Introduction:

In this chapter, we will deal with numbers following some fixed pattern. We will identify the pattern and classify the numbers as belonging to one group. Then we must understand how to sum given numbers.

Sequence:

A sequence is a succession of numbers or terms formed according to some rule.

For example, $3, 6, 9, \ldots$ is a sequence.

A sequence is a function whose domain is the set of natural numbers (N).

A sequence whose range is a subset of the set of real numbers (R) is called a real sequence.

The numbers are called terms of the sequence. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots., etc.$

a_1 is the term in the first position, a_2 is the term in the second position and so on.

In general, the term at nth position is denoted by a_n. It is called nth term of the sequence.

A sequence can be written as $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or (a_n) .

Representation of a Sequence:

There are several ways of representing a real sequence.

One way to represent a real sequence is to list its first few terms till the rule for writing down other terms becomes clear. For example, $1, 3, 5, 7, \ldots.$ Is a sequence whose nth term is $(2n - 1)$.

Another way to represent a real sequence is to give a rule of writing the nth term of the sequence. For example, the sequence $1, 3, 5, 7, \ldots$ can be written as $a_n = 2n - 1$.

Sometimes we represent a real sequence by using a recursive relation. For example, the Fibonacci sequence is given by

$a_1 = 1, a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}\ , n \geq 2$.

The terms of this sequence are $1, 1, 2, 3, 5, 8, \ldots$ .....

Finite and Infinite Sequences:

A sequence containing a finite number of terms is called a finite sequence. For example, $1, 3, 9, 27, 81$ is a finite sequence.

A sequence is called infinite, if it is not finite.

For example, $2, 5, 8, 11, \ldots$. Is an infinite sequence.

Example: Write the first three terms of the sequence whose nth terms are:

$(i)\ a_n = n(n+1)$

Solution: Putting $n = 1, 2, 3$, we get

$a_1 = 1.2 = 2\ ;\ a_2 = 2.3 = 6\ ;\ a_3 = 3.4 = 12$

$(ii)\ a_n = (-1)^{n-1}.2^n$

Solution: Putting $n = 1, 2, 3$, we get

$a_1 = (-1)^{1-1}.2 = 2\ ;\ a_2 = (-1)^{2-1}.2^2 = -4\ ;\ a_3 = (-1)^{3-1}.2^3 = 8$

Example: Find the 15th and 26th term of the sequence whose nth term is given by $a_n = \frac{n(n+2)}{n+4}$.

Solution: $a_{15} = \frac{15\,(15+2)}{15+4} = \frac{255}{19}$ and $a_{26} = \frac{26(26+2)}{26+4} = \frac{364}{15}$

Example: Write the first five terms of the sequence (a_n) defined by $a_1 = 2, a_n = a_{n-1} + 4$, for all $n > 1$.

Solution: We have $a_1 = 2$

$a_2 = a_1 + 4 = 2 + 4 = 6$

$a_3 = a_2 + 4 = 6 + 4 = 10$

$a_4 = a_3 + 4 = 10 + 4 = 14$

$a_5 = a_4 + 4 = 14 + 4 = 18$

Series:

If $\{a_n\}_{n=1}^{\infty}$ be a sequence, then the series is the sum of the terms of the sequence. Thus $a_1 + a_2 + a_3 + \cdots$ is the series corresponding to (a_n). Using the symbol $\sum(sigma)$, means summation, we can write $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

A series is finite or infinite according to as the number of terms in the corresponding sequence is finite or infinite.

Progressions:

The terms of a sequence don't need to always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.

Arithmetic Progression (A.P.):

A sequence is called an arithmetic progression if the difference between a term and the previous term is always the same.

$i.e.\ a_{n+1} - a_n = constant\ (= d)$ for all $n \in N$.

The constant difference d is called the common difference.

Example: Show that the sequence defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.

Solution: We have $a_n = 4n + 5$

Replacing n by $(n + 1)$, we get $a_{n+1} = 4(n + 1) + 5 = 4n + 9$

$\therefore\ a_{n+1} - a_n = (4n + 9) - (4n + 5) = 4$

Clearly, $a_{n+1} - a_n$ is independent of n and is equal to 4.

So, the given sequence is an A.P. with a common difference 4.

Point to Remember:

A sequence is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.

The general term of an A.P.:

Let $'a'$ be the first term and $'d'$ be the common difference of an A.P.

Then first term $= \ a_1 = a = a + (1 - 1)d$

Second term $= \ a_2 = a + d = a + (2 - 1)d$

Third term $= \ a_3 = a + 2d = a + (3 - 1)d$

Hence general term$=$ nth term $= \ a_n = a + (n - 1)d$

nth Term of an A.P. from the end:

Let $'a'$ be the first term and $'d'$ be the common difference of an A.P. having m terms.

Then nth term from the end $= (m-n+1)th$ term from the beginning

$= a_{m-n+1} = a + (m-n+1-1)d = a + (m-n)d$

Example: Show that the sequence 9, 12, 15, 18, … is an A.P. Find its 16th term and the general term.

Solution: $12-9 = 15-12 = 18-15 = 3$, so the given sequence is an A.P. with common difference $d = 3$ and first term $a = 9$.

Thus, the 16th term $a_{16} = a + 15d = 9 + 15 \times 3 = 54$.

The general term $= a_n = a + (n-1)d = 9 + (n-1) \times 3 = 3n + 6$

Example: Which term of sequence 72, 70, 68, 66, …. Is 40?

Solution: The given sequence is an A.P. with first term $a = 72$ and a common difference $d = -2$. Let its nth term be 40.

$i.e. a_n = 40$

$\Rightarrow a + (n-1)d = 40$

$\Rightarrow 72 + (n-1)(-2) = 40$

$\Rightarrow n = 17$

Hence, the 17th term of the given sequence is 40.

Example: How many terms are there in the sequence 3, 6, 9, 12, …, 111?

Solution: Clearly, the given sequence is an A.P. with first term $a = 3$ and a common difference $d = 3$.

Let there be n terms in the given sequence. Then,

nth term $= 111 \Rightarrow a + (n-1)d = 111$

$\Rightarrow 3 + (n-1) \times 3 = 111 \Rightarrow n = 37$

Thus, the given sequence contains 37 terms.

Example: Is 184 a term of sequence 3, 7, 11, …?

Solution: Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 4$. Let the nth term of the given sequence be 184.

Then, $a_n = 184 \Rightarrow a + (n-1)d = 184$

$\Rightarrow 3 + (n-1) \times 4 = 184 \Rightarrow n = 46\frac{1}{4}$

Since n is not a natural number, so, 184 is not a term of the given sequence.

Example: Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}$, ... is the first negative term?

Solution: The given sequence is an A.P. in which first term $a = 20$ and common difference $d = -\frac{3}{4}$. Let the nth term of the given A.P. be the first negative term.

Then $a_n < 0$

$\Rightarrow a + (n-1)d < 0$

$\Rightarrow 20 + (n-1)\left(-\frac{3}{4}\right) < 0 \Rightarrow n > 27\frac{2}{3}$

Since 28 is the natural number just greater than $27\frac{2}{3}$, so $n = 28$. Thus 28th term of the given sequence is the first negative term.

Example: If the 5th and 9th term of an A.P. are respectively 12 and 16, then find the 10th term.

Solution: Let a be the first term and d be a common difference.

So, $t_5 = a + 4d = 12$ and $t_9 = a + 8d = 16$

On subtraction, $4d = 4 \Rightarrow d = 1$

When $d = 1$, we obtain $a = 8$

Hence, the 10th term $= t_{10} = a + 9d = 8 + 9 = 17$.

Example: If m times the mth term of an A.P. is equal to n times its nth term, show that the $(m+n)th$ term of the A.P. is zero.

Solution: Let a be the first term and d be the common difference of the given A.P. Then,

m times mth term $= n$ times nth term

$\Rightarrow m\, a_m = n\, a_n$

$\Rightarrow m\{a + (m-1)d\} = n\{a + (n-1)d\}$

$\Rightarrow m\{a + (m-1)d\} - n\{a + (n-1)d\} = 0$

$\Rightarrow a\,(m-n) + \{m(m-1) - n(n-1)\}d = 0$

$\Rightarrow a(m-n) + \{\,(m^2 - n^2) - (m-n)\}d = 0$

$\Rightarrow a(m-n) + (m-n)(m+n-1)d = 0$

$\Rightarrow (m-n)\{a + (m+n-1)d\} = 0$

$\Rightarrow a + (m+n-1)d = 0 \;\Rightarrow\; a_{m+n} = 0$

Hence, the $(m+n)th$ term of the given A.P. is zero.

Example: If the pth term of an A.P. is q and the qth term is p, prove that its nth term is $(p + q - n)$.

Solution: Let a be the first term and d be the common difference of the given A.P. Then,

pth term $= q \;\Rightarrow a + (p-1)d = q$ …. (i)

qth term $= p \;\Rightarrow a + (q-1)d = p$ …. (ii)

Subtracting (ii) from (i), we get $(p-q)d = q - p \;\Rightarrow d = -1$

Putting $d = -1$ in (i), we get $a = p + q - 1$

Hence, nth term $= a + (n-1)d = (p + q - 1) + (n-1)(-1) = p + q - n$

Example: The sum of three numbers in A.P. is -3 and their product is 8. Find the numbers.

Solution: Let the numbers be $(a-d), a\,, (a+d)$. Then,

Sum $= -3 \;\Rightarrow a - d + a + a + d = -3 \;\Rightarrow a = -1$

Product $= 8 \;\Rightarrow (a-d)(a)(a+d) = 8$

$\Rightarrow a(a^2 - d^2) = 8 \;\Rightarrow (-1)(1 - d^2) = 8$

$\Rightarrow\; d^2 = 9$

$\Rightarrow d = \pm 3$

When $a = -1$ and $d = 3$, the numbers are $-4, -1, 2$.

When $a = -1$ and $d = -3$, the numbers are $2, -1, -4$.

So, the numbers are $-4, -1, 2$, or $2, -1, -4$.

Sum to n terms of an A.P.:

The sum of n terms of an A.P. with first term $'a'$ and common difference $'d'$ is

$S_n = \frac{n}{2}\{2a + (n-1)d\}$

Or, $S_n = \frac{n}{2}(a+l)$, where $l =$ the last term $= a + (n-1)d$

Example: Find the sum of 20 terms of A.P. 1, 4, 7, 10, ….

Solution: Let a be the first term and d be the common difference of the given A.P.

Here, $a = 1, d = 3$ and $n = 20$

We have $S_{20} = \frac{20}{2}\{2 \times 1 + (20-1) \times 3\} = 10 \times 59 = 590$

Example: Find the sum of the first 20 terms of an A.P., in which the 3rd term is 7 and 7th term is two more than thrice of its 3rd term.

Solution: Let a be the first term and d be the common difference of the given A.P. It is given that

$a_3 = 7$ and $a_7 = 3a_3 + 2$

$\Rightarrow a + 2d = 7$ and $a + 6d = 3 \times 7 + 2$

$\Rightarrow a + 2d = 7$ and $a + 6d = 23$

On subtraction, $4d = 16 \Rightarrow d = 4$

Also, $a = -1$

$\therefore S_{20} = \frac{20}{2}\{2 \times (-1) + (20-1) \times 4\} = 740$

Example: If the sum of n terms of an A.P. is $nP + \frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference.

Solution: Let $a_1, a_2, \dots, a_n$ be the given A.P. Then

$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = nP + \frac{1}{2}n(n-1)Q$

Therefore $S_1 = a_1 = P,\ S_2 = a_1 + a_2 = 2P + Q$

So, $a_2 = S_2 - S_1 = P + Q$

Hence, the common difference is given by $d = a_2 - a_1 = (P+Q) - P = Q$

Example: The sum of n terms of two arithmetic progressions are in the ratio $(3n+8):(7n+15)$. Find the ratio of their $12th$ terms.

Solution: Let a_1, a_2 be the first terms and d_1, d_2 the common differences between the two given A.P.'s. Then, the sums of their n terms are given by

$S_n = \frac{n}{2}\{2a_1 + (n-1)d_1\}$ and, $S_n' = \frac{n}{2}\{2a_2 + (n-1)d_2\}$

It is given that $\frac{S_n}{S_n'} = \frac{3n+8}{7n+15}$

$\Rightarrow \frac{\frac{n}{2}\{2a_1+(n-1)d_1\}}{\frac{n}{2}\{2a_2+(n-1)d_2\}} = \frac{3n+8}{7n+15}$

$\Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15}$

$\Rightarrow \frac{a_1+\frac{(n-1)}{2}d_1}{a_2+\frac{(n-1)}{2}d_2} = \frac{3n+8}{7n+15}$

Replacing $\frac{n-1}{2}$ by 11 $i.e. n$ by 23 on both sides, we get

$\frac{a_1+11\,d_1}{a_2+11\,d_2} = \frac{3\times23+8}{7\times23+15} = \frac{77}{176} = \frac{7}{16}$

Hence, the required ratio is 7: 16.

Properties of Arithmetic Progressions:

1. If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.
2. If each term of a given A.P. is multiplied or divided by a non -zero constant k, then the resulting sequence is also an A.P. with common difference kd or $\frac{d}{k}$, where d is the common difference of the given A.P.
3. In a finite A.P., the sum of the terms equidistant from the beginning and end is always the same and is equal to the sum of the first and last term.
4. A sequence is an A.P. iff its nth term is a linear expression in n $i.e.\, a_n = An + B$, where A, B are constants. In such a case the coefficients of n in a_n is the common difference of the A.P.

Example: If a, b, c are in A.P., prove that the following are also in A.P.

$(i)\, \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$

Solution: a, b, c are in A.P.

$\Rightarrow \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P.

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$(ii)\ b+c, c+a, a+b$

Solution: It is given that a, b, c are in A.P.

$\Rightarrow\ a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are in A.P.

$\Rightarrow\ -(b+c), -(c+a), -(a+b)$ are in A.P.

$\Rightarrow b+c, c+a, a+b$ are in A.P.

Arithmetic Mean:

If between two given quantities a and b we have to insert n quantities $A_1, A_2, \dots, A_n$ such that $a, A_1, A_2, \dots, A_n, b$ form an A.P., then we say that $A_1, A_2, \dots, A_n$ are arithmetic means between a and b.

If a, A, b are in A.P., we say that A is the arithmetic mean of a and b.

So, $A - a = b - A$

$\Rightarrow 2A = a + b\ \Rightarrow A = \frac{a+b}{2}$

Insertion of Arithmetic Means:

Let $A_1, A_2, \dots, A_n$ be n arithmetic means between two quantities a and b. Let d be the common difference of this A.P. Clearly, it contains $(n+2)$ terms.

$\therefore\ b = (n+2)th$ term $\Rightarrow b = a + (n+1)d\ \Rightarrow d = \frac{b-a}{n+1}$

Now, $A_1 = a + d = a + \frac{b-a}{n+1}$

$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$

…………………..

$A_n = a + nd = a + \frac{n(b-a)}{n+1}$

Example: Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution: Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that $3, A_1, A_2, A_3, A_4, A_5, A_6, 24$ are in A.P.

Here $a = 3, b = 24, n = 6$

So, $d = \frac{b-a}{n+1} = \frac{24-3}{6+1} = 3$

Thus, $A_1 = a + d = 3 + 3 = 6$

$A_2 = a + 2d = 3 + 2 \times 3 = 9$

$A_3 = a + 3d = 3 + 3 \times 3 = 12$

$A_4 = a + 4d = 3 + 4 \times 3 = 15$

$A_5 = a + 5d = 3 + 5 \times 3 = 18$

$A_6 = a + 6d = 3 + 6 \times 3 = 21$

Hence, six numbers between 3 and 24 are 6, 9, 12, 15, 18, and 21.

Example: If eleven A.M.'s is inserted between 10 and 28, then find the number of integral A.M.'s.

Solution: Let the 11 A.M.'s are $A_1, A_2, \ldots, A_{11}$

Here $a = 10, b = 28$ and $n = 11$

So, $d = \frac{b-a}{n+1} = \frac{28-10}{11+1} = \frac{3}{2}$

We have $A_n = a + nd = 10 + \frac{3}{2}n$

n should be a multiple of 2 for A.M. to be an integer.

So, $n = 2, 4, 6, 8, 10$

Thus, there are five integral arithmetic means between 10 and 28.

Applications of A.P.:

Example: The income of a person is ₹ 3,00,000 in the first year and he receives an increase of ₹ 10,000 to his income per year for the next 19 years. Find the total amount, he receives in 20 years.

Solution: Here, we have an A.P. with $a = 3{,}00{,}000, d = 10{,}000$, and $n = 20$.

Using the sum formula, we get

$S_n = \frac{20}{2}\,[600000 + 19\ \times 10000] = 79{,}00{,}000$

Hence, the person received ₹ 79,00,000 as the total amount at the end of 20 years.

Example: The digits of a positive integer, having three digits are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

Solution: Let the digits at ones, tens, and hundreds place be $(a-d), a,$ and $(a+d)$ respectively. Then the number is $(a+d)\times 100 + a\times 10 + (a-d) = 111a + 99d$

The number obtained by reversing the digits is $(a-d)\times 100 + a\times 10 + (a+d) = 111a - 99d$

It is given that $(a+d) + a + (a-d) = 15$ and $111a - 99d = 111a + 99d - 594$

$\Rightarrow 3a = 15$ and $198\,d = 594\ \Rightarrow a = 5$ and $d = 3$

So, the number is $111a + 99d\ = 111\times 5 + 99\times 3 = 852.$

Geometric Progression:

A sequence (a_n) of non – zero numbers is said to be a geometric progression (G.P.) or geometric sequence, if there exists a constant r such that $\frac{a_{n+1}}{a_n} = r\,, n \in N$ and the series $\sum_{n=1}^{\infty} a_n$ is called a geometric series. The above ratio is called the common ratio.

A geometric series is finite or infinite according to as the corresponding G.P. consists of a finite or infinite number of terms.

For example, sequence 5, 25, 125, … is a G.P. with first term 5 and common ratio 5.

Example: Show that the sequence given by $a_n = \frac{2}{3^n}, n \in N$ is a G.P.

Solution: We have, $a_n = \frac{2}{3^n}$

$\Rightarrow\ a_{n+1} = \frac{2}{3^{n+1}}$

So, $\frac{a_{n+1}}{a_n} = \frac{2}{3^{n+1}}\ \times \frac{3^n}{2} = \frac{1}{3}$, which is a constant for all $n \in N$.

So, the given sequence is a G.P. with a common ratio $\frac{1}{3}$.

General Term of a G.P.

Let $'a'$ be the first term and $'r'$ be the common ratio of a G.P.

Then the nth term or general term of the G.P. is $a_n = a\,r^{n-1}$

Thus the G.P. can be written as $a, ar, ar^2, ar^3, \ldots$

Example: Find the $9th$ term and the general term of the progression: $\frac{1}{4}, -\frac{1}{2}, 1, -2, \ldots$

Solution: The given progression is a G.P. with first term $a = \frac{1}{4}$ and common ratio $r = -2$.

So, the $9th$ term $= a_9 = ar^8 = \frac{1}{4}(-2)^8 = 64$

The general term $= a_n = a\, r^{n-1} = \frac{1}{4}(-2)^{n-1} = (-1)^{n-1}\, 2^{n-3}$

Example: Find $4th$ term from the end of G.P. 3, 6, 12, 24, …, 3072.

Solution: The given progression is a G.P. with last term $l = 3072$ and common ratio $r = 2$.

So, $4th$ term from the end $= l\left(\frac{1}{r}\right)^{4-1} = (3072)\left(\frac{1}{2}\right)^3 = 384.$

Example: Which term of the G.P., 2, 8, 32, … up to n terms is 131072?

Solution: Let 131072 be the nth term of the given G.P.

Here $a = 2$ and $r = 4$.

Therefore $131072 = a_n = 2\,(4)^{n-1} \Rightarrow 4^{n-1} = 65536 = 4^8$

$\Rightarrow n - 1 = 8 \Rightarrow n = 9$

Hence, 131072 is the $9th$ term of G.P.

Example: In a G.P., the $3rd$ term is 24 and the $6\,th$ terms are 192. Find the $10th$ term.

Solution: Here $a_3 = a\,r^2 = 24$ and $a_6 = a\,r^5 = 192$

On division, we get $r = 2$

Then $a = 6$

Hence $a_{10} = ar^9 = 6.2^9 = 3072.$

Example: The sum of the first three terms of a G.P. is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution: Let $\frac{a}{r}, a, ar$ be the first three terms of the G.P. Then

$\frac{a}{r} + a + ar = \frac{13}{12}$ … (1) and $\left(\frac{a}{r}\right)(a)(ar) = -1$ … (2)

From (2), we get $a^3 = -1, i.e., a = -1$

Substituting $a = -1$ in (1), we have $-\frac{1}{r} - 1 - r = \frac{13}{12}$ $or\ 12r^2 + 25r + 12 = 0$

Solving, we get $r = -\frac{3}{4}$ $or - \frac{4}{3}$

Thus, the three terms of G.P. are $\frac{4}{3}, -1, \frac{3}{4}$ for $r = -\frac{3}{4}$ and $\frac{3}{4}, -1, \frac{4}{3}$ for $r = -\frac{4}{3}$.

Example: The product of the first three numbers of a G.P. is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in A.P. Find the G.P.

Solution: Let the first three terms of the given G.P. be $\frac{a}{r}$, a, ar. Then,

$\frac{a}{r} \times a \times ar = 1000 \ \Rightarrow \ a^3 = 1000 \ \Rightarrow a = 10$

It is given that $\frac{a}{r}$, $a + 6, ar + 7$ are in A.P.

So, $2(a + 6) = \frac{a}{r} + ar + 7$

$\Rightarrow 32 = \frac{10}{r} + 10r + 7 \ \Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow (2r - 1)(r - 2) = 0 \ \Rightarrow r = \frac{1}{2}, 2$

Hence, the G.P. is 5, 10, 20, …. or 20, 10, 5, ….

Sum of the terms of a G.P.:

The sum of n terms of a G.P. with first term $'a'$ and common ratio $'r'$ is given by

$S_n = a\left(\frac{r^n - 1}{r - 1}\right), r \neq 1.$

If l is the last term of the G.P., then $S_n = \frac{lr - a}{r - 1}, r \neq 1.$

Example: Find the sum of first n terms and the sum of first 5 terms of the geometric series $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

Solution: Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$S_n = \frac{a\,(1 - r^n)}{1 - r} = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3\left[1 - \left(\frac{2}{3}\right)^n\right]$

In particular, $S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = \frac{211}{81}$

Example: How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}$, …. are needed to give the sum $\frac{3069}{512}$?

Solution: let n be the number of terms needed.

Given that $a = 3, r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$.

Since $S_n = \frac{a\,(1-r^n)}{1-r}$, so $\frac{3069}{512} = \frac{3\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$

$\Rightarrow \frac{3069}{3072} = 1 - \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = \frac{1}{1024}$

$\Rightarrow\ 2^n = 1024 = 2^{10} \Rightarrow n = 10.$

Example: Determine the number of terms in a G.P. if $a_1 = 3, a_n = 96$ and $S_n = 189$.

Solution: $a_n = a_1 r^{n-1} = 96 \Rightarrow 3\,r^{n-1} = 96$

$\Rightarrow\ r^{n-1} = 32$

Since, $S_n = 189 \Rightarrow \frac{a_1\,(1-r^n)}{1-r} = 189$

$\Rightarrow \frac{3\,(1-32\,r)}{1-r} = 189 \Rightarrow r = 2$

Hence $n = 6$.

Example: Find the sum of the following sequences:

(i) $7, 77, 777, 7777, \ldots$ to n terms.

Solution: Let $S_n = 7 + 77 + 777 + 7777 + \cdots$ to n terms

$= 7\{1 + 11 + 111 + 1111 + \cdots$ to n terms$\}$

$= \frac{7}{9}\{\,9 + 99 + 999 + 9999 + \cdots$ to n terms$\}$

$= \frac{7}{9}[(10-1) + (10^2-1) + (10^3-1) + (10^4-1) + \cdots n\ terms]$

$= \frac{7}{9}\ [(10 + 10^2 + 10^3 + 10^4 + \cdots n\ terms) - (1 + 1 + 1 + 1 + \cdots n\ terms)]$

$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{7}{9}\left[\frac{10}{9}\,(10^n-1) - n\right]$

$= \frac{7}{81}(10^{n+1} - 10 - 9n)$

(ii) 0.5 , 0.55, 0.555, 0.5555, ... to n terms.

Solution: Let $S_n = 0.5 + 0.55 + 0.555 + 0.5555 + \cdots to\ n\ terms$

$= 5\{0.1 + 0.11 + 0.111 + 0.1111 + \cdots to\ n\ terms\}$

$= \frac{5}{9}\{0.9 + 0.99 + 0.999 + 0.9999 + \cdots to\ n\ terms\}$

$= \frac{5}{9}\left\{\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \cdots to\ n\ terms\right\}$

$= \frac{5}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \cdots to\ n\ terms\right\}$

$= \frac{5}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \cdots + \left(1 - \frac{1}{10^n}\right)\right\}$

$= \frac{5}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n}\right)\right\}$

$= \frac{5}{9}\left\{n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right\} = \frac{5}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\} = \frac{5}{81}\left\{9n - 1 + \frac{1}{10^n}\right\}$

Example: A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution: Here $a = 2, r = 2$ and $n = 10$

We have $S_n = \frac{a\,(r^n - 1)}{r - 1}$

So, $S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

Sum of an Infinite G.P.:

The sum of an infinite G.P. with first term a and common ratio r $(-1 < r < 1\ i.e., |r| < 1)$ is

$S = \frac{a}{1-r}$

Remember: If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

Example: Find the sum to infinity of the G.P. $-\frac{5}{4}, \frac{5}{16}, -\frac{5}{64}, \ldots$

Solution: The given G.P. has first term $a = -\frac{5}{4}$ and the common ratio $r = -\frac{1}{4}$

Hence $S = \frac{a}{1-r} = \frac{-\frac{5}{4}}{1-(-\frac{1}{4})} = -1$

Example: The sum of an infinite G.P. is 8, its second term is 2, find the first term.

Solution: Let a be the first term and r the common ratio of the G.P. It is given that $S_\infty = 8$ and $ar = 2$

$\Rightarrow \frac{a}{1-r} = 8$ and $r = \frac{2}{a}$

$\Rightarrow \frac{a}{1-\frac{2}{a}} = 8 \Rightarrow a^2 - 8a + 16 = 0 \Rightarrow (a-4)^2 = 0 \Rightarrow a = 4$

Example: Prove that: $6^{\frac{1}{2}} \times 6^{\frac{1}{4}} \times 6^{\frac{1}{8}} \times \ldots = 6$

Solution: We have $6^{\frac{1}{2}} \times 6^{\frac{1}{4}} \times 6^{\frac{1}{8}} \times \ldots = 6^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)} = 6^{\frac{1/2}{1-\frac{1}{2}}}$

$= 6^1 = 6$

Example: If $x = 1 + a + a^2 + \cdots$ and $y = 1 + b + b^2 + \cdots$ where $|a| < 1$ and $|b| < 1$, prove that $1 + ab + a^2b^2 + \cdots = \frac{xy}{x+y-1}$

Solution: We have $x = 1 + a + a^2 + \cdots$

$\Rightarrow x = \frac{1}{1-a} \Rightarrow 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x}$

Also, $y = 1 + b + b^2 + \cdots$

$\Rightarrow y = \frac{1}{1-b} \Rightarrow 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y}$

Now, $1 + ab + a^2b^2 + \cdots = \frac{1}{1-ab} = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} = \frac{xy}{x+y-1}$

Properties of Geometric Progression:

1. If all the terms of a G.P. are multiplied or divided by the same non – zero constant, then it remains a G.P. with the same common ratio.
2. The reciprocals of the terms of a given G.P. form a G.P.
3. If each term of a G.P. is raised to the same power, the resulting sequence also forms a G.P.
4. In a finite G.P., the product of the terms equidistant from the beginning and the end is always the same and is equal to the product of the first and the last term.
5. Three non – zero numbers a, b, c are in G.P. iff $b^2 = ac$.

Example: If a, b, c, d are in G.P., prove that $a+b, b+c, c+d$ are in G.P.

Solution: Let r be the common ratio of the G.P. a, b, c, d.

Then $b = ar, c = ar^2$ and $d = ar^3$

$\therefore\ a+b = a+ar = a(1+r), b+c = ar+ar^2 = ar(1+r)$ and

$c+d = ar^2+ar^3 = ar^2(1+r)$

Now, $(b+c)^2 = \{ar(1+r)\}^2 = a^2r^2(1+r)^2 = \{a(1+r)\}\{ar^2(1+r)\} = (a+b)(c+d)$

Hence, $a+b, b+c, c+d$ are in G.P.

Example: If a, b, c are in G.P. and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, prove that x, y, z are in A.P.

Solution: We have, $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$(say) $\Rightarrow a = k^x, b = k^y$ and $c = k^z$

Now, a, b, c are in G.P.

$\Rightarrow\ b^2 = ac\ \Rightarrow\ (k^y)^2 = k^x \times k^y \Rightarrow\ k^{2y} = k^{x+y}$

$\Rightarrow 2y = x+z\ \Rightarrow x, y, z$ are in A.P.

Geometric Mean (G.M.):

Let a and b be two given numbers. If n numbers $G_1, G_2, \dots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \dots, G_n, b$ is a G.P. Then the numbers $G_1, G_2, \dots, G_n$ are known as n geometric means between a and b.

The sequence $a, G_1, G_2, \dots, G_n, b$ is a G.P. consisting of $(n+2)$ terms. Let r be the common ratio of this G.P.

Then, $b = (n+2)th$ term $= ar^{n+1}$

$\Rightarrow\ r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$\therefore\ G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \dots, G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

If a, G, b are in G.P., then G is the G.M. of a and b.
Then $G^2 = ab\ \Leftrightarrow G = \sqrt{ab}$
If a and b are two numbers of opposite signs, then geometric mean between them does not exist.
Example: Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.
Solution: Let G_1, G_2, G_3 be three numbers between 1 and 256 such that $1, G_1, G_2, G_3, 256$ is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$

For $r = 4$, we have $G_1 = ar = 4, G_2 = ar^2 = 16, G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are $-4, 16$, and -64.

Hence, we can insert $4, 16, 64$, or $-4, 16, -64$ between 1 and 256 so that the resulting sequences are in G.P.

Relationship Between A.M. and G.M.:

1. If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then $A \geq G$ $i.e. \frac{a+b}{2} \geq \sqrt{ab}$.
2. If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.
3. If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.

Example: If $A.M.$ and $G.M.$ of two positive numbers a and b are 10 and 8 respectively, find the numbers.

Solution: Given that $A.M. = \frac{a+b}{2} = 10$ and $G.M. = \sqrt{ab} = 8$

$\Rightarrow a + b = 20 \text{ and } ab = 64$ ... (1)

We have $(a - b)^2 = (a + b)^2 - 4ab = 400 - 256 = 144$

$\Rightarrow a - b = 12$

Solving $a + b = 20$ and $a - b = 12$, we get $a = 16$ and $b = 4$.

Hence, the required numbers are 16 and 4.

Example: If $x \in R$, find the minimum value of the expression $3^x + 3^{1-x}$.

Solution: We know that $A.M. \geq G.M.$

$\Rightarrow \frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}}$ for all $x \in R$.

$\Rightarrow \frac{3^x+3^{1-x}}{2} \geq \sqrt{3}$ for all $x \in R$.

$\Rightarrow 3^x + 3^{1-x} \geq 2\sqrt{3}$ for all $x \in R$.

Hence, the minimum value of $3^x + 3^{1-x}$ for any $x \in R$ is $2\sqrt{3}$.

Sum to n terms of Special Series:

We have

$(i)\ \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

$(ii)\ \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$(iii)\ \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Find the sum to n terms of the series, whose nth term is $n(n + 3)$.

Solution: Given that $a_n = n(n+3) = n^2 + 3n$

Thus, the sum to n terms is given by

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2+3k) = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$

$= \frac{n(n+1)(2n+1)}{6} + 3\,\frac{n(n+1)}{2} = \frac{n(n+1)(n+5)}{3}$

Example: Find the sum to n terms of the series $1^2 + 3^2 + 5^2 + \cdots$ to n terms.

Solution: Let a_n be the nth term of this series and S_n denote the sum of its n terms.

Then, $a_n = [1+(n-1)\times 2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$

So, $S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(4k^2 - 4k + 1)$

$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= 4\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + n$

$= \frac{n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}(4n^2 - 1)$

Example: Find the sum to n terms of the series: $5 + 11 + 19 + 29 + 41 + \cdots$

Solution: We have $S_n = 5 + 11 + 19 + 29 + 41 + \cdots + a_{n-1} + a_n$

Or, $S_n = 5 + 11 + 19 + 29 + \cdots + a_{n-2} + a_{n-1} + a_n$

On subtraction, we get

$0 = 5 + [\,6 + 8 + 10 + \cdots (n-1) \text{ terms}] - a_n$

Or, $a_n = 5 + \frac{(n-1)\,[\,12 + (n-2)\times 2\,]}{2} = 5 + (n-1)(n+4) = n^2 + 3n + 1$

Hence, $S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2 + 3k + 1)$

$= \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= \frac{n(n+1)(2n+1)}{6} + 3\,\frac{n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}$

Example: Sum the following series to n terms: $5 + 7 + 13 + 31 + 85 + \cdots$

Solution: The sequence of differences between successive terms is $2, 6, 18, 54, \ldots$.

Clearly, it is a G.P. Let a_n be the nth term and S_n be the sum of its n terms. Then,

$S_n = 5 + 7 + 13 + 31 + 85 + \cdots + a_{n-1} + a_n$

Also, $S_n = 5 + 7 + 13 + 31 + \cdots + a_{n-2} + a_{n-1} + a_n$

On subtraction, we get

$0 = 5 + \{2 + 6 + 18 + 54 + \cdots (n-1) terms\} - a_n$

$\Rightarrow 0 = 5 + \frac{2\,(3^{n-1} - 1)}{3-1} - a_n$

$\Rightarrow a_n = 5 + 3^{n-1} - 1 = 4 + 3^{n-1}$

$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(4 + 3^{k-1}) = \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$

$= 4n + (1 + 3 + 3^2 + \cdots + 3^{n-1})$

$= 4n + \frac{3^n - 1}{3-1} = \frac{1}{2}(3^n + 8n - 1)$

Example: Find the sum of the series $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \cdots$ to n terms

Solution: Let S be the sum of the given series. Then,

$S = (3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \cdots$ to n terms

$= \sum_{r=1}^{n}\{(2r+1)^3 - (2r)^3\}$

$= \sum_{r=1}^{n}\{(2r+1) - 2r\}\{(2r+1)^2 + (2r+1)(2r) + (2r)^2\}$

$= \sum_{r=1}^{n}(12r^2 + 6r + 1)$

$= 12 \sum_{r=1}^{n} r^2 + 6 \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= 12 \frac{n(n+1)(2n+1)}{6} + 6 \frac{n(n+1)}{2} + n = n(4n^2 + 9n + 6)$

Example: Find the sum to n terms of the series: $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$

Solution: Let a_r be the rth term of the given series. Then,

$a_r = \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

So, $S_n = \sum_{r=1}^{n} a_r$

$= \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

$= \frac{1}{2}\left\{\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$

$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$