique element in B.

Description of a Function:

Let $f: A \rightarrow B$ be a function such that the set A consists of a finite number of elements. Then, $f(x)$ be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1,\ 1,\ 2,\ 3\}$ and B is the set of real numbers, then a function $f: A \rightarrow B$ can be described as $f(-1) = 3,\ f(1) = 0,\ f(2) = \frac{3}{2}$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases, functions are generally described by a formula. For example $f: Z \rightarrow Z$ given by $f(x) = x^2 + 1$.

Number of Functions

If $n(A) = p$ and $n(B) = q$, then the total number of functions defined from A to B is q^p.

Example: If $x,\ y \in \{1,\ 2,\ 3,\ 4\}$, then which of the following are functions in the given set?

$a)\ f_1 = \{(x,\ y): y = x + 1\}$

Sol: We have $f_1 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\}$.

We observe that element 4 of the given set has no image. So f_1 is not a function from the given set to itself.

$b)\ f_2 = \{(x,\ y): x + y = 5\}$

Sol: We have $f_2 = \{(1,\ 4),\ (2,\ 3),\ (3,\ 2),\ (4,\ 1)\}$

We observe that each element of the given set has appeared as first components in one and only one ordered pair of f_2. So, f_2 is a function in the given set.

Domain, Co-domain, and Range of a Function:

Let $f: A \to B$. Then the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all $f - images$ of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

We have $f(A) \subseteq B\ i.e.\ \ range \subseteq co - domain.$

Example: Let N be the set of natural numbers and the relation R be defined on N such that $R = \{(x,\ y): y = 2x\ ,\ x,\ y \in N\}$. What are the domain, co-domain, and range of f ? Is this relation a function?

Sol: The domain of R is the set of natural numbers N. The codomain is also N. The range is the set of even natural numbers. Since every natural number n has one and only one image, this relation is a function.

Example: Let $A = \{-2,\ -1,\ 0,\ 1,\ 2\}$ and $f: A \to Z$ be given by $f(x) = x^2 - 2x - 3$. Find

(i) the range of f

(ii) pre-images of 6, -3, and 5.

Sol: (i) We have $f(x) = x^2 - 2x - 3$.

Now $f(-2) = (-2)^2 - 2(-2) - 3 = 5,\ f(-1) = (-1)^2 - 2(-1) - 3 = 0$

$f(0) = 0 - 2 \times 0 - 3 = -3, \quad f(1) = 1^2 - 2 \times 1 - 3 = -4,\ f(2) = 2^2 - 2 \times 2 - 3 = -3$

So, $rng\ - f = \{f(-2), f(-1),\ f(0),\ f(1),\ f(2)\} = \{-4,\ -3,\ 0,\ 5\}$

(ii) Let x be a pre-image of 6. Then $f(x) = 6$

$\Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = \frac{2\pm\sqrt{4+36}}{2} = 1 \pm \sqrt{10} \notin A.$

So there is no pre-image of 6.

Again let x be a pre-image of -3. Then $f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0$

$\Rightarrow x = 0,\ 2$. Since $0,\ 2 \in A$, so 0 and 2 are pre-images of -3.

Let x be a pre-image of 5. Then $f(x) = 5 \quad \Rightarrow x^2 - 2x - 3 = 5 \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x - 4)(x + 2) = 0 \Rightarrow x = 4,\ -2$

Since $-2 \in A$ but $4 \notin A$, so -2 is the pre-image of 5.

Equal Functions:

Two functions f and g are said to be equal if and only if

$(i)\ dom - f = dom - g$

$(ii)\ co - domain\ of\ f = co - domain\ of\ g$ and

$(iii) f(x) = g(x)$ for every x belongs to their common domain.

If two functions f and g are equal, then we write $f = g$.

Example: Find the domain for which the function $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

Sol: We have $f(x) = g(x)$

$\Rightarrow 2x^2 - 1 = 1 - 3x$

$\Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (x + 2)(2x - 1) = 0 \Rightarrow x = -2, \frac{1}{2}$

Thus, $f(x)$ and $g(x)$ are equal on the set $\left\{-2, \frac{1}{2}\right\}$

Example: Is $g = \{(1,\ 1),\ (2,\ 3),\ (3,\ 5),\ (4,\ 7)\}$ a function? If this is described by the formula, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β ?

Sol: Since no two ordered pairs in g have the same first component, so g is a function such that $g(1) = 1,\ g(2) = 3,\ g(3) = 5,\ g(4) = 7$.

It is given that $g(x) = \alpha x + \beta$

So $g(1) = 1$ and $g(2) = 3$

$\Rightarrow \alpha + \beta = 1$ and $2\alpha + \beta = 3$

Solving we get $\alpha = 2,\ \beta = -1$

REAL FUNCTIONS

A function $f: A \to B$ is called a real-valued function if B is a subset of R (set of real numbers)

If A and B both are subsets of R, then f is called a real function.

Example: $f: R \to R$ given by $f(x) = x^2 + 3x + 7$ is a real function.

Example: If for non –zero, $a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then find $f(x)$.

Sol: We have $a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ ..... (i)

Replacing x by $\frac{1}{x}$ we get $a\,f\left(\frac{1}{x}\right) + b\,f(x) = x - 5$ ..... (ii)

Multiplying (i) by a and (ii) by b and then subtracting, we get

$$a^2\,f(x) + ab\,f\left(\frac{1}{x}\right) - ab\,f\left(\frac{1}{x}\right) - b^2\,f(x) = a\left(\frac{1}{x} - 5\right) - b(x-5)$$

$$\Rightarrow \quad (a^2 - b^2)f(x) = \frac{a}{x} - 5a - bx + 5b$$

$$\Rightarrow \quad f(x) = \frac{\frac{a}{x} - 5a - bx + 5b}{(a^2 - b^2)} = \frac{1}{(a^2-b^2)}\left(\frac{a}{x} - bx\right) - \frac{5}{a+b}$$

Example: If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3\,f\left(\frac{1}{x}\right)$.

Sol: We have $f(x) = x + \frac{1}{x}$ So $f(x^3) = x^3 + \frac{1}{x^3}$ and $f\left(\frac{1}{x}\right) = \frac{1}{x} + x$

Now $[f(x)]^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = f(x^3) + 3\,f\left(\frac{1}{x}\right)$

Example: If f is a real function defined by $f(x) = \frac{x-1}{x+1}$, then prove that $f(2x) = \frac{3\,f(x)+1}{f(x)+3}$

Sol: We have, $f(x) = \frac{x-1}{x+1} \Rightarrow \frac{f(x)+1}{f(x)-1} = \frac{x-1+x+1}{x-1-x-1}$ [Applying componendo and dividendo]

$\Rightarrow \frac{f(x)+1}{f(x)-1} = -x \Rightarrow x = \frac{1+f(x)}{1-f(x)}$

Since $f(x) = \frac{x-1}{x+1}$ so $f(2x) = \frac{2x-1}{2x+1}$

$\Rightarrow f(2x) = \frac{2\left\{\frac{1+f(x)}{1-f(x)}\right\}-1}{2\left\{\frac{1+f(x)}{1-f(x)}\right\}+1} = \frac{2+2f(x)-1+f(x)}{2+2f(x)+1-f(x)}$

$\Rightarrow f(2x) = \frac{3\,f(x)+1}{f(x)+3}$.

Domain and Range of Real Functions

The domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ assumes real values only. In other words, the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful.

The range of a real function $y = f(x)$ is the set of all real values taken by $f(x)$ at points in its domain. To find the range of $f(x)$, we may use the following algorithm.

Step-I: Put $y = f(x)$

Step –II: Solve the equation $y = f(x)$ for x. Let $x = g(y)$

Step-III: Find the values of y for which x is real. The set of values of y is the range of f.

Example: Find the domain of the following functions:

$(i)\ f(x) = \dfrac{1}{x+2}$

Sol: Here $f(x)$ assumes real values for all real values of x except for the values of x satisfying $x + 2 = 0\ i.e.\ x = -2$.

Hence $dom - f = R - \{-2\}$

$(ii) f(x) = x^2 + 2x + 7$

Sol: Here $f(x)$ is real or can be defined for all real values of x.

So $dom - f = R$.

$(iii)\ f(x) = \dfrac{x^2+3x+5}{x^2-3x+2}$

Sol: Here $f(x)$ is a rational function of x. We get $f(x)$ assumes real values for all x except for all those values of x for which $x^2 - 3x + 2 = 0\ i.e.\ (x-1)(x-2) = 0\ i.e.\ x = 1,\ 2$.

Hence $dom - f = R - \{1,\ 2\}$.

$(iv) f(x) = \sqrt{x-2}$

Sol: Here $f(x)$ assumes real values for all x satisfying $x - 2 \geq 0 \Rightarrow x \geq 2$

$\Rightarrow x \in [2, \infty)$

Hence, $dom - f = [2, \infty)$

$(v) f(x) = \dfrac{1}{\sqrt{1-x}}$

Sol: Here $f(x)$ assumes real values for all x satisfying $1 - x > 0 \Rightarrow x < 1 \Rightarrow x \in (-\infty, 1]$

Hence, $dom - f = (-\infty, 1]$

$(vi)\ f(x) = \sqrt{4 - x^2}$

Sol: Here $f(x)$ assumes real values for all x satisfying $4 - x^2 \geq 0 \Rightarrow x^2 \leq 4$

$\Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$

Hence, $dom - f = [-2, 2]$

$(vii)\ f(x) = \dfrac{1}{\sqrt{x^2 - 9}}$

Sol: Here $f(x)$ assumes real values for all x satisfying $x^2 - 9 > 0 \Rightarrow x^2 > 9$

$\Rightarrow x < -3$ or $x > 3 \Rightarrow x \in (-\infty, -3)$ or $\in (3, \infty) \Rightarrow x \in (-\infty, -3) \cup (3, \infty)$

Hence $dom - f = (-\infty, -3) \cup (3, \infty)$

Example: Find the domain and range of the following functions:

$(i)\ f(x) = \dfrac{x-2}{3-x}$

Sol: Here $f(x)$ is defined for all x satisfying $3 - x \neq 0$ $i.e.$ $x \neq 3$.

Hence $dom - f = R - \{3\}$

Let $y = f(x) \Rightarrow y = \frac{x-2}{3-x} \Rightarrow 3y - xy = x - 2 \Rightarrow x(y+1) = 3y + 2$

$\Rightarrow x = \frac{3y+2}{y+1}$

Here x assumes real values for all y except $y + 1 = 0$ $i.e.$ $y = -1$.

Hence, $rng - f = R - \{-1\}$.

$(ii)\ f(x) = \dfrac{1}{\sqrt{x-5}}$

Sol: We have $f(x) = \frac{1}{\sqrt{x-5}}$

Here $f(x)$ takes real values for all x satisfying $x - 5 > 0 \Rightarrow x > 5 \Rightarrow x \in (5,\ \infty)$

So, $dom - f = (5,\ \infty)$

Again for any $x > 5$, we have $x - 5 > 0 \Rightarrow \frac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0 \Rightarrow y > 0$

Hence $rng - f = (0,\ \infty)$

$(iii) f(x) = \dfrac{x}{1+x^2}$

Sol: Here $f(x)$ takes real values for all $x \in R$. Hence, $dom - f = R$.

Let $y = f(x)$. Then

$y = f(x) \Rightarrow y = \frac{x}{1+x^2} \Rightarrow x^2y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$

So, x will assume real values, if $1 - 4y^2 \geq 0$ and $y \neq 0$

$\Rightarrow 4y^2 - 1 \leq 0$ and $y \neq 0$

$\Rightarrow y^2 - \frac{1}{4} \leq 0$ and $y \neq 0$

$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right] - \{\,0\,\}$

Also, $y = 0$ for $x = 0$

Hence $rng - f = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$(iv)\ f(x) = \dfrac{x^2-9}{x-3}$

Sol: We have $f(x) = \frac{x^2-9}{x-3}$

Here $f(x)$ is not defined for $x - 3 = 0\ i.e.\ x = 3$.

Therefore, $dom - f = R - \{3\}$

Let $f(x) = y$

Then $y = \frac{x^2-9}{x-3} \Rightarrow y = x + 3$

We get y takes all real values except 6 when x takes values in the set $R - \{3\}$.

Therefore $rng - f = R - \{6\}$

$(v)\ f(x) = \dfrac{3}{2 - x^2}$

Sol: For $f(x)$ to be real, we have $2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}$

So, $dom - f = R - \{-\sqrt{2},\ \sqrt{2}\ \}$

Let $f(x) = y$. Then $y = f(x) \Rightarrow y = \frac{3}{2-x^2} \Rightarrow 2y - x^2y = 3$

$\Rightarrow x^2y = 2y - 3 \Rightarrow x = \pm\sqrt{\frac{2y-3}{y}}$

We get x will take real values other than $-\sqrt{2}$ and $\sqrt{2}$, if $\frac{2y-3}{y} > 0$

$\Rightarrow y \in (-\infty,\ 0) \cup \left[\frac{3}{2},\ \infty\right)$

Hence $rng - f = (-\infty,\ 0) \cup \left[\frac{3}{2},\infty\right)$

$(vi)\ f(x) = \dfrac{1}{1 - x^2}$

Sol: We have$f(x) = \frac{1}{1-x^2}$

Here $f(x)$ is defined for all $x \in R$ except for which $1 - x^2 \neq 0 \Rightarrow x = \pm 1$.

So, $dom - f = R - \{-1,\ 1\}$

Let $f(x) = y$. Then

$f(x) = y \Rightarrow \frac{1}{1-x^2} = y \Rightarrow 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y} = \frac{y-1}{y} \quad \Rightarrow x = \pm\sqrt{\frac{y-1}{y}}$

Here x will take real values, if $\frac{y-1}{y} \geq 0 \Rightarrow y < 0$ or $y \geq 1$

$\Rightarrow y \in (-\infty,\ 0) \cup [1,\ \infty)$

Hence, $rng - f = (-\infty,\ 0) \cup [1, \infty)$

$(vii) f(x) = \dfrac{x^2}{1 + x^2}$

Sol: We have, $f(x) = \frac{x^2}{1+x^2}$

Clearly, $f(x)$ is defined for all $x \in R$ as $x^2 + 1 \neq 0$ for any $x \in R$.

So, $dom - f = R$.

Let $f(x) = y$. Then $y = \frac{x^2}{1+x^2} \Rightarrow y + x^2y = x^2$

$\Rightarrow x^2(1 - y) = y \Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$

Here x will take real values, if $\frac{y}{1-y} \geq 0 \Rightarrow 0 \leq y < 1$

$\Rightarrow\ y\ \in [0,\ 1)$

Hence, $rng - f = [0,\ 1)$

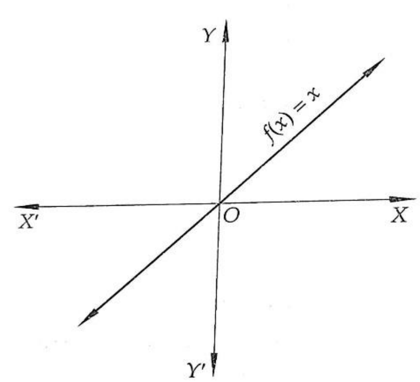
Different Real Functions and their Domain, Range with Graphs

1. Identity Function:

The function $f: R \to R$ defined by $f(x) = x$ for all $x \in R$ is called the identity function.

Here $dom - f = rng - f = R$.

The graph of the identity function is a straight line passing through the origin and equally inclined to the coordinate axes.

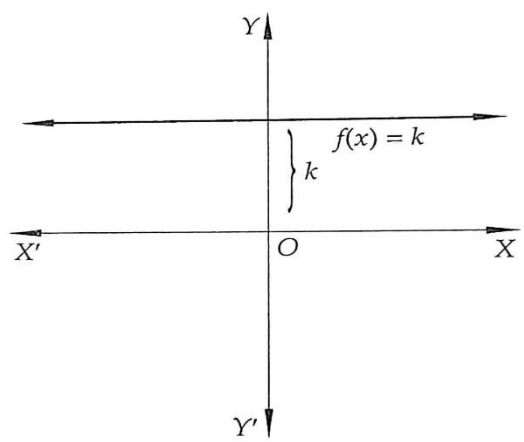


2. Constant Function:

If k is a fixed real number, then a function $f(x) = k$ for all $x \in R$ is called a constant function.

Here $dom - f = R$ and $rng - f = \{ k \}$

The graph of a constant function is a straight line parallel to $x - axis$, which is above or below $x - axis$ according to k is positive or negative. If $k = 0$, then the straight line is coincident to $x - axis$.



3. Polynomial Function:

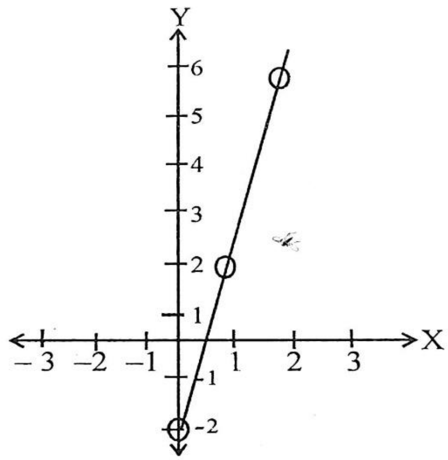
A function $f: R \rightarrow R$ defined by $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $n \in N$ and $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$ is called a polynomial function in x of degree n.

If $n = 1$, $f(x) = a_o + a_1x$, $a_0 \neq 0$ is called a linear function.

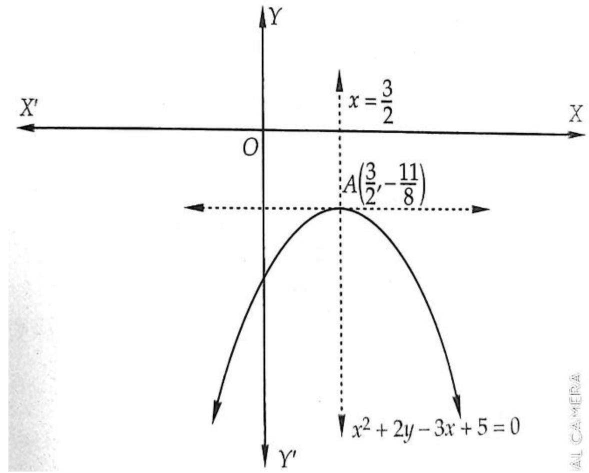
If $n = 2$, $f(x) = a_0 + a_1x + a_2x^2$, $a_2 \neq 0$ is called a quadratic function.

The functions defined by $f(x) = 2x^4 - x^2 + 5$, $g(x) = x^5 + \sqrt{3}x^2 - 3$ are examples of polynomial functions, whereas the function defined by $h(x) = x^{\frac{5}{2}} + 3x - 2$ is not a polynomial function.

Consider $f(x) = 4x - 2$



$f(x) = x^2 + 2y - 3x + 5$

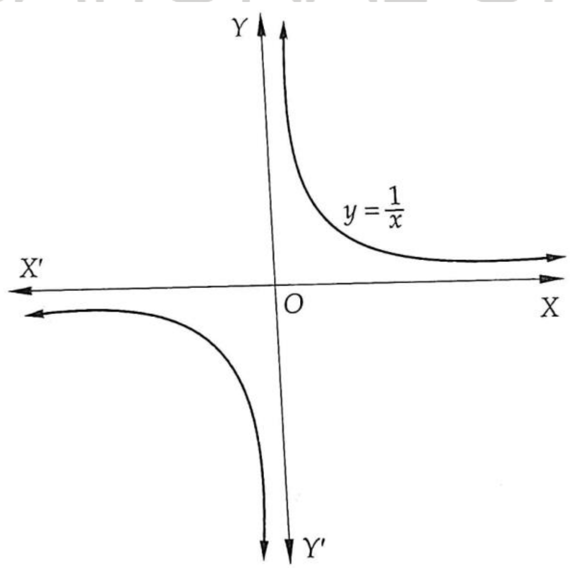


4. Rational Functions:

A function which can be expressed as the quotient of two polynomial functions is called rational function $i.e., r(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain and $g(x) \neq 0$, is a rational function.

For example, $f(x) = \frac{1}{x}(x \neq 0)$, $\frac{x^3+2x+3}{x^2+x+1}$, $\frac{2x+1}{x^2+4}$ are rational functions.

Graph of $f(x) = \frac{1}{x}$ is shown in the figure.

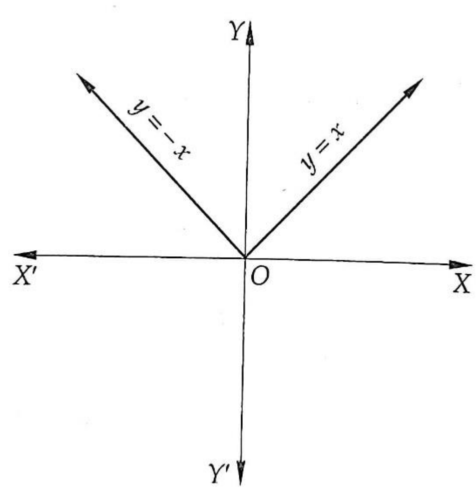


5. Modulus Function:

The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, & when\ x \geq 0 \\ -x, & when\ x < 0 \end{cases}$ is called the modulus function or absolute value function.

Here $dom - f = R$ and $rng - f = R^+ = \{x \in R : x \geq 0\}$.

The graph consists of two half-lines, one in the first quadrant bisecting the axes and the other in the second quadrant, bisecting the axes where the origin is included in the graph.



Properties of Modulus Function:

(i) For any $\in R$, $\sqrt{x^2} = |x|$.

(ii) If a is a positive real number then

$x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$ and $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$.

(iii) $|x \pm y| \leq |x| + |y|$

(iv) $|x \pm y| \geq \left||x| - |y|\right|$

(v) $|x|^2 = x^2$.

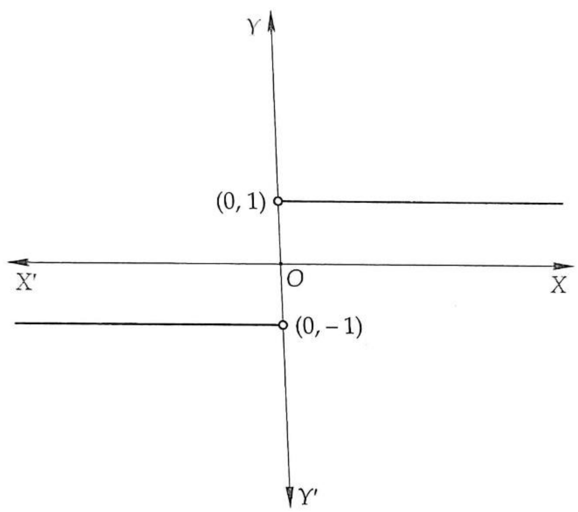
6. Signum Function:

The function $f: R \to R$ defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$ is called the signum function.

It is also denoted by $sgn\ (x)$.

Here $dom - f = R$ and $rng\ - f = \{-1,\ 0,\ 1\}$.

The graph of the signum function is as shown below.



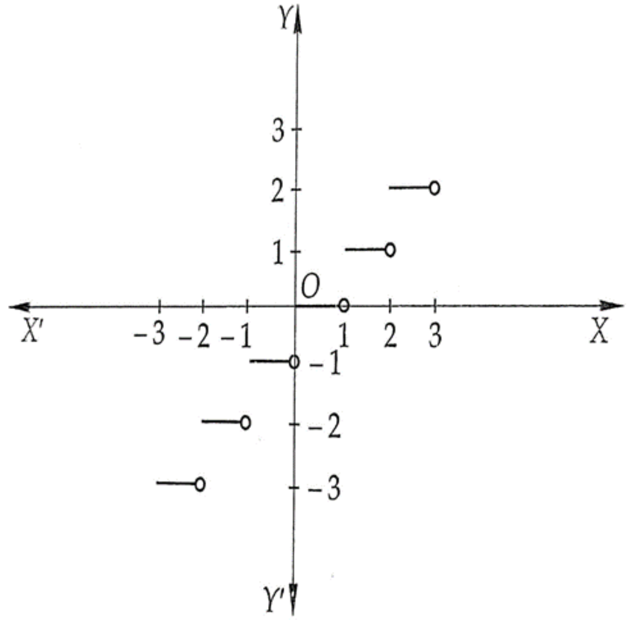
7. Greatest Integer Function:

A function $f: R \to R$ be defined by $f(x) = [x]$ is called the greatest integer function,

where $[x] = n\ ,\ n \leq x < n + 1,\ n \in Z$.

Here $dom - f = R$ and $rng - f = Z$

The graph of the greatest integer function is shown in the figure.



Properties:

(i) x and $[x]$ have the same sign for all x.

(ii) The graph lies within the first and third quadrant.

(iii) $[x] \leq x < [x] + 1,\ x \in R$

(iv) $n \leq x < n + 1 \Leftrightarrow [x] = n,\ n \in Z$, and $x \in R$.

(v) $x - 1 < [x] \leq x\ ,\ x \in R$.

$(vi)[-n] = -[n],\ n \in Z$

(vii) $[-x] = -[x] - 1,\ x \in R$

$(viii)[x + n] = [x] + n,\ n \in Z,\ x \in R$

(ix) $\big[[x]\big] = \ [x]$

(x) $[x] + [-x] = \begin{cases} 0, x \in Z \\ -1, x \notin Z \end{cases}$

Example: If f be a real-valued function defined by $f(x) = x - [x]$, then compute

(i) $f(2.5) = 2.5 - \ [2.5] = 2.5 - 2 = 0.5$

$(ii) f(-3.7) = -3.7\ -\ [-3.7] = -3.7\ - (-4) = -3.7 + 4 = 0.3$

Example: State true or false. If $[x]^2 - 5[x] + 6 = 0$, then x lies in the interval $[2,\ 3]$.

Sol: True

Example: Draw the graph of the following functions. Also, determine their domain and range.

$(i)\ f(x) = 2 \quad (ii) f(x) = -2 \quad (iii) f(x) = |x - 3|$

Example: Let $f: R \to R$ defined by $f(x) = 1 - x^2$ for all $x \in R^+$ .Find its domain and range. Also, draw its graph.

Example: Draw the graph of $f(x) = \frac{x^3}{2}$ .Also, find its domain and range.

Sum, Difference, Product, and Quotient of Functions

Let $f: A \to R$ and $g: B \to R$ be two real functions. Let $D = A \cap B$. Then, we define addition, subtraction, multiplication, and quotient of two real functions as follows:

(i) Addition: $(f + g): D \to R$ be defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in D$.

(ii)Subtraction: $(f - g): D \to R$ be defined by $(f - g)(x) = f(x) - g(x)$ for all $x \in D$.

(iii) Scalar Multiplication : $(\alpha f): A \to R$ be defined by $(\alpha f)(x) = \alpha f(x)$ for all $x \in A$.

(iv) Multiplication: $(f.g): D \to R$ be defined by $(f.g)(x) = f(x).g(x)$ for all $x \in D$.

(v)Reciprocal : $\left(\frac{1}{f}\right): C \to R$ be defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$ for all $x \in C = A - \{x: f(x) = 0\}$

(vi) Quotient: $\left(\frac{f}{g}\right): D_1 \to R$ be defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in D_1 = D - \{x: g(x) = 0\}$

Example: Let $f,\ g: R \to R$ be defined by $f(x) = 2x + 5,\ g(x) = x + 3$, find $f \pm g,\ 2f,\ \frac{f}{g},\ \frac{1}{f}, \frac{1}{g}$.

Sol: $(f + g)(x) = f(x) + g(x) = 2x + 5 + x + 3 = 3x + 8$

$(f - g)(x) = f(x) - g(x) = 2x + 5 - x - 3 = x + 2$

$(2f)(x) = 2\,f(x) = 2(2x + 5) = 4x + 10$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x + 5}{x + 3},\ x \neq -3\ i.e.,\ x \in R - \{-3\}$$

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{2x + 5}\ \forall\, x \in R - \left\{-\frac{5}{2}\right\}$$

$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} = \frac{1}{x+3}\ \forall x \in R - \{-3\}$

Example: Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then, find each of the following functions:

$(i) f+g \quad (ii)\ fg \quad (iii)\ \frac{f}{g} \quad (iv)\ ff$

Sol: Here $f(x)$ is defined for all x satisfying $x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$

So, $dom - f = [-2, \infty) = A$(say)

Again, $g(x)$ is defined for all x satisfying $4 - x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow x \in [-2, 2]$

So, $dom - g = [-2, 2] = B$ (say)

Now $D = A \cap B = [-2, 2]$

$(i)\ (f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}\ , x \in D$

$(ii)\ (fg)(x) = f(x).g(x) = \sqrt{x+2} \times \sqrt{4-x^2} = (x+2)\sqrt{2-x},\ x \in D$

$(iii)\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+2}}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2-x}}\ , x \in D - \{-2, 2\} = (-2, 2)$

$(iv)\ (ff)(x) = f(x)f(x) = \{f(x)\}^2 = \left(\sqrt{x+2}\right)^2 = x+2\ , x \in [-2, \infty)$

Example: Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non – negative real numbers. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Sol: We have $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

$(f-g)(x) = f(x) - g(x) = \sqrt{x} + x$

$(fg)(x) = f(x).\ g(x) = \sqrt{x}\,.x = x^{\frac{3}{2}}$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}},\ x \neq 0.$

Concept of Exponential and Logarithmic Functions

Exponential Function:

A function $f: R \to R$ defined by $f(x) = a^x\ (a > 0,\ a \neq 1)$ is called the exponential function.

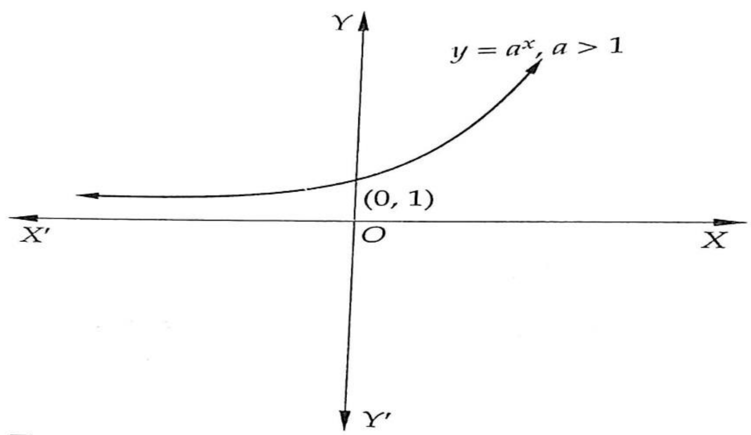
Here $dom - f = R$ and $rng - f = R_+ = (0, \infty)$.

2^x , 3^x, $\frac{1}{2^x}$, … etc. are examples of exponential functions.

As $a > 0$ and $a \neq 1$, so have the following cases.

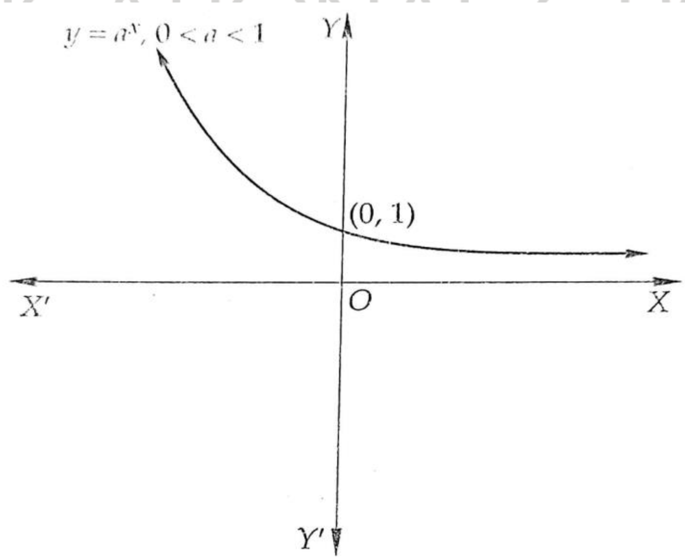
Case – I: When $a > 1$.

We get the values of $y = f(x) = a^x$ increase as the values of x increase.



Case II: When $0 < a < 1$.

In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.



Note: We have $2 < e < 3$. ($e = 2.718281$ …. , is an irrational number)

Therefore, the graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ for $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ for $0 < a < 1$.

Properties: Let $f(x) = a^x$, $(a > 0, a \neq 1)$

$(i)\ f(x+y) = a^{x+y} = a^x.a^y = f(x).\ f(y)$

$(ii) f(x-y) = a^{x-y} = \dfrac{a^x}{a^y} = \dfrac{f(x)}{f(y)}$

$(iii)\{f(x)\}^y = (a^x)^y = a^{xy} = f(xy)$

$(iv)\ f(x) = 1 \Leftrightarrow x = 0$

$(v) f(x)f(-x) = f(0) = 1$

Logarithmic Function:

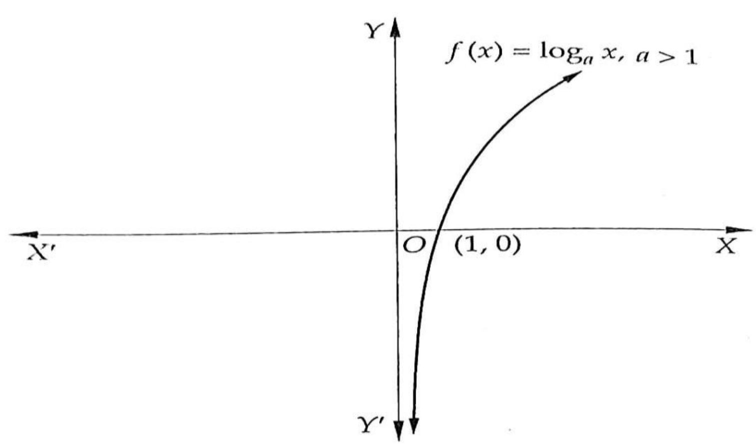
A function $f:(0,\infty) \to R$ defined by $f(x) = \log_a x$ $(a > 0, a \neq 1)$ is called a logarithmic function.

We have $\log_a x = y$ if and only if $a^y = x$.

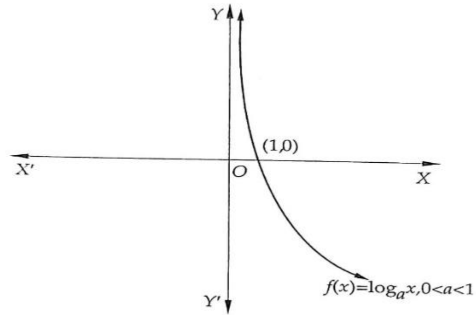
Here $dom - f = R_+$ and $rng - f = R$.

As $a > 0$ and $a \neq 1$, so we have the following cases.

Case I: When $a > 1$



Case II: When $0 < a < 1$



Properties:

(i) $\log_a 1 = 0$, where $a > 0,\ a \neq 1$

(ii) $\log_a a = 1$, where $a > 0,\ a \neq 1$

(iii) $\log_a(xy) = \log_a|x| + \log_a|y|$, where $a > 0,\ a \neq 1$ and $xy > 0$

(iv) $\log_a\left(\frac{x}{y}\right) = \log_a|x| - \log_a|y|$, where $a > 0,\ a \neq 1$ and $\frac{x}{y} > 0$

(v) $\log_a(x^n) = n\ \log_a|x|$, where $a > 0,\ a \neq 1$ and $x^n > 0$

(vi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$, where $a > 0,\ a \neq 1$ and $x > 0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $a > 0,\ a \neq 1,\ x > 0,\ y > 0$

$(viii)$ $\log_a x = \frac{1}{\log_x a}$, where $a > 0,\ a \neq 1$ and $x > 0,\ x \neq 1$.

Notes:

(i) Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.

(ii) If $a = e$, then $\log_e x$ (written as lnx) is the inverse of an exponential function to the base e.

$i.e.,\ lnx = y \Leftrightarrow x = e^y$.

Example: If $f(x) = 2^x$, then what is the range of f?

Sol: Here $rng - f = R_+$

Example: Fill in the blank. If $f(x) = \log_2 x$, then $f(x) < 0$ for x lies in the interval ______ .

Sol: $(0,\ 1)$.