Chapter- 9

Sequences and Series

Numbers are is a determine order forms a sequence.

Sequence

A sequence is a function whose domain is the set of natural numbers and range is a subset of a set of real numbers.

i.e $f:N\to S$ (where $S\subseteq R$) be defined by $f(n)=a_n$, where $n\in N$

A sequence is represented by $\{a_n\}$ or (a_n)

We have $\{a_n\}=\{a_1,a_2,a_3,.........a_n,.......\}$ here a_1, a_2 are called the terms or enumerations of the sequences.

a_n is the nth term and is called the general term of the sequence

If $\{a_1,a_2,a_3,.....\}$ are real number, the $\{a_n\}$ is called a real sequence.

Types of Sequence:

1. Finite sequence

A sequence is said to be finite if it contains a finite number of terms

2. Infinite sequence

A sequence which is not a finite sequence is known as an infinite sequence.

Representation of a sequence

A real sequence can be represented in different ways.

(i)A real sequence can be represented by listing its few terms until the rule for writing down other terms becomes clear.

e.g 3,5,7,...is a sequence and rule for writing down other terms is (2n+1)

(ii)A real sequence can be represented in terms of a rule or an algebraic formula of writing the nth term of the sequence

e.g.The sequence 1,3,5,7,... can be written as a_n=2n-1

(iii)Sometimes the sequence i.e an arrangement of numbers has no visible pattern but the sequence can be represented by the recurrence relation.

e.g.The sequence 1,1,2,3,5,8,.... has no visible pattern but its recurrence relation is $a_1=a_2=1$ and $a_{n+1}=a_n+a_{n-1}$, $n\geq 2$ This sequence is called Fibonacci sequence.

Example:-1

Find the first three terms of the sequence defined by, $a_n = \frac{n}{n^2+1}$

Solution:-

$$a_1 = \frac{1}{1+1} = \frac{1}{2}$$

$$a_2 = \frac{2}{4+1} = \frac{2}{5}$$

$$a_3 = \frac{3}{9+1} = \frac{3}{10}$$

Example:-2

If the nth term, 'a_n' is given by $a_n = n^2 - n + 1$ write down its first five terms

Solution:-

$$a_1 = 1-1+1 = 1$$

$$a_2 = 4-2+1 = 3$$

$$a_3 = 9-3+1 = 7$$

$$a_4 = 16-4+1 = 13$$

$$a_5 = 25-5+1 = 21$$

Example:-3

Find the first four terms of the sequence defined by $a_1 = 3$ and $a_n = 3\,a_{n-1} + 1$ for all n > 1

Solution:-

$$a_2 = 3a + 1 = 3(3) + 1 = 10$$

$$a_3 = 3a_2 + 1 = 3(10) + 1 = 31$$

$$a_4 = 3a_3 + 1 = 3(31) + 1 = 94$$

Series:-

The sum of terms of a sequence is called a series. If $a_1, a_2, a_3.........$ are terms of a sequence, then

$a_1 + a_2 + a_3 + ... = \sum_{n=1}^{\infty} a_n$ is a series

A series is said to be finite or infinite according to the corresponding sequence is finite or infinite.

Example:-4

Let the sequence a_n is defined as $a_1=2, a_n=a_{n-1}+3$ for $n \geq 2$. Find the first five terms and write the corresponding series.

Solution:-

We have $a_1=2$ and $a_n=a_{n-1}+3$,so $a_2=5, a_3=8, a_4=11, a_5=14$

Thus first five terms of given sequence are 2,5,8,11 and 14.

Also, the corresponding series is 2+5+8+11+14+...

Arithmetic Progression (A.P)

A sequence $a_1, a_2, a_3, \ldots\ldots$ is called as A.P of $a_2 - a_1 = a_3 - a_2 = \ldots\ldots = a_n - a_{n-1} = \ldots\ldots = d$ where d is called the common difference.

i.g 1, 4, 7, 10, 13, …….. is an A.P

Example:-1

Show that the sequence defined by $a_n = 4n+5$ is an AP. Also, find its common difference.

Solution:-

$a_n = 4n+5$

$\Rightarrow a_{n-1} = 4(n-1)+5$

$= 4n+1$

Now, $a_n - a_{n-1}$

$= (4n+5)-(4n+1) = 4$

Which is a constant and is independent of n. Thus the given sequence is an A.P and the common difference is 4.

Example:-2

Show that the sequence defined by $a_n = 2n^2+3$ is not an AP.

Solution:-

$a_n = 2n^2+3$

$\Rightarrow a_{n-1} = 2(n-1)^2+3$

$= 2n^2+2-4n+3$

$= 2n^2-4n+5$

Now, $a_n - a_{n-1}$

$= (2n^2+3)-(2n^2-4n+5)$

$= 4n-2$

Which is not independent of n

Thus the given sequence is not an AP; we cannot determine the common difference.

Properties:-

01. If a constant is added or subtracted to each term of as AP, then the resulting sequence is also an A.P

02. If each term of an AP is multiplied by a constant, then its resulting sequence is also as AP

03. If each term as AP is divided by a non-zero constant, then the resulting sequence is also as AP

04. If a_1, a_2, a_3,…… are in AP and b_1, b_2, b_3 ……… are in AP, then the sequence obtained by terminals addition is also an AP. i.e $a_1+b_1, a_2+b_2, a_3+b_3$....... is an A.P

Note:- If d is a common difference and a is the first term, then the terms of the sequence which is in AP are $a, a+d, a+2d, a+3d, ...$

The general term of an AP:-

If *a* be the first term and *d* be the common difference of an AP, then its nth term is $a_n = a+(n-1)d$.

Example: - 3

Show that the sequence 9, 12, 15, 18, ……….. is an AP. Find its 16th term.

Solution: - Here $12-9=15-12=18-15=...=3$

So, the given sequence is as AP

Here, $a=9, d=3, n=16$

Now, $a_{16}=9+15\times3=54$

Example:-4

Which term of the sequence 72, 70, 68, 66, …….. is 40?

Solution:- Here, $70-72=68-70=....=-2$

a = 72, let the nth term is 40, i.e a_n = 40

Now, $a_n = a+(n-1)d$

$\Rightarrow 40 = 72+(n-1)(-2)$

$\Rightarrow (-2)(n-1) = -32$

$\Rightarrow n-1=16$

$\Rightarrow n=17$

Hence, the 17th term of the sequence is 40.

Example:- 5

Which term of the sequence 4, 9, 14, 19, …… is 124?

Solution:- Here, $a=4, d=5, a_n=124$

Now, $a_n=a+(n-1)d$

$\Rightarrow 124=4+(n-1)5$

$\Rightarrow 5(n-1)=120$

$\Rightarrow n-1=24$

$\Rightarrow n=25$

Hence, the 25th term of the AP is 124.

Example:- 6

The sum of three numbers in AP is -3 and their product is 8. Find the AP

Solution:- Let the three numbers be a - d, a, a+d,

According to question sum of three numbers in AP is -3

$\Rightarrow a-d+a+a+d=-3$

$\Rightarrow a=-1$

Also, their product is 8

$\Rightarrow (a-d)a(a+d)=8$

$\Rightarrow a(a^2-d^2)=8$

$\Rightarrow (-1)(1-d^2)=8$

$\Rightarrow 1-d^2=-8$

$\Rightarrow d^2=9$

$\Rightarrow d=\pm 3$

When d = 3, the numbers are -4, -1, 2

When d = -3, the numbers are 2, -4, -4

The Sum to n terms of A.P:-

Let a,a+d,a+2d,... a+(n-1)d be an A.P

Then l=a+(n-1)d

The sum S_n of n terms with first terms

$$S_n=\frac{n}{2}[2a+(n-1)d$$

$=\frac{n}{2}[a+a+(n-1)d]$

$=\frac{n}{2}$ [First term + last term]

Example:- 1

The sum of n terms of two arithmetic progression is in the ratio $(3n+8):(7n+15)$. Find the ratio of their 12th terms.

Solution:-Let a_1,a_2, and d_1,d_2 be the first term and common difference between the first and second arithmetic progression, respectively. According to the given condition we have

$$\frac{Sum\ to\ n\ terms\ of\ first\ AP}{Sum\ to\ n\ terms\ of\ second\ AP}=\frac{3n+8}{7n+15}$$

$$\Rightarrow\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n+8}{7n+15}$$

$$\Rightarrow\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n+8}{7n+15}$$ .... (1)

Now $\frac{12th\ term\ of\ first\ AP}{12th\ term\ of\ second\ AP}=\frac{a_1+11d_1}{a_2+11d_2}$

$$\frac{2a_1+22d_1}{2a_2+22d_2}=\frac{3\times23+8}{7\times23+15}$$ (By putting n=23 in (1))

$$\therefore\frac{a_1+11d_1}{a_2+11d_2}=\frac{12th\ terms\ of\ first\ AP}{12th\ terms\ of\ second\ AP}=\frac{7}{16}$$

Hence, the required ratio is 7:16

Example:- 2

The income of a person is RS 3, 00,000 in, first-year and he receives an increase of RS 10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution:-Here ,we have an AP with a=3,00,000,d=10,000 and n=20

Using the sum formula, we get $S_{20}=\frac{20}{2}[600000+19\times10000]$=79,00,000

The arithmetic mean:-

Let a and b be any two number

Then $A_1,A_2,\ldots\ldots,A_n$ are called n

AMS of a and b if

$a,A_1,A_2,\ldots\ldots\ldots A_n,b$ are in AP

Here, $(n+2)$th term = b

If d is a common difference then $b = a + (n+2-1)d$

$\Rightarrow b = a + (n+1)d$

$\Rightarrow b - a = (n+1)d$

$\Rightarrow d = \frac{b-a}{n+1}$

So, $A_1 = a + d = a + \frac{b-a}{n+1}$

$A_2 = a + 2d = a + 2\frac{(b-a)}{n+1}$

$A_n = a + n\left(\frac{b-a}{n+1}\right)$

A number A is called the AM of a and b if a, A, b are in AP

$A - a = b - A$

$\Rightarrow 2A = a + b$

$\Rightarrow A = \frac{a+b}{2}$

Example:- 3

Insert 6 numbers between 3 and 24 such that the resulting sequence is an AP.

Solution:-Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that

3, $A_1, A_2, A_3, A_4, A_5, A_6$,24 are in A.P.Here a=3,b=24,n=8

Therefore 24=3+(8-1)d,so that d=3

Thus

A_1=a+d=3+3=6; A_4=a+4d=15;

A_2=a+2d=9 A_5=a+5d=18

A_3=a+3d=12 A_6=a+6d=21

Hence six numbers between 3 and 24 are 6,9,12,15,18 and 21.

Geometric Progressions:-

A sequence a_1, a_2, a_3, ………. of positive numbers is called a geometric progression if $\frac{a_n}{a_{n-1}} = r, \forall n$

Here, r is called the common ratio

Example:- 1

The sequence $\frac{1}{3},\frac{-1}{2},\frac{3}{4},\frac{-9}{8}$ is a GP since

$$\frac{-\frac{1}{2}}{\frac{1}{3}}=\frac{-3}{2} \qquad \frac{\frac{3}{4}}{-\frac{1}{2}}=-\frac{3}{2},.......$$

Properties of Geometric progression:-

01. If all the terms of a G.P are multiplied or divided by the same non-zero constant, then it remains a G.P

02. The reciprocal of the terms of a given G.P form a G.P

03. If each term of a G.P is raised to the same power, then the resulting sequence also forms a G.P.

04. If the terms of a given G.P are chosen at regular intervals, then the new sequence so formed also forms a G.P

05. if $a_1, a_2, a_3,$ .... is a G.P of non zero non-negative terms, then $\log a_1, \log a_2, \log a_3$ .... are in AP and vice versa.

General Term of a G.P:-

The nth term of a G.P is called the general term. If a is the first term and r is the common ratio, then nth term $=a_n=a\,r^{n-1}$.

Example:- 2

Find the 5th term of the progression $\frac{1}{4},\frac{-1}{2},1,-2,..........$

Solution:- Here, $a=\frac{1}{4}$

$$r=\frac{-\frac{1}{2}}{\frac{1}{4}}=-2$$

S0, 5th term, $a_5=a\,r^{5-1}$

$$=\frac{1}{4}\times(-2)^4$$

$$=\frac{1}{4}\times16=4$$

Example:- 3

Find the 5^{th} term of the progression $1, \frac{\sqrt{2}-1}{2\sqrt{3}}, \frac{3-2\sqrt{2}}{12}, \frac{5\sqrt{2}-7}{24\sqrt{2}}$

Solution:- Here, $a_1 = 1$

$$r = \frac{\frac{\sqrt{2}-1}{2\sqrt{3}}}{1} = \frac{\sqrt{2}-1}{2\sqrt{3}}$$

5^{th} term, $a_5 = a\, r^{n-1} \quad = 1\times\left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \frac{\left(2+1-2\sqrt{2}\right)^2}{16\times 9}$

$$= \frac{\left(3-2\sqrt{2}\right)^2}{144} = \frac{9+8-12\sqrt{2}}{144} = \frac{17-12\sqrt{2}}{144}$$

Note:-

If a is the first term and r is the common ratio of a GP, then the GP can be written as a, ar, ar^2, ...

n^{th} term from the end:-

The nth term from the end of a term GP consisting m terms is the $(m-n+1)^{th}$, term from the beginning. So, $a_{m-n+1} = a\, r^{m-n}$

Note:-

The nth term from the end of a GP with last term l and common ratio r is given by $a_n = \ell\times\left(\frac{1}{r}\right)^{n-1}$

Example:-4

Find the fifth term from the end of the GP 3, 6, 12, 24, ..., 3072

Solution:- Here, r = 2, l = 3072

So, the fifth term from the end $= \ell\left(\frac{1}{r}\right)^4$

$$= 3012\times\left(\frac{1}{2}\right)^4 = 3072\times\frac{1}{16} = 192$$

Example:- 5

Which term of the G.P $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots\ldots\ldots$ is $\frac{1}{512}$?

Solution:- Here, a = 2, r = ½

Let the nth term is $\frac{1}{512}$

i.e $a_n = \frac{1}{512}$

$\Rightarrow a\ r^{n-1} = \frac{1}{512}$

$\Rightarrow 2\left(\frac{1}{2}\right)^{n-1} = \frac{1}{512}$

$\Rightarrow \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1024}$

$\Rightarrow \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{10}$

$\Rightarrow n-1=10$

$\Rightarrow n=11$

Example:-6

The fourth, seventh, and last term of a GP are 10, 80, 2560 respectively. Find the first term and the number of terms in G.P

Solution:-

Let a be the first term and r be the common ratio of the given G.P

Here, $a_4 = 10,\ a_7 = 80$ and $a_n = 2560$

$\Rightarrow ar^3 = 10,\ ar^6 = 80,\ ar^{n-1} = 2560$

Now, $\frac{ar^6}{ar^3} = \frac{80}{10}$

$\Rightarrow r^3 = 8$

$\Rightarrow r = 2$

Since, $a\ r^3 = 10$

$\Rightarrow a = \frac{10}{8}$

$\Rightarrow a = \frac{5}{4}$

Since, $a\ r^{n-1} = 2560$

$\Rightarrow \frac{5}{4} \times (2)^{n-1} = 2560$

$\Rightarrow 2^{n-1} = \frac{2560 \times 4}{5}$

$\Rightarrow 2^{n-1} = 512 \times 4$

$\Rightarrow 2^{n-1} = 2048$

$\Rightarrow 2^{n-1} = 2^{11} = n-1 = 11 = n = 12$

Example:- 7

If the sum of three numbers is G.P is 38 and their product is 1728. Find the numbers

Solution:- Let the numbers are $\frac{a}{r}, a, ar$

$\Rightarrow \frac{a}{r} \times a \times ar = 1728$

$\Rightarrow a^3 = 1728$

$\Rightarrow a^3 = 12^3 = a = 12$

Also, sum = 38

$\Rightarrow \frac{a}{r} + a + ar = 38$

$\Rightarrow \frac{12}{r} + 12 + 12r = 38$

$\Rightarrow \frac{12}{r} + 12 + 12r = 38$

$\Rightarrow 12\left(\frac{1}{r} + 1 + r\right) = 38$

$\Rightarrow 12\left(\frac{1+r+r^2}{r}\right) = 38$

$\Rightarrow 12 + 12r + 12n^2 = 38r$

$\Rightarrow 12n^2 - 26r + 12 = 0$

$\Rightarrow 6r^2 - 9r - 4r + 6 = 0$

$\Rightarrow 6r^2 - 9r - 4r + 6 = 0$

$\Rightarrow 3r(2r-3) - 2(2r-3) = 0$

$\Rightarrow (3r-2)(2r-3) = 0$

$\Rightarrow r = 2/3 \text{ or } r = 3/2$

If $r = 2/3$ the terms are 18, 12, 8

If $r = 3/2$, the terms are 8, 12, 18

Sum to n-terms of a G.P:-

Let 'a' is the first term and r is the common ratio of the G.P. Then the sum of n terms is given by

$$S_n = a + ar + ar^2 + ... + ar^{n-1}$$

$$= \begin{cases} \frac{a(1-r^n)}{1-r} & \text{if} \quad r<1 \\ na & \text{if} \quad r=0 \\ \frac{a(r^n-1)}{r-1} & \text{if} \quad r>1 \end{cases}$$

Note:- If l is the last term of a G.P this $\ell = \frac{a - ar^n}{1-r}$

Now, $S_n = \frac{a - a\,r^n}{1-r} = \frac{a - a\,r^{n-1}.r}{1-r} = \frac{a - \ell r}{1-r}$

$$\Rightarrow S_n = \frac{a - \ell r}{1-r}$$

Example:- 1

Find the sum of eight terms of the G.P 3, 6, 12,................

Solution:- Here, a = 3, r = 2, n = 8

So, $S_n = \frac{3(2^8 - 1)}{2-1}$

$= 3 \times 255 = 765$

Example:-2

Find the sum to 7 terms of the sequence $\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right),\left(\frac{1}{5^4}+\frac{2}{5^5}+\frac{3}{5^6}\right),\left(\frac{1}{5^7}+\frac{2}{5^6}+\frac{3}{5^9}\right)$

Solution:-

Here, $a = \left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right)$

$\Rightarrow a = \frac{25+10+3}{125} = \frac{38}{125}$

Here, $r = \frac{1}{5^3} = \frac{1}{125} < 1$

So, $S_7 = \frac{a(1-r^7)}{1-r}$

$$=\frac{\frac{38}{125}\left(1-\left(\frac{1}{125}\right)^{7}\right)}{1-\frac{1}{125}}$$

$$=\frac{38}{125}\left[\frac{1-\left(\frac{1}{125}\right)^{7}}{\frac{124}{125}}\right]=\frac{19}{62}\left[1-\left(\frac{1}{125}\right)^{7}\right]$$

Example:-3

Find the sum of the series $2+6+18+.....+4374$

Solution:-

Here, a = 2, $\ell=4374, r=3$

So, $S=\frac{a-\ell r}{1-r}$

$$=\frac{2-4374\times 3}{1-3}$$

$$=\frac{-13120}{-2}=6560$$

Example:- 4

Find the sum of the following series $5+55+555+5555+.............$ to n term

Solution:- Let 5 be the sum of the series

$S=5+55+5555+5555+..................$ to n terms

$$=5(1+11+111+1111+............\text{to n term})$$

$$=\frac{5}{9}[9+99+999+999+..............\text{to n terms}]$$

$$=\frac{5}{9}\left[\left(10^{1}-1\right)+\left(10^{2}-1\right)+\left(10^{3}-1\right)+............+\left(10^{n}-1\right)\right]$$

$$=\frac{5}{9}\left\{\left(10^{1}+10^{2}+10^{3}+...+10^{n}\right)-n\right\}$$

$$=\frac{5}{9}\left[\frac{10\left(10^{n}-1\right)}{10-1}-n\right]$$

$$=\frac{5}{81}\left[10^{n+1}-9n-10\right]$$

Example:-5

Find the sum to n-terms of the sequence gives by $a_n = 2^n + 3n,\ n \in N$

Solution:- We have $S_n = a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n$

$\Rightarrow S_n = (2+3\times1)+(2^2+3\times2)+(2^3+3\times3)+\ldots\ldots\ldots+(2^n+3\times n)$

$=(2^1+2^2+2^3+\ldots\ldots\ldots\ldots 2^n)+3(1+2+3+\ldots\ldots\ldots+n)$

$=\frac{2(2^n-1)}{2-1}+3\frac{n(n+1)}{2}$

$=2(2^n-1)+\frac{3}{2}n(n+1)$

Example:-6

Find the sum to n terms of the series $11+103+1005+\ldots\ldots\ldots\ldots\ldots$

Solution:-

Let $S_n = 11+103+1005+\ldots\ldots\ldots\ldots\ldots\ldots$

$=(10+1)+(10^2+3)+(10^3+5)+\ldots\ldots\ldots\ldots+(10^n+(2n-1))$

$=(10+10^2+10^3+\ldots\ldots\ldots\ldots+10^n)+[1+3+5+\ldots\ldots\ldots(2n-1)]$

$=\frac{10(10^n-1)}{10-1}+n^2$

$\frac{10}{9}(10^n-1)+n^2$

Sum of an Infinite G.P:-

If the first term is a and the common ratio is r, where $|r|<1$, i.e $-1<r<1$ then sum of as infinite G.P is

$S=\frac{a}{1-r}$

Example:-1

Find the sum to infinity of the G.P $-\frac{5}{4},\frac{5}{16},\frac{-5}{64},\ldots\ldots\ldots\ldots$

Solution:-

Here, $a=\frac{-5}{4}, r=\frac{-1}{4}$

So, $S = \frac{a}{1-r}$

$$= \frac{-5/4}{1-\left(-1/4\right)} = \frac{-5/4}{1+1/4}$$

$$= \frac{-1/4}{5/4} = -1$$

Geometric Means:-

Let a and b two gives numbers, then the numbers $G_1, G_2, G_3 \ldots\ldots\ldots\ldots G_n$ are n geometric means of a and b, if the sequence $a, G_1, G_2 \ldots\ldots\ldots\ldots, G_n$, b is a G.P

Let r is the common ratio

Here, $(n+2)^{th}$ term = b

$$\Rightarrow a\,r^{n+2-1} = b$$

$$\Rightarrow r^{n+1} = \frac{b}{a}$$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Now, $G_1 = a\,r = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

If 3 number a, G, b are is G.P this $\frac{G}{a} = \frac{b}{G}$

$$\Rightarrow G^2 = ab$$

$$\Rightarrow G = \sqrt{ab}$$

Example:- 2

Insert five numbers between 576 and 9 so that the resulting sequence is a GP.

Solution:-

Here, a = 576, b = 9, n = 5

$$r = \left(\frac{b}{a}\right)^{1/n+1}$$

$$= \left(\frac{9}{576}\right)^{1/6} = \left(\frac{1}{64}\right)^{1/6} = \frac{1}{2}$$

$$G_1 = ar = 576 \times \frac{1}{2} = 288 \qquad G_2 = G_1 r = 288 \times \frac{1}{2} = 144$$

$$G_3 = G_2 r = 144 \times \frac{1}{2} = 72 \qquad G_4 = G_3 r = 72 \times \frac{1}{2} = 36$$

$$G_5 = G_4 r = 36 \times \frac{1}{2} = 18$$

Example:- 3

Insert five numbers between 16 and1/4so that the resulting sequence is a GP.

Solution:- Here, a = 16, b = ¼, n = 5

$$r = \left(\frac{b}{a}\right)^{1/n+1} = \left(\frac{1/4}{16}\right)^{1/5+1} = \left(\frac{1}{66}\right)^{1/6} = \frac{1}{2}$$

$$G_1 = ar = 16 \times \frac{1}{2} = 8 \qquad G_2 = G_1 r = 8 \times \frac{1}{2} = 4$$

$$G_3 = G_2 r = 4 \times \frac{1}{2} = 2$$

$$G_4 = G_3 r = 2 \times \frac{1}{2} = 1$$

$$G_5 = G_4 \times 5 = 1 \times \frac{1}{2} = \frac{1}{2}$$

The relation between AM and G.M

(a) If A and G are respectively AM and GM between two positive numbers a and b then A > G.

Here $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Now, $A - G = \frac{a+b}{2} - \sqrt{ab}$

$$= \frac{a + b - 2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} > 0$$

$\Rightarrow A > G$

(b) If A and G are respectively AM and GM between two positive quantity, a and b then the quadratic equation having a and b as the roots are $x^2 - 2Ax + G^2 = 0$

(c) if A and G is the AM and GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$

Example:-4

If, AM and GM of two positive numbers a and b are 10 and 8 respectively. Find the numbers.

Solution:- Here A = 10, G = 8

If a and b are two numbers, then

$a = A + \sqrt{A^2 - G^2}$

$= 10 + \sqrt{100 - 64}$

$= 10 + 6 = 16$

$b = A - \sqrt{A^2 - G^2}$

$= 10 - 6 = 4$

Hence, a = 16, b = 4

Some special series:-

(a) The sum of 1st n natural number

$S_n = 1 + 2 + 3 + ... + n$

$= \sum_{k=1}^{n} K = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$

(b) The sum of the squares of first n natural numbers

$S_n = 1^2 + 2^2 + 3^2 + ... + n^2$

$= \sum_{k=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$

Consider, $(K+1)^3 - K^3$

$= K^3 + 3K^2 + 3K + 1 - K^3$

$= 3K^2 + 3K + 1$

Putting K = 1, $2^3 - 1^3 = 3.1^2 + 3.1 + 1$

$K = 2, 3^3 - 2^3 = 3.2^2 + 3.2 + 1$

$K = 3, 4^3 - 3^3 = 3.3^2 + 3.3 + 1$

$K = n, (n+1)^3 - n^3 = 3.n^2 + 3.n + 1$

Adding these by term wise

$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) + n$

$\Rightarrow n^3 + 3n^2 + 3n + 1 - 1 = 3.S_n + 3\frac{n(n+1)}{2} + n$

$\Rightarrow 3S_n = n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n$

$= n^3 + 3n^2 + 3n - \frac{3n^2}{2} - \frac{3n}{2} - n$

$= \frac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{2}$

$= \frac{2n^3 + 3n^2 + n}{2}$

$= \frac{n(2n^2 + 3n + 1)}{2} \Rightarrow S_n = \frac{n(n+1)(2n+1)}{6}$

(c) Sum of cubes of first n natural numbers

$S_n = \sum_{K=1}^{n} K^3$

$= 1^3 + 2^3 + 3^3 + ... + n^3$

$= \left[\frac{n(n+1)}{2}\right]^2$

(d) $4n^3 + 6n^2 + 2n$

Here, $a_n = 4n^3 + 6n^2 + 2n$

$S_n = \sum_{K=1}^{n} K$

$= \sum_{K=1}^{n} (4K^3 + 6K^2 + 2K)$

$= 4\sum_{K=1}^{n} K^3 + 6\sum_{K=1}^{n} K^2 + 2\sum_{K=1}^{n} K$

$= 4\left[\frac{n(n+1)}{2}\right]^2 + 6\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}$

$$=n^2(n+1)^2+n(n+1)(2n+1)+n(n+1)$$

$$=n(n+1)[n(n+1)+2n+1+1]$$

$$=n(n+1)[n^2+n+2n+2]$$

$$=(n^2+n)[n^2+3n+2]$$

$$=n(n+1)(n+1)(n+2)$$

$$=n(n+1)^2(n+2)$$

Example:-1

Find the sum of the series $(1\times2\times3)+(3.3.4)+(3.4.5)+......$ to n terms

Solution:-

Here nth term, $a_n=n(n+1)(n+2)$

$$=(n^2+n)(n+2)$$

$$=n^3+3n^2+2n$$

$$\Rightarrow S_n=\sum_{K=i}^{n}a_r=\sum_{K=1}^{n}(K^3+3K^2+2K)$$

$$=\frac{3.n(n+1)(2n+1)}{6}+\left[\frac{n(n+1)}{2}\right]^2+\frac{2.n(n+1)}{2}$$

$$=\frac{n(n+1)(2n+1)}{2}+\frac{n^2(n+1)^2}{4}+n(n+1)$$

$$=n(n+1)\left[\frac{2n+1}{2}+\frac{n(n+1)}{4}+1\right]$$

$$=n(n+1)\left[\frac{4n+2+n^2+n+4}{4}\right]$$

$$=\frac{n(n+1)(4n+2+n^2+n+4)}{4}$$

$$=\frac{n(n+1)(n^2+5n+6)}{4}$$

$$=\frac{n(n+1)(n+2)(n+3)}{4}$$

Example:-2

Find the sum of the series $3.8+6.11+9.14+..........$ to n terms

Solution:-

Here, a_n= nth term of $(3.8+6.11+9.14+..........)$

$$=3n(3n+5)=9n^2+15n$$

$$\Rightarrow S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(9k^2+15k)$$

$$=\frac{9n(n+1)(2n+1)}{6}+\frac{15n(n+1)}{2}$$

$$=\frac{3n(n+1)(2n+1)}{2}+\frac{15n(n+1)}{2}$$

$$=\frac{3n(n+1)}{2}[2n+1+5]$$

$$=\frac{3n(n+1)}{2}(2n+6)$$

$$=3n(n+1)(n+3)$$

Example:-3

Find the sum of n terms of the series 5+11+19+29+41+........... to n term

Solution:-

Let $S_n=5+11+19+29+14+.........$

$$\Rightarrow S_n=5+11+19+29+41+..........+a_{n-1}+a_n$$

Also $S_n=\quad 5+11+19+29+41+..........+a_{n-2}+a_{n-1}+a_n$

On subtraction, we get

$$0=5+6+8+10+12+...........+(a_n-a_{n-1})-a_n$$

$$\Rightarrow a_n=5+\left[6+8+10+12+..........+(n-1)\text{terms}\right]$$

$$\Rightarrow a_n=5+\frac{n-1}{2}\left[2\times6+(n-10-1)2\right]$$

$$=5+\frac{n-1}{2}[12+2n-4]$$

$$=5+\frac{n-1}{2}[8+2n]$$

$$=5+(n-1)(n+4)$$

$$= n^2 + 3n + 1$$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(k^2 + 3k + 1\right)$$

$$= \sum_{k=1}^{n} K^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + 3.\frac{n(n+1)}{2} + n$$

$$= n\left[\frac{(n+1)(2n+1)}{6} + \frac{3(n+1)}{2} + 1\right]$$

$$= n\left[\frac{(n+1)(2n+1) + 9(n+1) + 6}{6}\right]$$

$$= \frac{n}{6}\left[(n+1)(2n+1) + 9(n+1) + 6\right]$$

$$= \frac{n}{6}\left(2n^2 + 12n + 16\right)$$

$$= \frac{n}{3}\left(n^2 + 6n + 8\right)$$

$$= \frac{n}{3}(n+4)(n+2)$$