Chapter- 10

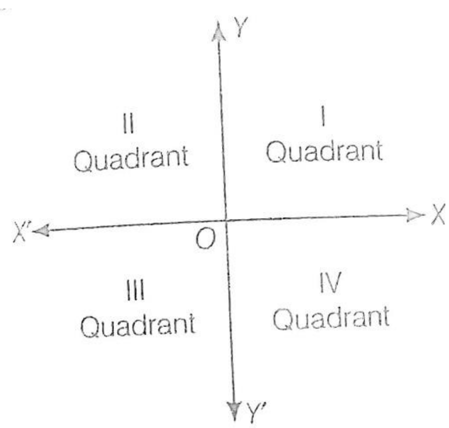
Straight Lines

Introduction to Two Dimensional Geometry

The system which helps us to locate the position of an object is called the coordinate system.

The two-dimensional coordinate system is introduced on a plane with the help of two straight lines namely $X'OX$ and $Y'OY$. These two lines are known as $x-axis$ and $y-axis$ respectively and the two lines taken together are called the coordinate axes or the axes of coordinates.

The $x-axis$ is horizontal and is directed from left to right and $the\ y-axis$ is vertical, which is directed from bottom to top. These two axes are mutually perpendicular to each other at a point O called the origin.



Coordinate Plane:

The coordinate axes divide the plane into four parts. These four parts are called quadrants, $\left(\frac{1}{4}\ th\ part\right)$ numbered $I,\ II,\ III,\ IV$ anticlockwise from OX. Thus, the plane consists of the axes and four quadrants, is known as $XY-plane$ or Rectangular Cartesian coordinate plane or Euclidean plane and is denoted by $R \times R$ or R^2.

$$i.e.\ R \times R = \{(x,\ y): x,\ y \in R\}$$

Coordinates of a Point in the Cartesian plane

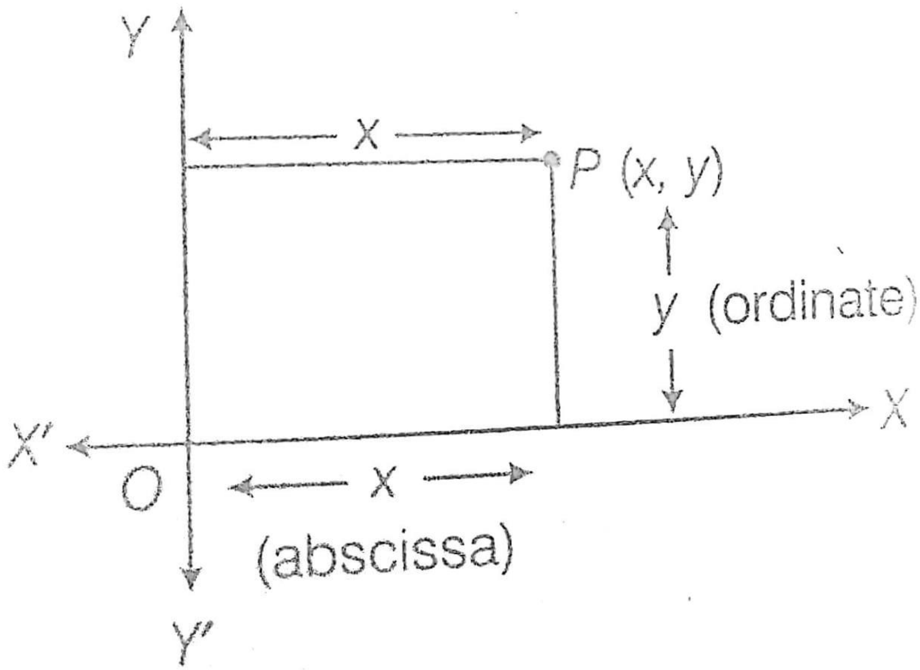
Let P be any point in a plane. If the distance of P from $the\ y-axis$ is x and the distance of P from the $x-axis$ is y, the coordinates of a point P are $(x,\ y)$. Here x is called the $x-$coordinate or abscissa and y are called $y-$coordinate or ordinate.

Thus, for a given point, the abscissa and ordinate are the distances of the given point from $the\ y-axis$ and $x-axis$ respectively.

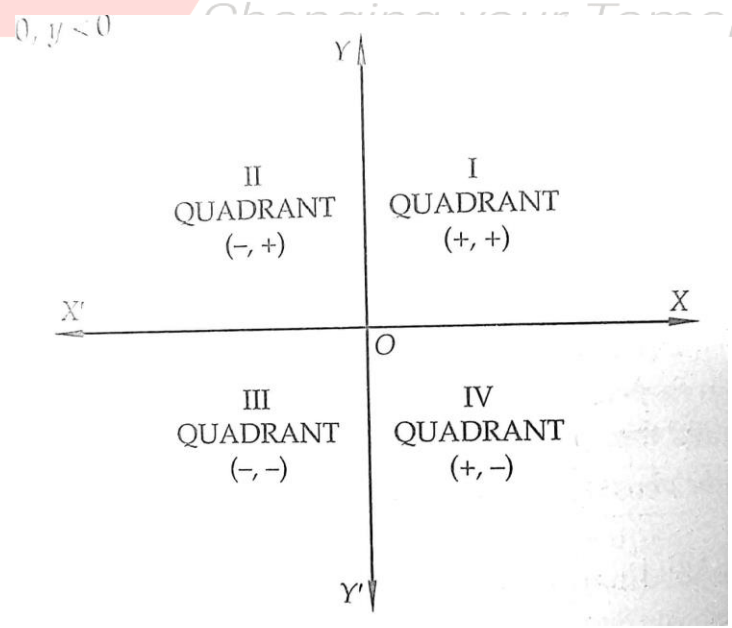
The coordinates of a point on the $x - axis$ are of the form $(x,\ 0)$ and a point on the $y - axis$ are of the form $(0,\ y)$. Thus, if the abscissa of a point is zero, it would lie somewhere on the $y - axis$ and if its ordinate is zero it would lie on $the\ x - axis$.

The coordinates of origin O are $(0,\ 0)$.

There is one to one correspondence between the set of points and the set of ordered pairs of real numbers.



Sign Convention of Coordinates:



Distance between two Points

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance of any point $P(x,\ y)$ from the origin is $|OP| = \sqrt{x^2 + y^2}$

The distance of any point $P(x,\ y)$ from $the\ x - axis$ is $|y|$.

The distance of any point $P(x,\ y)$ from the $\mathrm{y} - axis$ is $|x|$.

Example: Find the distance between the following points.

$(i)(2,\ -3)$ and $(-7,\ 0)$

Sol: The given points are $P(2,\ -3)$ and $Q(-7,\ 0)$.

So, $|PQ| = \sqrt{(-7-2)^2 + (0+3)^2} = 3\sqrt{10}\ units$

$(ii)\ P(acos\alpha\ ,\ a\ sin\alpha)$ and $Q(a\ cos\beta,\ a\ sin\beta)$

Sol: $|PQ| = \sqrt{(a\ cos\beta\ - a\ cos\alpha)^2 + (a\ sin\beta\ - a\ sin\alpha)^2}$

$= \sqrt{a^2\ (cos\beta\ - cos\alpha)^2 + a^2(sin\beta\ - sin\alpha)^2}$

$= a\sqrt{cos^2\beta + cos^2\alpha - 2\cos\alpha\cos\beta + sin^2\beta + sin^2\alpha - 2\ sin\alpha\ sin\beta}$

$= a\sqrt{(cos^2\beta + sin^2\beta) + (cos^2\alpha + sin^2\alpha) - 2(\ cos\alpha\ cos\beta + sin\alpha\ sin\beta)}$

$= a\sqrt{1 + 1 - 2\cos(\alpha - \beta)}$

$= a\sqrt{2\{1 - \cos(\alpha - \beta)\}}$

$= a\sqrt{2\ \times 2\ sin^2\left(\frac{\alpha - \beta}{2}\right)}$

$= 2a\sin\left(\frac{\alpha - \beta}{2}\right)$

Section or Division Formulae

Internal Division:

The coordinates of the point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

External Division:

The coordinates of the point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Example: Find the coordinates of the point which divides the join of $(-1,\ 7)$ and $(4,\ -3)$ in the ratio 2 : 3 (i) internally (ii) externally.

Sol: Let the given points are $A(-1,\ 7)$ and $B(4,\ -3)$.

(i) Here, P divides AB internally in the ratio 2 : 3. So, the coordinates of P are

$\left(\frac{2\times4+3\times(-1)}{2+3}, \frac{2\times(-3)+3\times7}{2+3}\right)$ $i.e.,\ (1,\ 3)$

(ii) Here, P divides AB externally in the ratio 2 : 3. So, the coordinates of P are

$\left(\frac{2\times4-3\times(-1)}{2-3}, \frac{2\times(-3)-3\times7}{2-3}\right)$ $i.e.,\ (-11,\ 27)$

Midpoint Formula

If P be the mid-point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the coordinates of P are

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example: Without using distance formula, show that the points $(-2,\ -1)$, $(4,\ 0)$, $(3,\ 3)$ and $(-3,\ 2)$ are the vertices of a parallelogram.

Sol: Let the given points are $A(-2,\ -1)$, $B(4,\ 0)$, $C(3,\ 3)$ and $D(-3,\ 2)$.

Now, mid-point of AC is $\left(\frac{-2+3}{2}, \frac{-1+3}{2}\right) = \left(\frac{1}{2}, 1\right)$

and mid-point of BD is $\left(\frac{4-3}{2}, \frac{0+2}{2}\right) = \left(\frac{1}{2}, 1\right)$

We get mid-point of $AC =$ mid-point of BD. Thus mid-points of both diagonals are coinciding with each other. Hence, the points A, B, C, D are vertices of a parallelogram.

The centroid of a Triangle

If G is the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) , then coordinates of G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

The centroid divides each median in the ratio 2: 1.

Example: If the vertices of a triangle are $P(1,\ 3),\ Q(2,\ 5)$ and $R(3,\ -5)$, then find the centroid of a $\Delta\ PQR$.

Sol: The coordinates of the centroid G are $\left(\frac{1+2+3}{3},\ \frac{3+5-5}{3}\right) = \left(\frac{6}{3}, \frac{3}{3}\right) = (2,\ 1)$

Area of a Triangle

If $A(x_1, y_1),\ B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of an $\Delta\ ABC$, then the area of the triangle is

$$\frac{1}{2}\ |[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\ |$$

NOTE:

- The area of any triangle is always non –negative
- If the points A, B, C are collinear, then the area of the triangle is zero.
  $i.e.\ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$
- Using area formula, we can determine the area of any polygon of n sides.

Example: Find the area of an ΔABC, whose vertices are $A(6,\ 3),\ B(-3,\ 5)$ and $C(4,\ -2)$.

Sol: Area of the triangle $ABC = \frac{1}{2}\ |6(5+2) + (-3)(-2-3) + 4(3-5)|$

$$= \frac{1}{2}|6 \times 7\ - 3(-5) + 4(-2)| = \frac{49}{2}\ sq.\ units$$

Example: For what value of k are points $(k,\ 2-2k),\ (-k+1,\ 2k),\ (-4-k,\ 6-2k)$ are collinear.

Sol: Since the points are collinear, so $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

$\Rightarrow\ k(2k - 6 + 2k) + (-k+1)(6 - 2k - 2 + 2k) + (-4-k)(2 - 2k - 2k) = 0$

$\Rightarrow\ 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$

$\Rightarrow\ 8k^2 + 4k - 4 = 0$

$\Rightarrow\ 2k^2 + k - 1 = 0$

$\Rightarrow\ (2k-1)(k+1) = 0$

$\Rightarrow\ k = \frac{1}{2}\ ,\ -1$

Hence, the given points are collinear for $k = \frac{1}{2}$ or $k = -1$.

The slope of a Line and Angle between two Lines

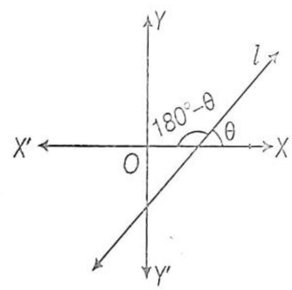
The inclination of a Line

A straight line is a curve, such that all the points on the line segment joining any two points of it lies on it.

An angle θ made by the line with positive $x-axis$ in an anti-clockwise direction is called the angle of inclination of a line. A line in coordinate plane forms two angles with the $x-axis$, which are supplementary. Thus $0 \le \theta \le 180°$.

The inclination of $the\ x-axis$ or any line parallel to the $x-axis$ is $0°$.

The inclination of $the\ y-axis$ or any line parallel to $the\ y-axis$ is $90°$.



Slope or Gradient of a Line

If θ is the angle of inclination of a line l, then $tan\theta$ is called the slope or gradient of the line l and it is denoted by m.

$i.e.\ m = \tan\theta$

The slope of $the\ x-axis$ is $m = tan0° = 0$

The slope of a line when $\theta = 90°$ is not defined $i.e.,$ the slope of $the\ y-axis$ is not defined.

The slope of a line is positive or negative according to the line leans to right or left.

Example: The slope of a line whose inclination is $150°$ is $tan150° - \frac{1}{\sqrt{3}}$

Example: Find the slope of the line making inclination of $60°$ with the positive direction of $the\ x-axis.$

Sol: Here inclination of the line is $\theta = 60°$. Therefore, the slope of the line is $m = tan60° = \sqrt{3}$

The slope of a Line Joining two Points

Let $A(x_1,\ y_1)$ and $B(x_2, y_2)$ be two given points. Then the slope of the line segment joining AB is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope of a line joining two points $A(1,\ 2)$ and $B(3,\ -4)$.

Sol: The slope of the line = $\frac{-4-2}{3-1} = -3$

Example: Find the angle between the $x - axis$ and the line joining the points $(3,\ -1)$ and $(4,\ -2)$.

Sol: The slope of the line $= \frac{-2+1}{4-3} = -1.$

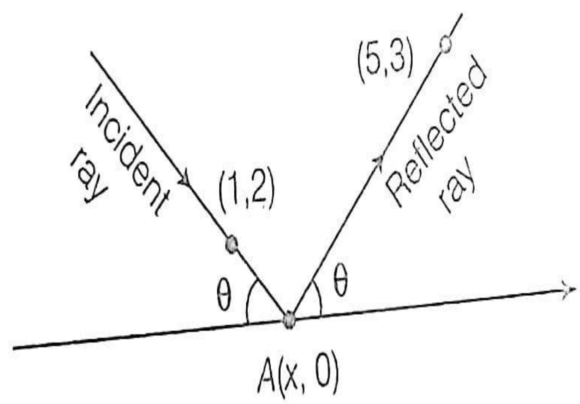
If θ is the angle between $the\ x - axis$ and the given line, then $tan\theta = -1$

$\Rightarrow \theta = 135°$

Example: A ray of light coming from the point $(1,\ 2)$ is reflected at a point A on the $x - axis$ and then passes through the point $(5,\ 3)$. Find the coordinates of point A.

Sol: Let the coordinates of point A be $(x, 0)$. From the figure, the slope of the reflected ray is given by $tan\theta = \frac{3}{5-x}$ … (i)

Again, the slope of the incident ray is given by $\tan(\pi - \theta) = \frac{-2}{x-1}$



$\Rightarrow\ -tan\theta = -\frac{2}{x-1} \Rightarrow tan\theta = \frac{2}{x-1}$ … (ii)

From (i) and (ii), we get $\frac{3}{5-x} = \frac{2}{x-1}$

$\Rightarrow\ 3x - 3 = 10 - 2x \Rightarrow x = \frac{13}{5}$

Therefore, the required coordinates of the point A are $\left(\frac{13}{5}\ ,\ 0\right)$.

The angle between Two Lines

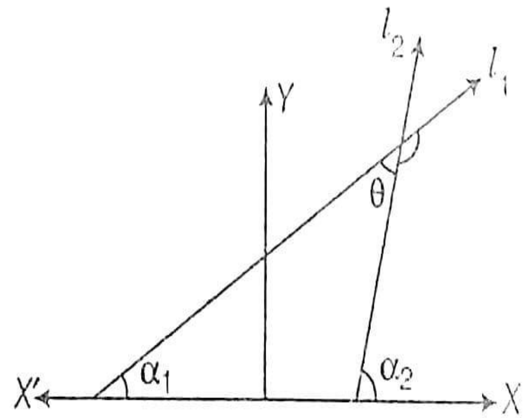
Let l_1 and l_2 be two lines and their inclination are θ_1 and θ_2 respectively. Then, their slopes are $m_1 = tan\theta_1$ and $m_2 = \tan\theta_2$.

If θ is the angle between l_1 and l_2, then $tan\theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2}$

For acute angle, we take $tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

For obtuse angle, we take $\varphi = \pi - \theta$

2.



Example: Find the angle between the lines joining the points $(0, 0)$, $(2, 3)$ and $(2, -2)$, $(3, 5)$.

Sol: We have $m_1 =$ slope of the line joining $(0, 0)$ and $(2, 3) = \frac{3-0}{2-0} = \frac{3}{2}$

$m_2 =$ slope of the line joining $(2, -2)$ and $(3, 5) = \frac{5+2}{3-2} = 7$

If θ is the angle between them then, $tan\theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2} = \pm \left(\frac{7 - \frac{3}{2}}{1 + 7 \times \frac{3}{2}}\right) = \pm \left(\frac{11}{23}\right)$

Example: If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Sol: We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

Let $m_1 = \frac{1}{2}, m_2 = m, \theta = \frac{\pi}{4}$

Now, putting these values, we get

$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \Rightarrow 1 = \left|\frac{2m-1}{2+m}\right| \Rightarrow \pm 1 = \frac{2m-1}{2+m} \Rightarrow \frac{2m-1}{2+m} = 1$ or $\frac{2m-1}{2+m} = -1$

$\Rightarrow m = 3$ or $m = -\frac{1}{3}$

Condition of Parallelism of Lines

If two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is $0°$.

$\therefore \tan\theta = \tan 0° = 0$

$\Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \Rightarrow m_1 = m_2$

Thus, two lines are parallel if and only if their slopes are equal.

Example: What is the value of y so that the line through $(3,\ y)$ and $(2,\ 7)$ is parallel to the line through $(-1,\ 4)$ and $(0,\ 6)$?

Sol: Let $A(3,\ y)$, $B(2,\ 7)$, $C(-1,\ 4)$ and $D(0,\ 6)$ be the given points.

Then $m_1 =$ slope of the line $AB = \frac{7-y}{2-3} = y - 7$

and $m_2 =$ slope of the line $CD = \frac{6-4}{0-(-1)} = 2$

Since, AB and CD are parallel, so, $m_1 = m_2 \Rightarrow y - 7 = 2 \Rightarrow y = 9$.

Condition of Perpendicularity of Two Lines

If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is $90°$.

$\therefore \tan\theta = \tan 90° = \frac{1}{0}$

$\Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{1}{0} \Rightarrow m_1 m_2 = -1$

Thus, two lines are perpendicular if and only if the product of their slopes is -1.

Example: Line through the points $(-2,\ 6)$ and $(4,\ 8)$ is perpendicular to the line through $(8,\ 12)$ and $(x,\ 24)$. Find the value of x.

Sol: Slope of the line through the points $(-2,\ 6)$ and $(4,\ 8)$ is $m_1 = \frac{8-6}{4-(-2)} = \frac{1}{3}$

Slope of the line through the points $(8,\ 12)$ and $(x,\ 24)$ is $m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$

Since, the two lines are perpendicular, so $m_1 m_2 = -1$

$\Rightarrow \frac{1}{3} \times \frac{12}{x-8} = -1 \Rightarrow x = 4$

Collinearity of Three Points

If $A,\ B,\ C$ are three points in XY $-$plane, then they will be collinear $i.e.$ will lie on the same line if and only if the $slope\ of\ AB = slope\ of\ BC$.

Example: Prove that the points $A(1,\ 4)$, $B(3,\ -2)$, $C(4,\ -5)$ are collinear.

Sol: Now, slope of $AB = \frac{-2-4}{3-1} = -3$

Also, slope of $BC = \frac{-5+2}{4-3} = -3$

Since, the slope of $AB =$ slope of BC, so points A, B and C are collinear.

Various Forms of Equation of a Line

Locus and Equation to a Locus

Locus: The curve described by a moving point under given geometrical condition(s), is called locus of that point.

Equation of the Locus of a Point:

The equation of the locus of a point is the relationship that is satisfied by the coordinates of every point on the locus of the point.

A straight line is a curve, such that all the points on the line segment joining any two points of it lies on it.

The Straight Lines

Various Forms of Equation of Line

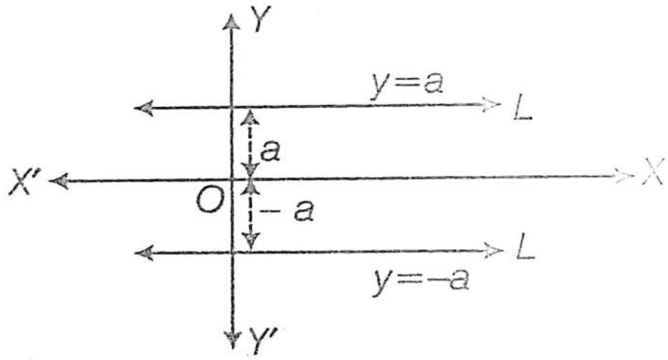
The equation of a straight line is the relation between x(the abscissa) and y (the ordinate) which is satisfied by the coordinates of each and every point on the line and is not satisfied by the coordinates of any point which does not lie on the line.

Equation of Line Parallel to $x - axis$ or Equation of a horizontal line

Let L be a straight line parallel to the $x - axis$ at a distance a from it, then the equation of the line L is $y = a$ or $y = -a$.

The choice of the sign will depend upon the position of the line according to the line is on the positive or negative side of the $-axis$.

The equation of $the\ x - axis$ is $y = 0$.



Example: Write down the equation of a line parallel to the $x - axis$ and at a distance of 6 units above the $x - axis$.

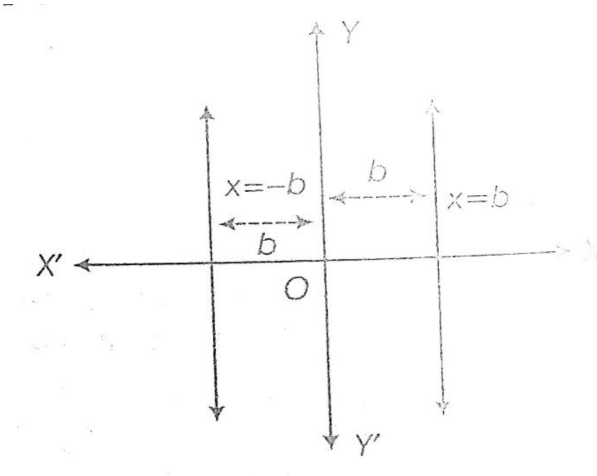
Sol: The equation of a line parallel to the $x - axis$ and at a distance of 6 units above the $x - axis$ is $y = 6$.

Equation of Line Parallel to $y - axis$ or Equation of vertical Line

Let L be a straight line parallel to $the\ y - axis$ at a distance b from it, then the equation of the line L is $x = b$ or $x = -b$.

The choice of the sign will depend upon the position of the line according to the line is on the positive or negative side of the $-axis$.

The equation of the $y - axis$ is $x = 0$.



Example: Find the equation of the lines parallel to the axes and passing through the point $(-3,\ 5)$.

Sol: The equation of a line parallel to the $x - axis$ and passing through $(-3,\ 5)$ is $y = 5$.

The equation of a line parallel to the $y - axis$ and passing through $(-3,\ 5)$ is $x = -3$.

Example: Find the equation of a line that is equidistant from the lines $x = -4$ and $x = 8$.

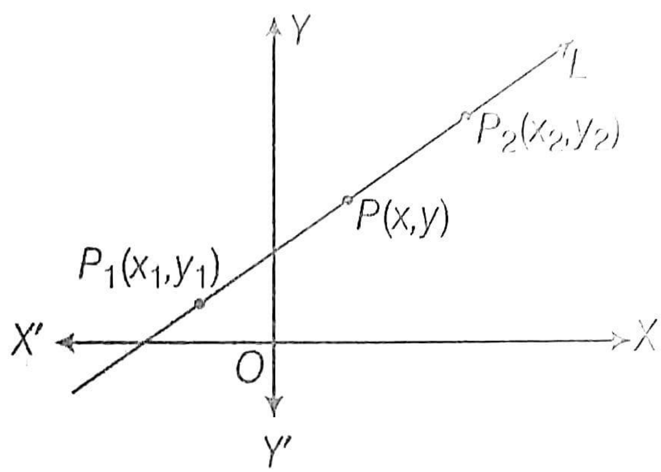
Sol: Since the given lines are both parallel to $the\ y-axis$ and the required line is equidistant from these lines, so it is also parallel to $the\ y-axis$ and its distance from the $y-axis$ is $\frac{1}{2}\ (-4+8)=2\ units$.

Hence, its equation is $x=2$.

Two-point Form of a Line

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y-\ y_1=\ \frac{y_2-y_1}{x_2-x_1}\ (x-x_1)$$



Example: Find the equation of the line joining the points $(-1,\ 3)$ and $(4,\ -2)$.

Sol: By two –point form, the equation of the line is

$$y-3=\ \frac{-2-3}{4-(-1)}(x-(-1))$$

$\Rightarrow\ y-3=-(x+1)$

$\Rightarrow\ y-3=-x-1$

$\Rightarrow\ x+y-2=0$

Example: If $A(2,\ 1)$, $B(-2,\ 3)$ and $C(4,\ 5)$ are the vertices of a ΔABC, then find the equation of the median through the vertex C.

Sol: Let F be the midpoint of side AB and CF is a median through C.

Now, coordinates of F are $\left(\frac{2+(-2)}{2},\ \frac{1+3}{2}\right)\ i.e.\ (0,\ 2)$.

So, the equation of the median CF is given by $y-5=\ \frac{2-5}{0-4}(x-4)$

$\Rightarrow 3x - 4y + 8 = 0.$

Point Slope Form

The equation of the straight line having slope m and passes through the point $(x_0,\ y_0)$ is

$$y - \ y_0 = m(x - x_0)$$

Example: Find the equation of the line passing through $(-4,\ 3)$ and having slope $\frac{1}{2}$.

Sol: Given,m = slope of the line $= \frac{1}{2}$ and $x_0 = -4$ and $y_0 = 3$.

Therefore, the equation of the line is $y - 3 = \frac{1}{2}\ (x - (-4))$

$\Rightarrow\ 2y - 6 = x + 4 \Rightarrow x - 2y + 10 = 0.$

Example: If the line joining two points $A(2,\ 0)$ and $B(3,\ 1)$ is rotated about A in an anti-clockwise direction through an angle of $15°$. Find the equation of the line in a new position.

Sol: The slope of line AB is given by $m = \ \frac{1-0}{3-2}$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°$

After rotation of the line AB about A is anticlockwise direction through an angle of $15°$, the slope of the line AC in a new position is given by $m_1 = \tan(45° + 15°) = \tan 60° = \ \sqrt{3}$.

Therefore the equation of the new line AC is $y - 0 = \ \sqrt{3}(x - 2)$

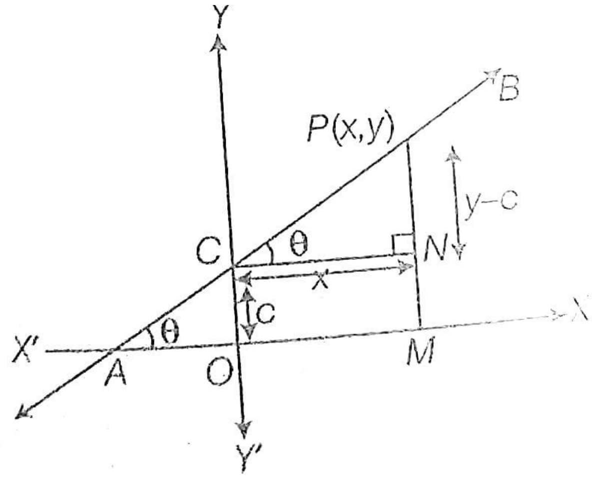
$\Rightarrow \sqrt{3}x\ - y - 2\sqrt{3} = 0.$

Slope Intercept Form

Suppose a line L with slope m cuts the $y - axis$ at a distance c from the origin. (distance c is called the y –intercept of the line).

Then the equation of the line is $y = mx + c$.

If a line L with slope m cuts the $x - axis$ at a distance d from the origin $i.e.$ Makes x –intercept d, then the equation of line L is $y = m(x - d)$.



NOTE: If the line passing through the origin then $c = 0$.

Therefore, the equation of the line passing through the origin is $y = mx$, where m is the slope of the line.

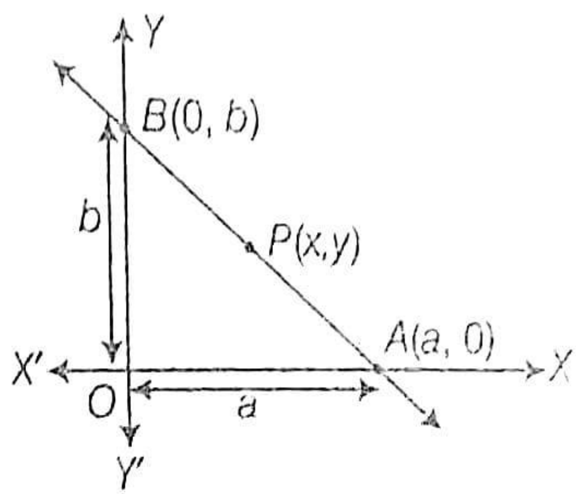
Example: Find the equation of the line which has slope $\frac{1}{2}$ and cuts – off an intercept -5 on $the\ y - axis$.

Sol: Given, $m =$ slope of the line $= \frac{1}{2}$ and $c = -5$.

Hence, the required equation of the line is $y = \frac{1}{2}x - 5 \Rightarrow x - 2y - 10 = 0$.

Intercept Form

The equation of a line which cuts – off intercepts a and b on the $x - axis$ and $y - axis$ respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.



Example: Find the equation of the lines which cuts –off intercepts on the axes whose sum and product are 1 and -6 respectively.

Sol: Let a and b be the intercepts of the line $x - axis$ and $-axis$, respectively.

Then the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1 \ldots (i)$

Given, sum of intercepts $a + b = 1 \ldots (ii)$ and product of intercepts, $ab = -6 \ldots (iii)$

On putting the value of b from (ii) in (iii), we get $a(1 - a) = -6$

$\Rightarrow a^2 - a - 6 = 0 \Rightarrow a = 3$ or -2.

From (iii), when $a = 3,\ b = -2$ and when $a = -2,\ b = 3$. Hence the required lines are $\frac{x}{3} + \frac{y}{-2} = 1$ and $\frac{x}{-2} + \frac{y}{3} = 1 \Rightarrow 2x - 3y - 6 = 0$ and $3x - 2y + 6 = 0$.

Example: Find the equation of the line which passes through the point $(-4,\ 3)$, the portion of the line intercepted between the axes is divided internally in the ratio 5 : 3 by this point.

Sol: Let the line intersects $x - axis$ and $y - axis$ respectively at $A(x,\ 0)$ and $B(0,\ y)$

Then $-4 = \frac{5\times 0+3x}{5+3} \Rightarrow \frac{3x}{8} = -4 \Rightarrow x = -\frac{32}{3}$

and $3 = \frac{5\times y+3\times 0}{5+3} \Rightarrow \frac{5y}{8} = 3 \Rightarrow y = \frac{24}{5}$

Thus the intercepts on the coordinate axes are $-\frac{32}{3}$ and $\frac{24}{5}$ respectively.

Hence, the required equation of the line is $\frac{x}{-\frac{32}{3}} + \frac{y}{\frac{24}{5}} = 1$

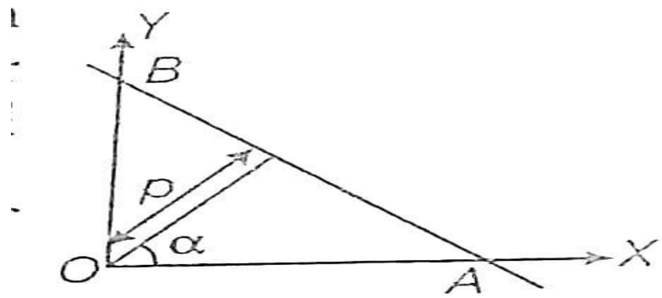
$\Rightarrow 9x - 20y + 96 = 0$

Normal Form or Perpendicular Form of a Line

The equation of a straight line upon which the length of the perpendicular $i.e$. Normal from the origin is p and this perpendicular makes an angle α with the positive direction of $the\ x - axis$ is

$$x\ cos\alpha + y\ sin\alpha = p$$

Here slope of line $= -\frac{1}{tan\alpha} = -\frac{cos\alpha}{sin\alpha}$



Example: The perpendicular distance of a line from the origin is 7 cm and its slope is -1. Find the equation of the line.

Sol: Here slope$(m) = -\frac{1}{tan\alpha} = -1 \Rightarrow tan\alpha = 1 \Rightarrow \alpha = 45°$

Thus the required equation of the line is $x\, cos\alpha + y\, sin\alpha = p$

$\Rightarrow \quad x \cos 45° + y\, sin 45° = 7$

$\Rightarrow \quad x \frac{.1}{\sqrt{2}} + y \,.\frac{1}{\sqrt{2}} = 7 \Rightarrow x + y - 7\sqrt{2} = 0.$

General Equation of a Line

An equation of the form $Ax + By + C = 0$, where $A,\ B,\ C \in R, \sqrt{A^2 + B^2} \neq 0$ is called general linear equation or general equation of the line.

Different Forms of $\boldsymbol{Ax + By + C = 0}$

The general equation of a line can be reduced into various forms of the equation of the line, which are given below.

Slope Intercept Form

If $B \neq 0$, then $Ax + By + C = 0$ can be written as $y = -\frac{A}{B}\, x + \left(-\frac{C}{B}\right)$

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$

Intercept Form

If $C \neq 0$, then $Ax + By + C = 0$ can be written as $\frac{x}{-C/A} + \frac{y}{-C/B} = 1$, where $a = -\frac{C}{A}$, $b = -\frac{C}{B}$

Normal Form

Let $x\, cos\omega + y \sin \omega = p$ be the normal form of the line represented by the equation

$$Ax + By + C = 0$$

or $Ax + By = -C$

Thus, both the equations are same and therefore $\frac{A}{cos\omega} = \frac{B}{sin\omega} = -\frac{C}{p}$

$\Rightarrow \; cos\omega = -\frac{Ap}{C}$ and $sin\omega = -\frac{Bp}{C}$

Since $sin^2\omega + cos^2\omega = 1$, so, $\left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$

$\Rightarrow \; \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2} = 1 \Rightarrow p^2 = \frac{C^2}{A^2+B^2} \Rightarrow p = \pm \frac{C}{\sqrt{A^2+B^2}}$

Therefore, $cos\omega = \pm \frac{A}{\sqrt{A^2+B^2}}$ and $sin\omega = \pm \frac{B}{\sqrt{A^2+B^2}}$

Thus the normal form of the equation $Ax + By + C = 0$ is $cos\omega + y \sin\omega = p$, where

$cos\omega = \pm \frac{A}{\sqrt{A^2+B^2}}$, $sin\omega = \pm \frac{B}{\sqrt{A^2+B^2}}$ and $p = \pm \frac{C}{\sqrt{A^2+B^2}}$ (p should be positive)

To transform the general equation of a line to the normal form, we use the following steps.

( )I Shift the constant term to the RHS and make it positive.

(ii) Divide both sides by $\sqrt{(coefficient\ of\ x)^2 + (coefficient\ of\ y)^2}$

The equation so obtained is in the normal form.

Example: Transform the equation of the line $3x + 2y - 7 = 0$ to

(i) slope-intercept form and also find the slope and y −intercept.

(ii) intercept form and the intercepts on the coordinate axes.

(iii) normal form and also find the inclination of the perpendicular segment from the origin on the line with the axis and its length.

Sol: (i) Given equation is $3x + 2y - 7 = 0$

It can be written as $2y = -3x + 7 \Rightarrow y = -\frac{3}{2}x + \frac{7}{2}$

which is the required slope-intercept form where slope $m = -\frac{3}{2}$ and y −intercept $c = \frac{7}{2}$.

(ii) Given equation can be rewritten as $3x + 2y = 7$

$\Rightarrow \; \frac{x}{7/3} + \frac{y}{7/2} = 1$, which is the required intercept form of the given line.

Here x −intercept $= \frac{7}{3}$ and y −intercept $= \frac{7}{2}$

(iii) Given equation is $3x + 2y - 7 = 0 \Rightarrow 3x + 2y = 7$

On dividing both sides by $\sqrt{3^2 + 2^2} = \sqrt{13}$, we get

$\frac{3}{\sqrt{13}} x + \frac{2}{\sqrt{13}} y = \frac{7}{\sqrt{13}}$, which is the required normal form of the given line.

Here $cos\omega = \frac{3}{\sqrt{13}}, sin\omega = \frac{2}{\sqrt{13}}$, and $p = \frac{7}{\sqrt{13}}$. Since $cos\omega$ and $sin\omega$ both are positive, so ω is in the first quadrant and is obtained from $\tan\omega = \frac{2}{3}$.

Example: Equation of a line is $3x - 4y + 10 = 0$. Find it's (i) slope (ii) x-intercepts (iii) y −intercepts.

Sol: (i) Given equation $3x - 4y + 10 = 0$ can be written as $y = \frac{3}{4}x + \frac{5}{2} \dots (1)$

Comparing (1) with $y = mx + c$, we have slope of the given line is $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as $3x - 4y = -10$

$\Rightarrow \frac{x}{-10/3} + \frac{y}{5/2} = 1 \dots (2)$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have $x - intercept$ as $a = -\frac{10}{3}$ and $y - intecept$ as $b = \frac{5}{2}$

The angle between Two Lines, having General Equations

Let, general equations of lines be $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.

Then the slope of given lines are $m_1 = -\frac{A_1}{B_1}$ and $m_2 = -\frac{A_2}{B_2}$

Let θ be the angle between two lines, then $tan\theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2} = \pm \left(\frac{-\frac{A_1}{B_1} + \frac{A_2}{B_2}}{1 + \frac{A_1}{B_1}\frac{A_2}{B_2}} \right)$

Example: Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}\,y - x + 6 = 0$.

Sol: Given lines are $y = \sqrt{3}x + 5 \dots (1)$ and $y = \frac{1}{\sqrt{3}} x - 2\sqrt{3} \dots (2)$

The slope of a line (1) is $m_1 = \sqrt{3}$ and slope of the line (2) is $m_2 = \frac{1}{\sqrt{3}}$

The acute angle θ between two lines is given by $tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

Putting the value of m_1 and m_2, we get $tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}}\right| = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = 30°$

Hence, the angle between the two lines is either $30°$ or $180° - 30° = 150°$.

Condition for two lines to be Parallel

If the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel, then their slopes are equal

$i.e.\ m_1 = m_2 \Rightarrow -\frac{A_1}{B_1} = -\frac{A_2}{B_2} \Rightarrow \frac{A_1}{B_1} = \frac{A_2}{B_2}$

Condition for two lines to be Perpendicular

If the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are perpendicular, then the product of their slopes is -1.
$i.e.\ m_1 \times m_2 = -1 \Rightarrow \left(-\frac{A_1}{B_1}\right) \times \left(-\frac{A_2}{B_2}\right) - 1 \Rightarrow A_1A_2 + B_1B_2 = 0.$

Example: A line passing through the points $(a,\ 2a)$ and $(-2,\ 3)$ is perpendicular to the line $4x + 3y + 5 = 0$. Find the value of a.

Sol: Let m_1 be the slope of the line joining $A(a,\ 2a)$ and $B(-2,\ 3)$.

Then $m_1 = \frac{3-2a}{a+2}$

Let m_2 be the slope of line $4x + 3y + 5 = 0$. Then $m_2 = -\frac{4}{3}$.

Since, the given lines perpendicular so, $m_1.m_2 = -1$

$\Rightarrow\ \frac{3-2a}{a+2} \times \left(-\frac{4}{3}\right) = -1$

$\Rightarrow\ 8a - 12 = 3a + 6$

$\Rightarrow\ a = \frac{18}{5}$

Point of Intersection of Two Lines

Lines Parallel and Perpendicular to a Given Line

Line Parallel to a Given Line

The equation of a line parallel to a given line $Ax + By + C = 0$ is $Ax + By + k = 0$, where k is a constant.

Example: Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5,\ -6)$.

Sol: The equation of any line parallel to the line $3x - 2y + 5 = 0$ $3x - 2y + k = 0$ ... (1)

This passes through $(5,\ -6)$, so $3(5) - 2(-6) + k = 0$

$\Rightarrow\ k = -27.$

Putting $k = -27$ in (1), we obtain $3x - 2y - 27 = 0$ as the required equation.

Line Perpendicular to a Given Line

The equation of a line perpendicular to a given $Ax + By + C = 0$ is $Bx - Ay + k = 0$, where k is a constant.

Example: Find the equation of the straight line that passes through the point $(3,\ 4)$ and perpendicular to the line $3x + 2y + 5 = 0$.

Sol: The equation of a line perpendicular to $3x + 2y + 5 = 0$ is $2x - 3y + k = 0$ … (1)

This passes through the point $(3,\ 4)$

So, $3 \times 2\ - 3 \times 4 + k = 0 \Rightarrow k = 6$

Putting $k = 6$ in (1), we obtain $2x - 3y + 6 = 0$ as the required equation.

Example: Find the equation of the line perpendicular to $x - 7y + 5 = 0$ and having x −intercept 3.

Sol: The equation of a line perpendicular to $x - 7y + 5 = 0$ is $7x + y + k = 0$ … (1)

Its x −intercept is 3. So it passes through the point $(3,\ 0)$ on the $x - axis$

Thus $7 \times 3 + 0 + k = 0 \Rightarrow k = -21$

Putting $k = -21$ in (1) we obtain $7x + y - 21 = 0$ as the equation of the required line.

Point of Intersection of Two Lines

Let the equations of two lines be $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.

Suppose these two lines intersect at a point $\mathrm{P}(x_1, y_1)$. Then (x_1, y_1) satisfies each of the given equations.

So $A_1x_1 +\ B_1y_1 + C_1 = 0$ and $A_2x_2 + B_2y_2 + C_2 = 0$.

By cross – multiplication we get

$$x_1 = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}\ ,\ y_1 = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}$$

Hence, the coordinates of the point of intersection of the given lines are

$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right)$$

Concurrent Lines

Three lines are said to be concurrent if they pass through a common point $i.e.$ They meet at a point.

Thus, if three lines are concurrent, the point of intersection of two lines lies on the third line.

Let $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ and $A_3x + B_3y + C_3 = 0$ be three lines.

Then the condition for which these three lines are concurrent is

$A_1(B_2C_3 - B_3C_2) + B_1(C_2A_3 - C_3A_2) + C_1(A_2B_3 - A_3B_2) = 0.$

Example: Show that the lines $x - y - 6 = 0$, $4x - 3y - 20 = 0$ and $6x + 5y + 8 = 0$ are concurrent. Also, find their common point of intersection.

Sol: The given lines are $x - y - 6 = 0 \ldots (1)$

$4x - 3y - 20 = 0 \ldots (2)$ and $6x + 5y + 8 = 0 \ldots (3)$

Solving (1) and (2) by cross – multiplication, we get

$\frac{x}{20-18} = \frac{y}{-24+20} = \frac{1}{-3+4} \Rightarrow x = 2$ and $y = -4$

Thus, the first two lines intersect at the point $(2, -4)$.

Putting $x = 2$ and $y = -4$ in (3), we get $6 \times 2 + 5 \times (-4) + 8 = 0$

Thus the point $(2, -4)$ lies on the line (3).

Hence, the given lines are concurrent and their common point of intersection is $(2, -4)$.

The distance of a Point from a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line.

The distance of the point (x_1, y_1) from the line $Ax + By + C = 0$ is

$$d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$$

The distance of origin from the line $Ax + By + C = 0$ is $d = \left|\frac{C}{\sqrt{A^2+B^2}}\right|$

Example: Find the distance of the point $(2, -3)$ from line $2x - 3y + 6 = 0$.

Sol: Required distance of the point from the line

$=$ The perpendicular distance from a point to the line

$$= \left|\frac{2 \times 2 - 3(-3) + 6}{\sqrt{2^2 + (-3)^2}}\right| = \frac{19}{\sqrt{13}}$$

Example: Find the points on $the\ x - axis$ whose perpendicular distance from the line $4x + 3y = 12$ is 4.

Sol: Given points are on $the\ x - axis$. So, let its coordinates be $(\alpha, 0)$

Then, the length of the perpendicular from $(\alpha,\ 0)$ to the line $4x+3y-12=0$ is 4.

So, $\left|\frac{4\alpha+3\times0-12}{\sqrt{4^2+(-3)^2}}\right| = 4$

$\Rightarrow\ \left|\frac{4\alpha-12}{5}\right| = 4$

$\Rightarrow\ |4\alpha-12| = 20$

$\Rightarrow\ 4\alpha - 12 = \pm 20$

$\Rightarrow\ 4\alpha = 12 \pm 20 = -8,\ 32$

$\Rightarrow\ \alpha = -2, 8$

Hence the required points are $(8,\ 0)$ and $(-2,\ 0)$

Equations of lines passing through a given point and making a given angle with a line

The equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Example: Find the equations of two straight lines through $(7,\ 9)$ and making an angle of $60°$ with the line $x - \sqrt{3}y - 2\sqrt{3} = 0$.

Sol: We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle α with $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here, $x_1 = 7, y_1 = 9,\ \alpha = 60°$ and $m = \left(Slope\ of\ the\ line\ x - \sqrt{3}y - 2\sqrt{3} = 0\right) = \frac{1}{\sqrt{3}}$

So, equations of required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60°}{1 - \frac{1}{\sqrt{3}}\tan 60°}(x-7) \text{ and } y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60°}{1 + \frac{1}{\sqrt{3}}\tan 60°}(x-7)$$

or, $(y-9)\left(1 - \frac{1}{\sqrt{3}}\tan 60°\right) = \left(\frac{1}{\sqrt{3}} + \tan 60°\right)(x-7)$

and, $(y-9)\left(\frac{1}{\sqrt{3}} + \tan 60°\right) = \left(1 - \frac{1}{\sqrt{3}}\tan 60°\right)(x-7)$

or, $0 = \left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)(x-7)$ and $(y-9)(2) = \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)(x-7)$

or, $x - 7 = 0\ \ and,\ \ x + \sqrt{3}y = 7 + 9\sqrt{3}$.

Hence, the required lines are $x = 7$ and $x + \sqrt{3}$y=7+9$\sqrt{3}$.

Distance between Two Parallel Lines

The distance between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$d = \left|\frac{c_1 - c_2}{\sqrt{1+m^2}}\right|$.

If the lines are given in the general form

$i.e.$ Given lines are $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$,

then distance between these lines is $d = \left|\frac{c_1 - c_2}{\sqrt{A^2+B^2}}\right|$

Example: Find the distance between the parallel lines $3x - 4y + 9 = 0$ and $6x - 8y - 15 = 0$.

Sol: Given lines are $3x - 4y + 9 = 0$ and

$6x - 8y - 15 = 0 \Rightarrow 3x - 4y - \frac{15}{2} = 0$

So, the required distance $= \left|\frac{c_1 - c_2}{\sqrt{A^2+B^2}}\right| = \left|\frac{9+\frac{15}{2}}{\sqrt{3^2+(-4)^2}}\right| = \frac{33}{10}\ units.$

Example: Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.

Sol: Equation of any line parallel to $3x - 4y - 5 = 0$ is

$3x - 4y + l = 0$ ....(i)

Putting $x = -1$ in $3x - 4y - 5 = 0$, we get $y = -2$. Therefore, $(-1, -2)$ is a point on $3x - 4y - 5 = 0$. Since the distance between the two lines is one unit. Therefore, the length of perpendicular from $(-1, -2)$ to $3x - 4y + l = 0$ is one unit.

$i.e.\ \frac{/3\times-1-4\times-2+\lambda/}{\sqrt{3^2+(-4)^2}} = 0$

$\Rightarrow \frac{|5+\lambda|}{5} = 1 \Rightarrow$

$\Rightarrow |5+\lambda| = 5 \Rightarrow 5 + \lambda = \pm 5 \Rightarrow \lambda = 0\ or\ -10.$

Substituting the values of λ in (i), we get $3x - 4y = 0$ and, $3x - 4y - 10 = 0$ as the equations of the required lines.

Family of Lines passing through the intersection of Two Lines

The equation of the family of lines passing through the intersection of the lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where λ is a parameter.

Example: Find the equation of the straight line which passes through the point $(2, -3)$ and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.

Sol: Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$$(x + y + 4) + \lambda(3x - y - 8) = 0 \qquad ....(1)$$

This will pass through $(2, -3)$, if

$$(2 - 3 + 4) + \lambda(6 + 3 - 8) = 0$$

$\Rightarrow\ 3 + l = 0$ Þ $l = -3.$

Putting the value of λ in (1), the equation of the required line is $2x - y - 7 = 0$.

Example: Find the equation of the straight line which passes through the point of intersection of the straight lines $x + 2y = 5$ and $3x + 7y = 17$ and is perpendicular to the straight line $3x + 4y = 10$.

Sol: The equation of any line through the intersection of the lines

$x + 2y - 5 = 0$ and $3x + 7y - 17 = 0$ is

$$(x + 2y - 5) + \lambda(3x + 7y - 17) = 0$$

or, $x(3\lambda + 1) + y(7\lambda + 2) - (17\lambda + 5) = 0 \qquad ....(1)$

This is perpendicular to the line $3x + 4y = 10$.

$\therefore\ \textit{Product of their slopes} = -1$

Þ $-\left(\frac{3l+1}{7l+2}\right)\left(-\frac{3}{4}\right) = -1 \Rightarrow$□$\lambda$=□$\frac{11}{37}$

Putting this value of λ in (1), the equation of the required line is $4x - 3y + 2 = 0$.

Distance Form of a Line

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of the x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, where r is the distance of the point (x, y) on the line from the point (x_1, y_1).

NOTE 1 The equation of line is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$\Rightarrow x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$ $\Rightarrow x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If P is on the right side of (x_1, y_1), then r is positive and if P is on the left side of (x_1, y_1), then r is negative.

Since different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

NOTE 2 In the above form one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point (x_1, y_1) on the line $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}$ there are two points $viz.$ $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Example: In what direction a line be drawn through the point $(1,2)$ that its point of intersection with the line $x + y = 4$ is at a distance $\frac{\sqrt{6}}{3}$ from the given point.

Sol: Let the line drawn through $A(1,2)$ makes an angle θ with the positive direction of the x-axis and intersects the line $x + y = 4$ at P such that $AP = \frac{\sqrt{6}}{3}$. Then, the coordinates of P are given by

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \frac{\sqrt{6}}{3} \Rightarrow x = 1 + \sqrt{\frac{2}{3}}\cos\theta, y = 2 + \sqrt{\frac{2}{3}}\sin\theta$$

So, the coordinates of P are $\left(1 + \sqrt{\frac{2}{3}}\cos\theta, 2 + \sqrt{\frac{2}{3}}\sin\theta\right)$.

Clearly, point P lies on the line $x + y = 4$.

$$\therefore\ 1 + \sqrt{\frac{2}{3}}\cos\theta + 2 + \sqrt{\frac{2}{3}}\sin\theta = 4$$

$$\Rightarrow \cos\theta + \sin\theta = \sqrt{\frac{3}{2}}$$

$$\Rightarrow (\cos\theta + \sin\theta)^2 = \frac{3}{2}$$

$$\Rightarrow 1 + \sin 2\theta = \frac{3}{2}$$

$$\Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6} \text{ or, } 2\theta = \frac{5\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Hence, the line drawn makes an angle whose measure is either $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ with the x-axis.

Shifting of Origin

Let O be the origin and let $x'Ox$ and $y'Oy$ be the axis of x and y respectively. Let O' and P be two points in the plane having coordinates (h, k) and (x, y) respectively referred to $X'OX$ and $Y'OY$ as the coordinate axes. Let the origin be transferred to O' and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of Preferred to new axes as the coordinate axes be (X, Y).

Then,

$O'N = X$, $PN = Y$, $OM = x$, $PM = y$, $OL = h$ and, $O'L = k$.

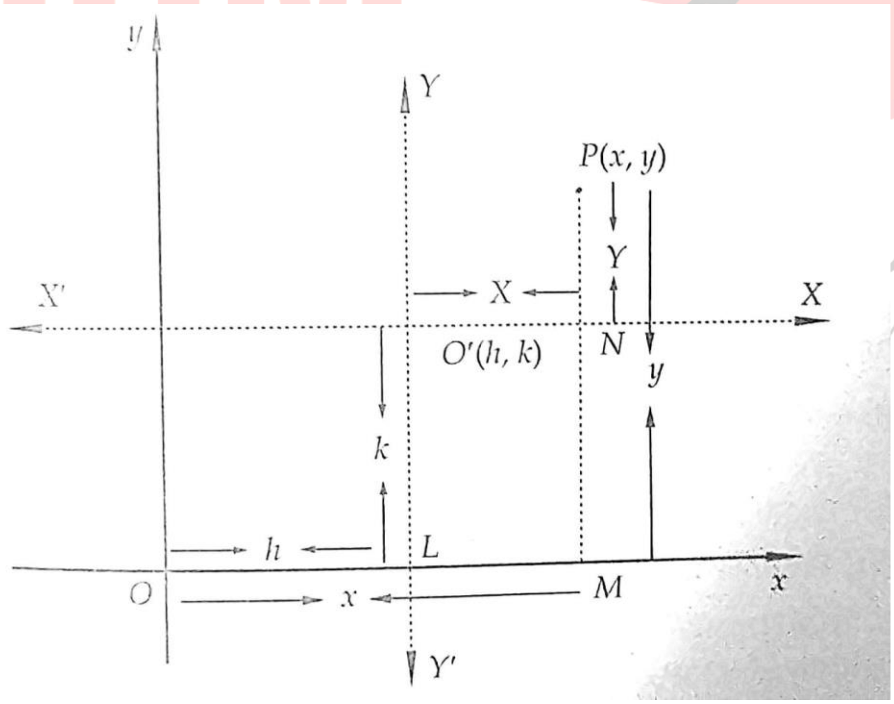
Now, $x = OM = OL + LM = OL + O'N = h + X$

and, $y = PM = PN + NM = PN + O'L = Y + k$

$\therefore$ $x = X + h$ and $y = Y + k$.

Thus, if (x, y) are the coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to the new axis, then

$x = X + h$ and $y = Y + k$



$i.e.,\ (Old\ x - coordinate) = (New\ x - coordinate) + h$

and, $(Old\ y - coordinate) = (New\ y - coordinate) + k$

If therefore the origin is shifted at a point (h, k) we must substitute $X + h$ and $Y + k$ for x and y respectively.

The transformation formula from new axes to old axes is: $X = x - h,\ \ Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

Example: If the axes are shifted to the point $(1, -2)$ without rotation, what do the following equations become?

$(i)\ 2x^2 + y^2 - 4x + 4y = 0$ $\qquad$ $(ii)\ y^2 - 4x + 4y + 8 = 0$

Sol: (i) Substituting $x = X + 1,\ \ y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$,

we get

$$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0 \quad \Rightarrow 2X^2 + Y^2 = 6$$

(ii) Substituting $x = X + 1,\ y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0 \quad \Rightarrow \ Y^2 = 4X$$

Example: At what point the origin be shifted, if the coordinates of a point $(4,\ 5)$ become $(-3,\ 9)$?

Sol: Let $(h,\ k)$ be the point to which origin is shifted. Then,

$x = 4,\ \ y = 5,\ \ X = -3,\ \ Y = 9$

$\therefore \quad x = X + h$ and $y = Y + k \ \Rightarrow 4 = -3 + h$ and $5 = 9 + k$

$\Rightarrow \ h = 7\ \ and\ k = -4$

Hence, the origin must be shifted to $(7,\ -4)$.