Chapter-8

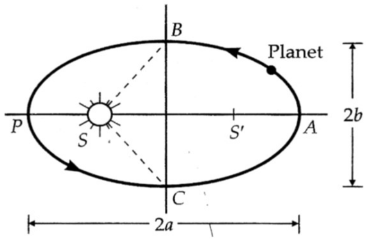
Gravitation

STUDY NOTE

Class - 1

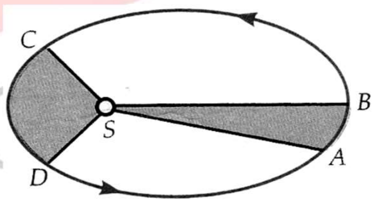
Kepler's laws of planetary motion:-

To explain the motion of the planets, Kepler formulated the following three laws.



Law of orbital (First Law):- "*Each planet revolves around the sun in an elliptical orbit with the sun situated at one of the two foci.*"

As shown in the figure, the planets move around the sun in an elliptical orbit. An ellipse has two foci S and S', the sun remains at one focus S. The points P and A on the orbit are called the perihelion and the aphelion and represent the closest and farthest distances from the sun respectively. The orbits of Mercury are highly elliptical. The orbits of Neptune and Venus are circular. The orbits of other planets have slight elasticity and may be taken as nearly circular.



Law of area (Second Law):- '*The radius vector drawn from the sun to a planet sweeps out equal areas in equal intervals of time i.e, the areal velocity (area covered per unit time) of a planet around the sun is constant.*'

Suppose a planet takes some time to go from position A to B as in going from C to D. From Kepler's second law, the area ASB and CSD (covered in equal time) must be equal. The planet covers a large distance CD when it is near the sun than AB when it is farther away in the same interval of time. Hence the **linear velocity of a planet is more when it is closer to the sun then its linear velocity when away from the sun.**

Law of periods (Third law):- "*The square of the period of revolution of a planet around the sun is proportional to the cube of the semimajor axis of its elliptical orbit.*"

If T is the period of revolution of a planet and R is the length of the semi-major axis of its elliptical orbit, then

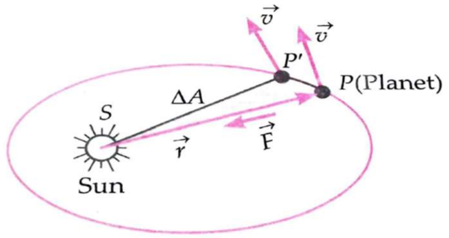
$T^2 \propto R^3$ Or $T^2 = KR^3$

Where K is proportionality constant.

For two different planets, we can write $\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$

Kepler's second law is under conservation of Angular Momentum-

As shown in figure consider a planet moving in an elliptical orbit with the sun at focus S. Let $\vec{r}$ be the position vector of the planet w.r.t the sum and $\vec{F}$ be the gravitational force on the planet due to the sun. The torque exerted on the planet by this force about the sun is



$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

But $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

$$\therefore \frac{d\vec{L}}{dt} = 0 \text{ or } \vec{L} = \text{ constant}$$

Suppose the planet moves from position p to p' in time Δt. The area swept by the radius vector $\vec{r}$ is

$\Delta\vec{A} =$ Area of triangular region SPP'.

$$= \frac{1}{2}\vec{r} \times \overrightarrow{PP'}$$

But $\vec{P}p' = \Delta\vec{r} = \vec{v}\,\Delta t = \dfrac{\vec{p}}{m}\Delta t$

$$\therefore \Delta\vec{A} = \frac{1}{2}\vec{r} \times \frac{\vec{p}}{m}\Delta t$$

Or $\dfrac{\Delta\vec{A}}{\Delta t} = \dfrac{1}{2m}(\vec{r} \times \vec{p}) = \dfrac{\vec{L}}{2m}$

Or $\dfrac{\Delta\vec{A}}{\Delta t} = $ constant

Thus the areal velocity of the planet remains constant. i.e the radius vector joining planet to the sun sweeps out equal areas in equal intervals of time. This proves Kepler's second law of planetary motion.

Example:- Calculate the period of revolution of Neptune around the sun, given that diameter of its orbit is 30 times the diameter of earth's orbit around the sun, both orbits being assumed to be circular.

Solution:- According to Kepler's law of periods.

$$\left[\frac{T_N}{T_E}\right]^2 = \left[\frac{r_N}{r_E}\right]^3$$

But $\frac{r_N}{r_E} = 30$ and $T_E = 1\,\text{year}$

$$\therefore\ T_N^2 = T_E^2\left[\frac{r_N}{r_E}\right]^3 = (1)^2 \times (30)^3 = 27000$$

$$T_N = \sqrt{27000} = 164.3\,\text{years}$$

Example:-In Kepler's law of periods: $T^2 = kr^3$, the constant $k = 10^{-13}s^2m^{-3}$. Express the constant k in days and kilometres. The moon is at a distance of 3.84×10^5km from earth. Obtain its period of revolution in days.

Solution:-

Given $k = 10^{-13}\frac{s^2}{m^3}$

As $1s = \frac{1}{24\times60\times60}$day and $1m = \frac{1}{1000}$km

$$\therefore\quad k = 10^{-13}\times\frac{1}{(24\times60\times60)^2}d^2\frac{1}{(1/1000)^3}km^{-3}$$

$$= 1.33\times10^{-14}d^2km^{-3}$$

For the moon, $r = 3.84\times10^5$km

$$\therefore\ T^2 = kr^3 = 1.33\times10^{-14}\times\left(3.84\times10^5\right)^3 = 27.3\,\text{days}$$

Example:-{NCERT} The planet Mars has two moons, Phobos and Deimos.

(i) Phobos has a period of 7 hours, 39 minutes and an orbital radius of 9.4×10^3km. Calculate the mass of Mars.

(ii) Assume that earth and Mars move in circular orbits around the Sun, with the Martian orbit being 1.52 times the orbital radius of the Earth. What is the length of the Martian year in days?

Solution:-

(i) Here $r = 9.4\times10^6$m, T = 7 h 39 min = 459min = 459×60s

Mass of Mars, $M_M = \frac{4\pi^2r^3}{GT^2} = \frac{4\times9.87\times(9.4)^3\times10^{18}}{6.67\times10^{-11}\times(459\times60)^2} = 6.48\times10^{23}$kg

(ii) Here $r_m = 1.52 r_e, T_E = 365 \text{days}$

According to Kepler's law of periods, $\frac{T_M^2}{T_E^2} = \frac{R_M^3}{R_E^3}$

$$\therefore \quad T_M = \left(\frac{R_M}{R_E}\right)^{3/2} T_E = (1.52)^{3/2} \times 365 = 684 \text{ days}$$

Example:- The distances of two planets from the sun are 10^{13} m and 10^{12} m respectively. Find the ratio of periods and speeds of the two planets.

Solution:- Here $r_1 = 10^{13} \text{m}, r_2 = 10^{12} \text{m}$

From Kepler's third law, $\frac{T_1}{T_2} = \left[\frac{r_1}{r_2}\right]^{3/2} = \left[\frac{10^{13}}{10^{12}}\right]^{3/2} = 10\sqrt{10}$

If v_1 and v_2 are the orbital speeds of the planets, then $v_1 = \frac{2\pi r_1}{T_1}$ and $v_2 = \frac{2\pi r_2}{T_2}$

$$\therefore \quad \frac{v_1}{v_2} = \frac{r_1}{r_2} \cdot \frac{T_2}{T_1} = \frac{r_1}{r_2}\left[\frac{r_2}{r_1}\right]^{3/2} = \left[\frac{r_2}{r_1}\right]^{1/2}$$

$$= \left[\frac{10^{12}}{10^{13}}\right]^{1/2} = 1/\sqrt{10}$$

Problems for Practice:-

01. If the earth is one half its present distance from the sun, how many days will the present one year on the surface of the earth change?

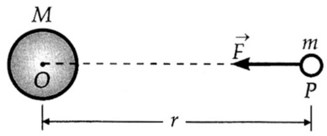
02. The distance of planet Jupiter from the sun is 5.2 times that of the earth. Find the period of revolution of Jupiter around the sun.

03. The planet Neptune travels around the sun with a period of 165 years. Show that the radius of its orbit is approximately 30 times that of earth's orbit, both being considered as circular.

04. A geostationary satellite is orbiting the earth at a height 6R above the surface of the earth, where R is the radius of the earth. Find the period of another satellite at a height of 2.5 R from the surface of the earth in hours.

05. The radius of earth's orbit is $1.5 \times 10^8 \text{km}$ and that of Mars is $2.5 \times 10^{11} \text{m}$. In how many years, do Mars completes its one revolution?

Class – 2



Statement of Newton's law of gravitation:- In 1967, Newton published the universal law of gravitation in his book Principia which can be stated as follows.

"Every particle in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them and this force acts along the line joining the two particles."

Consider two bodies of masses m_1 and separated by distance r. According to the law gravitation, the force of attraction F between them is such that.

$$F \propto m_1 m_2 \text{ and } F \propto \frac{1}{r^2}$$

$$\therefore \quad F \propto \frac{m_1 m_2}{r^2} \text{ or } F = G\frac{m_1 m_2}{r^2}$$

Where G is a constant called universal gravitational constant.

Definition of G:- If $m_1 = m_2 = 1$ and $r = 1$, then $F = G$

G is defined as the force of attraction between two bodies of unit mass each and placed a unit distance apart.

The dimension of G:- As $F = G\frac{m_1 m_2}{r^2}$

$$\therefore \quad G = \frac{Fr^2}{m_1 m_2}$$

$$\text{Dimensions of } G = \frac{MLT^{-2} \times L^2}{M \times M} = \left[M^{-1}L^3T^{-2}\right]$$

Unit of G:- As $G = \frac{Fr^2}{m_1 m_2}$

$$\therefore \quad \text{SI unit of } G = \frac{Nm^2}{kg \times kg} = Nm^2kg^{-2}$$

Similarly, cgs unit of G = dyn cm^2 g^{-2}

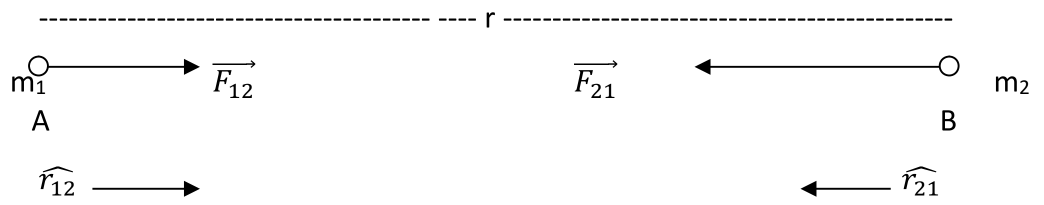
Value of G:- In SI, $G = 6.67 \times 10^{-11} Nm^2\ kg^{-2}$

In the cgs system, $G = 6.67 \times 10^{-8} dyn\ cm^2\ g^{-2}$

The value of G does not depend on the nature and size of the bodies. It also does not depend on the nature of the medium between the two bodies. That is why G is called universal gravitational constant.

Vector Form Of Law of Gravitation—

Let two bodies A and B of masses m_1 and m_2 are separated by distance r.



Let $\widehat{r_{12}}$ = unit vector from A to B

$\widehat{r_{21}}$ = unit vector from B to A

$\overrightarrow{F_{12}}$ = Force exerted on A by B

$\overrightarrow{F_{21}}$ = Force exerted on B by A

$\overrightarrow{F_{12}} = -\frac{Gm_1m_2}{r^2}\overrightarrow{r_{21}}$ (negative sign is due to $\overrightarrow{F_{12}}$ is opposite to $\widehat{r_{21}}$). Also, Gravitational force is Attractive in nature.

Similarly $\overrightarrow{F_{21}} = -\frac{Gm_1m_2}{r^2}\overrightarrow{r_{12}}$

But $\widehat{r_{21}} = -\widehat{r_{12}}$.

Hence $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

The gravitational force acting between two particles form action and reaction pair. As these two forces are directed towards the centre of two bodies, so Gravitational force is a central force.

Principle of superposition of gravitational forces:- According to the principle of superposition, the gravitational force between two masses acts independently and uninfluenced by the presence of other bodies. Hence the resultant gravitational force acting of a particle due to some masses is the vector sum of the gravitational forces exerted by the individual masses on the given particle.

Mathematically $\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i$

Shell Theorem- To find out the gravitational force between an extended object like earth and a Point mass., this theorem is applied.

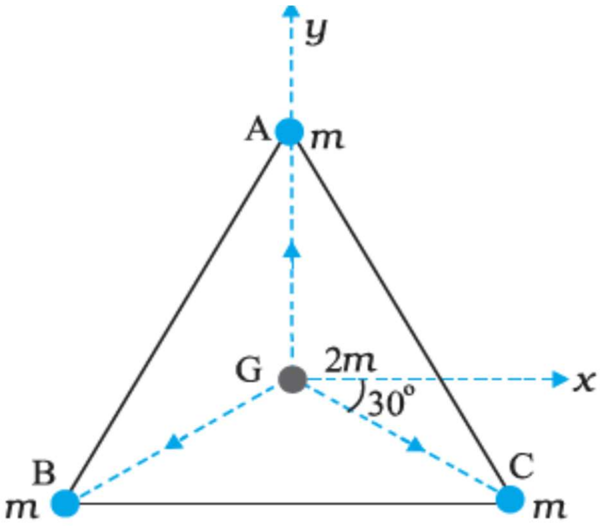
(1) **The force of attraction between a hollow spherical shell of uniform density and a point mass situated outside is just as if the entire mass of the shell is concentrated at the centre of the shell.**

(2) The force of attraction due to a hollow spherical shell of uniform density, on a point mass situated inside it, is Zero.

NCERT EXAMPLE:-Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.

(a)What is the force acting on a mass 2m placed at the centroid G of the triangle?

(b)What is the force if the mass at vertex A is doubled?



Take AG=BG=CG=1m.

SOLUTION:- The angle between GC and the + X-axis is 30^0 and so is the angle between GB and negative X-axis. The individual forces in vector notation are

$\vec{F}_{GA} = \frac{Gm(2m)}{1}\vec{j}$

$\vec{F}_{GB} = \frac{Gm(2m)}{1}(-\vec{\imath}\cos 30^0 - \vec{j}\sin 30^0)$

$\vec{F}_{Gc} = \frac{Gm(2m)}{1}(\vec{\imath}\cos 30^0 - \vec{j}\sin 30^0)$

From the principle of superposition and law of vector addition, the resultant gravitational Force $\overrightarrow{F_R}$ On (2m) is

$\overrightarrow{F_R} = \vec{F}_{GA} + \vec{F}_{GB} + \vec{F}_{GC}$

$=2Gm^2\vec{j} + 2Gm^2(-\vec{\imath}\cos 30^0 - j\sin 30^0) + 2Gm^2(\vec{\imath}\cos 30^0 - j\sin 30^0)$

$=0$

That is the resultant force is Zero.

(b)By symmetry, the x component of force cancels out. The Y component survives.

$\overrightarrow{F_R} = 4Gm^2\vec{j} - 2Gm^2\vec{j} = 2Gm^2\vec{j}$

Example:- A mass M is broken into two parts of masses m_1 and m_2. How are m_1 and m_2 related so that force of gravitational attraction between the two parts is maximum?

Solution:- Let m_1 = m, then m_2 = M-m

Force of gravitation between the two parts when they are placed distance r apart is

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(Mm - m^2)$$

Differentiating w.r.t m, we get $\frac{dF}{dm} = \frac{G}{r^2}(M-2m)$

For f to be maximum, $\frac{dF}{dm} = 0$ or $\frac{G}{r^2}(M-2m) = 0$

Or M= 2m or m= M/2

$\therefore \quad m_1 = m_2 = M/2$

Example:- Two particles, each of mass m, go round a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.

Solution:- The force on each particle is directed along the radius of the circle. The two particles will always lie at the ends of a diameter so that distance between them is 2R.

$$\therefore \quad F = G\frac{m \times m}{(2R)^2} = \frac{Gm^2}{4R^2}$$

As this force provides the centripetal force, so $\frac{Gm^2}{4R^2} = \frac{mv^2}{R}$ or $v = \sqrt{\frac{Gm}{4R}}$

Example:- Calculate the force of attraction between two balls each of mass 1kg each when their centres are 10 cm apart. Given $G = 6.67 \times 10^{-11} Nm^2kg^{-2}$.

Solution:- Here, $m_1 = m_2 = 1kg, r = 10cm = 0.10m$

$$\therefore \quad F = \frac{Gm_1m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 1 \times 1}{(0.10)^2} = 6.67 \times 10^{-9} N$$

Example:- The mean orbital radius of the earth around the sun is 1.5×10^8 km. Calculate the mass of the sun if $G = 6.67 \times 10^{-11} Nm^2kg^{-2}$.

Solution:- Here $r = 1.5 \times 10^8 km = 1.5 \times 10^{11} m$

$T = 365 day = 365 \times 24 \times 3600$ s

∴ Centripetal force required = Force of gravitation

$$\therefore \quad \frac{mv^2}{r} = \frac{GMm}{r^2} \quad \text{or} \quad \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{GMm}{r^2}$$

$$\text{Or} \quad M = \frac{4\pi^2 r^3}{GT^2} \qquad = \frac{4\times 9.87\times\left(1.5\times 10^{11}\right)^3}{6.67\times 10^{-11}\times\left(365\times 24\times 3600\right)^2} = 2.01\times 10^{30}\,\text{kg}$$

Problems for Practice:-

01. Aspherical mass of 20 kg lying on the surface of the earth is attracted by another spherical mass of 150 kg with a force equal to the weight of 0.25 mg. The centres of the two masses are 30 cm apart. Calculate the mass of the earth. The radius of the earth = 6×10^6 m

02. Assuming the earth to be a uniform sphere of radius 6400 km and density 5.5 g cm^{-3}, find the value of g on its surface. Given $G = 6.66\times 10^{-11}\,Nm^2\,kg^{-2}$

03. The value of g on the surface of the earth is 9.81 ms^{-2}. Find its value on the surface of the moon. Given the mass of earth = 6.4×10^{24} kg, a radius of the earth $= 6.4\times 10^6$ m, a mass of moon = $= 6.4\times 10^{22}$ kg, a radius of moon = 1.76×10^6 m.

04. An astronaut on the moon measures the acceleration due to gravity to be 1.7 ms^{-2}. He knows that the radius of the moon is about 0.27 times that of the earth. Find the ratio of the mass of the earth to that of the moon, if the value of g on the earth's surface is 9.8 ms^{-2}.

05. The acceleration due to gravity on the surface of the earth is 10ms^{-2}. The mass of the planet Mars as compared to earth is 1/10 and the radius is 1/2. Determine the gravitational acceleration of a body on the surface of Mars.

Class – 3

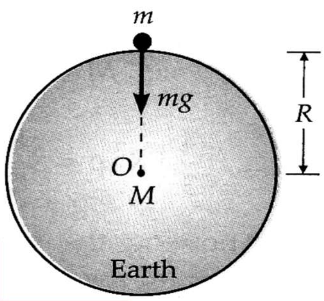
Acceleration due to gravity:-

When a body falls freely towards the surface of the earth, its velocity continuously increases. The acceleration developed in its motion is called acceleration due to gravity.

The acceleration produced in a freely falling body under the gravitational pull of the earth is called acceleration due to gravity. **It is denoted by g.**

It is a vector having direction towards the centre of the earth. It does not depend on the mass, size and shape of the body. The value of g is constant at a given place. However, it varies from place to place to place on the surface of the earth.

It depends on altitude, depth, rotation of the earth and shape of the earth. Near the **surface of the earth,** $g = 9.8\,ms^{-2}$ or $32\,ft s^{-2}$.



The relation between g and G:- Consider the earth to be a sphere of mass M and radius R. Suppose a body of mass m is lying on its surface., as shown in the figure. According to the law of gravitation, the force of attraction between the earth and the body is $F = \frac{GMm}{R^2}$.

The force of gravity F produces an acceleration g (called acceleration due to gravity) in the body of mass m. From Newton's second law of motion, we get. F = mg

From the above two equations, we have $mg = \frac{GMm}{R^2}$ or $g = \frac{GM}{R^2}$

This gives acceleration due to gravity on the surface of the earth. The value of g is independent of the mass, size and shape of the body falling under gravity.

Example:-{NCERT} You are given the following data: $g = 9.81\,ms^{-2}, R_E = 6.37 \times 10^6\,m$, the distance to the moon $r = 3.84 \times 10^8 m$ and the period of the moon's revolution is 27.3 days. Obtain the mass of the earth M_E in two different ways.

Solution:- (i) $M_E = \frac{gR_E^2}{G} = \frac{9.81 \times (6.37 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.97 \times 10^{24} kg$

(ii) From Kepler's law of periods, $M_E = \frac{4\pi^2 r^2}{GT^2} = \frac{4 \times 3.14 \times 3.14 \times (3.84)^3 \times 10^{24}}{6.67 \times 10^{-11} \times (27.3 \times 24 \times 60 \times 60)^2} = 6.02 \times 10^{24} kg$

Both the methods give almost the same mass, the difference being less than 1%.

Example:- The acceleration due to gravity at the moon's surface is $1.67ms^{-2}$. If the radius of the moon is $1.74\times10^{6}m$. Calculate the mass of the moon. Use the known value of G.

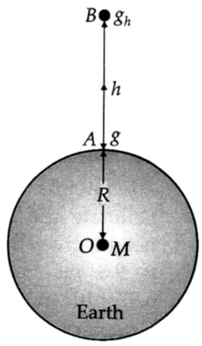
Solution:- Here $G=6.67\times10^{-11}Nm^{2}kg^{-2}$

$g=1.67ms^{-2}$, $R=1.74\times10^{6}m$

$$M=\frac{gR^2}{G}=\frac{1.67\times\left(1.74\times10^{6}\right)^2}{6.67\times10^{-11}}=7.58\times10^{22}kg$$

<u>Variation of g with altitude (Height)</u>:-

Consider the earth to be a sphere of mass M, radius R and centre O. Then the acceleration due to gravity at a point A on the surface on the earth will be.



$$g=\frac{GM}{R^2} \text{ .... (i)}$$

It g_h is the acceleration due to gravity at a point B at a height h from the earth's surface, then

$$g_h=\frac{GM}{(R+h)^2} \text{ .... (ii)}$$

Dividing equation (ii) by (i) we get. $\frac{g_h}{g}=\frac{GM}{(R+h)^2}\times\frac{R^2}{GM}$

Or $\frac{R_h}{g}=\frac{R^2}{(R+h)^2}$ .... (iii)

Or $\frac{g_h}{g}=\frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2}=\left(1+\frac{h}{R}\right)^{-2}$

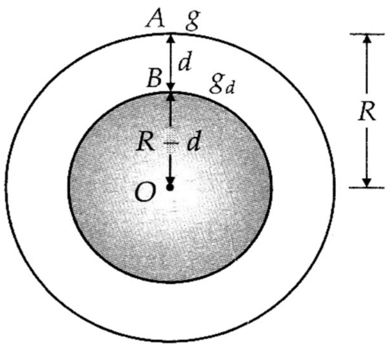
Expanding RHS by using binomial theorem, we get $\frac{g_h}{g}=1-\frac{2h}{R}+$ terms containing higher power of $\frac{h}{R}$.

If h << R, then $\frac{h}{R}<<1$, so that higher powers $\frac{h}{R}$ can be neglected, we get. $\frac{g_h}{g}=1-\frac{2h}{R}$

Or $g_h=g\left(1-\frac{2h}{R}\right)$.............................. (iv)

Both equations (iii) and (iv) show that the value of acceleration due to gravity decreases with the increase in height h.

Variation of g with depth:-



Consider the earth to be a sphere of mass M, radius R and centre O. The acceleration due to gravity at any point A on the surface at the earth will be.

$$g=\frac{GM}{R^2}$$

Assuming the earth to be a homogeneous sphere of average density ρ, then its total mass will be

M = volume X density = $\frac{4}{3}\pi R^3\rho$

$$\therefore \qquad g=\frac{G\times\frac{4}{3}\pi R^3\rho}{R^2}$$

Or $\qquad g=\frac{4}{3}\pi GR\rho$

Let g_d be the acceleration due to gravity at a point B at depth d below the surface of the earth. A body at B is situated at the surface of the inner solid sphere and lies inside the spherical shell of thickness d. The gravitational force of attraction on a body inside a spherical shell is always zero. Therefore, a body at B experiences a gravitational force due to the inner shaded sphere of radius $(R-d)$ and mass M, where

$$M'=\frac{4}{3}\pi(R-d)^3\rho$$

$$\therefore \qquad g_d=\frac{GM'}{(R-d)^2}=\frac{G}{(R-d)^2}\times\frac{4}{3}\pi(R-d)^3\rho$$

Or $\qquad g_d=\frac{4}{3}\pi G(R-d)\rho$

$$\frac{g_d}{g}=\frac{\frac{4}{3}\pi G(R-d)\rho}{\frac{4}{3}\pi GR\rho}=\frac{R-d}{R}=1-\frac{d}{R}$$

Or $\qquad g_d=g\left(1-\frac{d}{R}\right)$

The acceleration due to gravity decreases with the increase in-depth d. That is why the acceleration due to gravity is less in mines than on earth's surface.

The relation between height h and depth d for the same charge in g:- Acceleration due to gravity at a height h above the earth's surface,

$$g_h = g\left(1-\frac{2h}{R}\right) \text{for } h << R$$

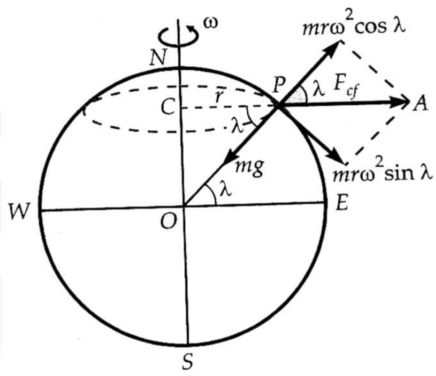
Acceleration due to gravity at a depth d below the earth's surface,

$$g_d = g\left(1-\frac{d}{R}\right)$$

For the same change in g, we have

$$\therefore \quad 1-\frac{2h}{R} = 1-\frac{d}{R} \text{ or } \frac{2h}{R} = \frac{d}{R} \text{ or } d = 2h$$

Hence the acceleration due to gravity at a height h above the earth's surface will be same as that at depth $d = 2h$, below the earth's surface. But this fact holds only when h << R.



Effect of latitude or rotation of the earth on g:- As the earth rotates about its polar axis, every particle lying on its surface also revolves along a horizontal circle with the same angular velocity ω.

Let a particle of mass m lying at point p, whose latitude is λ. The particle P describes a horizontal circle of radius, $r = PC = R\cos\lambda$.

The centrifugal force acting on the particle is

$F_{cf} = mr\omega^2$, acting along with PA.

This force has two rectangular components $mr\omega^2\cos\lambda$ and $mr\omega^2\sin\lambda$. The component $mr\omega^2\sin\lambda$ acts perpendicular to mg and does not affect it. The component $mr\omega^2\cos\lambda$ acts opposite to mg. So the apparent weight of the particle P is

$$mg_\lambda = mg - mr\omega^2\cos\lambda$$

Or $\quad g_\lambda = g - r\omega^2\cos\lambda$

Or $\quad g_\lambda = g - R\omega^2\cos^2\lambda$

As λ increases, $\cos\lambda$ decreases and g_λ increases. So as we move from equator to pole, the acceleration due to gravity increases.

(**Note--g is maximum at Poles, minimum at equator and Zero at the centre of the earth.**)

Example:- At what height above the earth's surface, the value of g is the same as in a mine 80 km deep?

Solution:- Let h be the height at which 'g' is the same as that at depth d. Now

$g_h = g_d$ or $g\left(1-\frac{2h}{R}\right) = g\left(1-\frac{d}{R}\right)$ Or $\frac{2h}{R} = \frac{d}{R}$ $\therefore$ $h = \frac{d}{2} = \frac{80}{2} = 40$ km

Example:- At what height above the earth's surface, the value of g is half of its value on the earth's surface? Given its radius is 6400 km.

Solution:- Here $g_h = g/2$

But $g_h = g\left(\frac{R}{R+h}\right)^2$

$\therefore$ $\frac{g}{2} = g\left(\frac{R}{R+h}\right)^2$ or $\left(\frac{R}{R+h}\right)^2 = \frac{1}{2}$

Or $\frac{R+h}{R} = \sqrt{2}$

Or $h = \left(\sqrt{2}-1\right)R = 0.414R = 0.414 \times 6400 = 2649.6$ km

Example:- Find the percentage decrease in the weight of u body when taken to a height of 32 km above the surface of the earth. The radius of the earth is 6400 km.

Solution:- Here h = 32 km, R = 6400 km

As h << R, so $g_h = g\left(1-\frac{2h}{R}\right) = g - \frac{2gh}{R}$

Or $g - g_h = \frac{2gh}{R}$

Per cent decrease in weight $= \frac{mg - mg_h}{mg} \times 100 = \frac{g - g_h}{g} \times 100$

$= \frac{2gh}{g \times R} \times 100 = \frac{2h}{R} \times 100$

$= \frac{2 \times 32}{6400} \times 100 = 1\%$

Problems for Practice:-

01. Find the value of g at a depth of 640 km from the surface of the earth. Given the radius of the earth. R = 6400 km and the value of g on the surface of the earth is 9.8 ms^{-2}.

02. Calculate the depth below the surface of the earth where the acceleration due to gravity becomes half of its value at the surface of the earth. The radius of the earth = 6400 km.

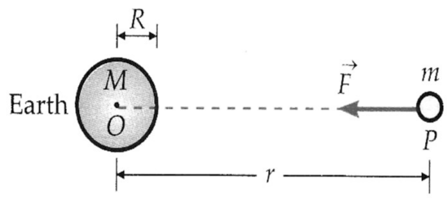
03. How much below the surface of the earth does the acceleration due to gravity become 70% of its value at the surface of the earth? The radius of the earth is 6400km.

4. The radius of the earth is 6000 km. What will be the weight of a 120 kg body if it is taken to a height of 2000 km above the surface of the earth?

5. How far away from the surface of the earth does the acceleration due to gravity become 4% of its value on the surface of the earth? The radius of earth =6400Km.

Lecture – 4

<u>The intensity of gravitational field due to a body</u>:- Consider a body of mass M. To determine its gravitational field intensity at a point P at distance r from its centre O, place a test mass m(m << M) at the point P.



Let $\vec{F}$ be the force of gravitation experienced by test mass m. The gravitational field intensity at point P will be.

$$\vec{E} = \frac{\vec{F}}{m} \text{ .... (i)}$$

The direction $\vec{E}$ is the same as that of $\vec{F}$.

According to Newton's law of gravitation,

$$F = \frac{GMm}{r^2}$$

$$\therefore \therefore E = \frac{F}{m} = \frac{GM}{r^2} \text{ .... (ii)}$$

At $r = \infty$, E = 0. Thus, gravitational field intensity decreases as distance r increases and becomes zero at infinity.

If the test mass m is free to move, it will move towards the mass M with acceleration a under the force F, so

$$a = \frac{F}{m} \text{ .... (iii)}$$

From equation (ii) and (iii) we get a = E.

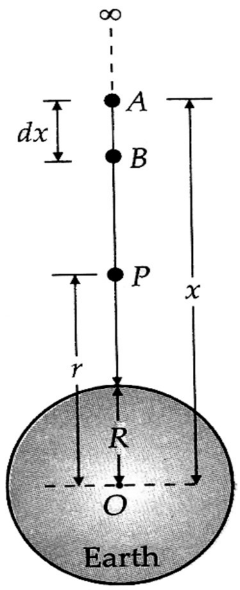
Units of E:- As gravitational field intensity is force per unit mass, so its SI unit is Nkg^{-1} and cgs unit is $dyng^{-1}$

Dimensions of E:-

As $E = \frac{F}{m}$

$\therefore$ Dimension of $E = \frac{MLT^{-2}}{M} = \left[LT^{-2}\right]$

<u>Gravitational potential energy</u>:- The gravitational potential energy of a body is *the energy associated with it due to its position in the gravitational field of another body* and is measured *by the amount of work done in bringing a body from infinity to a given point in the gravitational field of the other.*



The expression for gravitational potential energy:- suppose the earth is a uniform sphere of mass M and radius R. We will calculate the potential energy of a body of mass m located at point P such that $OP = r$ and $r > R$.

Suppose at any instant the body is at point A such that OA = x

The gravitational force of attraction on the body at A is $F = \frac{GMm}{x^2}$

The small work done in moving the body through small distance $AB(= dx)$ is given by

$$dW = Fdx = \frac{GMm}{x^2}dx$$

The total work done in bringing the body from infinity $(x = \infty)$ to the point $P(x = r)$ will be.

$$W = \int dW = \int_{\infty}^{r} \frac{GMm}{x^2}dx = GMm\int_{\infty}^{r} x^{-2}dx$$

$$= GMm\left[-\frac{1}{x}\right]_{\infty}^{r} = -GMm\left[\frac{1}{r} - \frac{1}{\infty}\right] = -\frac{GMm}{r}$$

This work done is the gravitational potential energy U of the body of mass m located at distance r from the centre of the earth.

$$\therefore \quad U = -\frac{GMm}{r}$$

Gravitational Potential:- *The gravitational potential at a point is the potential energy associated with a unit mass due to its position in the gravitational field of another body.*

The gravitational potential at a point in the gravitational field of a body is defined **as the amount of work done in bringing a body of unit mass from infinity to that point.**

Gravitational Potential, $V = \frac{\text{work done}}{\text{Mass}} = \frac{W}{m}$

The gravitational potential is a **scalar quantity**. Its **SI unit** is Jkg^{-1} and cgs unit is $erg\ g^{-1}$.

Dimensional formula -: $\left[M^0L^2T^{-2}\right]$.

Gravitational Potential at a point due to the earth:- The work is done in bringing a body of mass m from infinity to a point at distance r from the centre of the earth is

$$W = \frac{GMm}{r}$$

Hence the **gravitational potential due to the earth at distance r** from its centre is $V=\frac{W}{m}=-\frac{GM}{r}$

At the surface of the earth, r = R, therefore $V_{surface}=-\frac{GM}{R}$

The relation between gravitational potential energy and gravitational potential:-

From the above equations, we find that.

$$U=-\frac{GMm}{r}=\left(-\frac{GM}{r}\right)\times m$$

$\therefore$ **Gravitational potential energy = Gravitational potential $\times$ mass**

Example:- Find the intensity of gravitational field when a force of 100 N acts on a body of mass 10 kg in the gravitational field.

Solution:- Here F = 100 N, m = 10 kg

Intensity of gravitation field, $E=\frac{F}{m}=\frac{100\,N}{10\,kg}=10\,Nkg^{-1}$

Example:- Two bodies of masses 10 kg and 1000 kg are at a distance 1 m apart. At which point on the line joining them will the gravitational field intensity be zero?

Solution:- Let the resultant gravitational intensity be zero at distance x from the mass of 10 kg on the line joining the centres of the two bodies. At this point, the gravitational intensities due to the two bodies must be equal and opposite.

$$\therefore \frac{G\times 10}{x^2}=\frac{G\times 1000}{(1-x)^2}$$

Or $100x^2=(1-x)^2$ Or $10x=1-x$

Or $11x=1$ Or $x=1/11\,m$

Example:- At a point above the surface of the earth, the gravitational potential is $-5.12\times 10^7\ Jkg^{-1}$ and the acceleration due to gravity is $6.4\ ms^{-2}$. Assuming the mean radius of the earth to 6400 km, calculate the height of this point above the earth's surface.

Solution:- Let r be the distance of the given point from the centre of the earth. Then

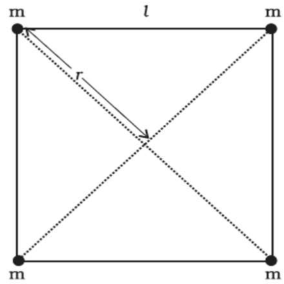
Gravitational potential $V=-\frac{GM}{r}=-5.12\times 10^7\ Jkg^{-1}$ .... (i)

Acceleration due to gravity, $g=\frac{GM}{r^2}=6.4\ ms^{-2}$ .... (ii)

Dividing (i) by (ii) $r=\frac{5.12\times 10^7}{6.4}=8\times 10^6 m=8000km$

Height of the point from the earth's surface $=8000-6400=1600km$

Example-**Find the potential energy of a system of four particles placed at the vertices of a square of side L. Find the potential at the Centre of the square. (NCERT)**



Solution- Consider four masses each of mass m at the corners of a square of side l as in the figure. We have four mass pairs at distance l and two diagonal pairs at distance √2 l. Hence

$W(r) = -4\frac{Gm^2}{l} - 2\frac{Gm^2}{\sqrt{2}l}$

$= -2\frac{Gm^2}{l}\left(2+\frac{1}{\sqrt{2}}\right)$

$= -5.41\frac{Gm^2}{l}$

The gravitational Potential at the centre of the square ($r=\frac{\sqrt{2}l}{2}$) is

$U(r) = -4\sqrt{2}\frac{Gm}{l}$

Problems for Practice:-

01. The gravitational field intensity at a point 10,000 km from the centre of the earth is $4.8 Nkg^{-1}$. Calculate the gravitational potential at that point.

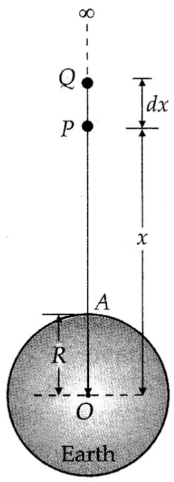
02. The distance between the earth and the moon is 3.85×10^{8} metre. At what point in between the two will the gravitational field intensity be zero? Mass of the earth $= 6.0\times10^{24}$ kg, a mass of the moon = 7.26×10^{22} kg.

03. Two bodies of masses 100 kg and 1000 kg are at a distance 1.00 metre apart. Calculate the gravitational field intensity and the potential at the middle-point of the line joining them.

04. The radius of the earth is R and the acceleration due to gravity at its surface is g. Calculate the work required in raising a body of mass m to a height h from the surface of the earth.

05. Find the work done to bring 4 particles each of mass 100 gram from large distances to the vertices of a square of side 20 cm.

Lecture – 5



Escape Velocity:-. *The minimum velocity with which a body must be projected vertically upwards so that it may just escape the gravitational field of the earth.*

Consider the earth to be a sphere of mass M and radius R with centre O. Suppose a body of mass m lies at point P at distance x from its centre, as shown in the figure. The gravitational force of attraction on the body at P is $F = \frac{GMn}{x^2}$. The small work done in moving the body through small distance $PQ = dx$ against the gravitational force is given by $dW = Fdx = \frac{GMm}{x^2}dx$

The total work done in moving the body from the surface of the earth (x = R) to a region beyond the gravitational field of the earth $(x = \infty)$ will be

$$W = \int dW = \int_R^\infty \frac{GMn}{x^2}dx$$

$$= GMm\int_R^\infty x^{-2}dx = GMm\left[-\frac{1}{x}\right]_R^\infty$$

$$= GMm\left[-\frac{1}{\infty} + \frac{1}{R}\right] = \frac{GMm}{R}$$

It v_e is the escape velocity of the body, then the kinetic energy $\frac{1}{2}mv_e^2$ imparted to the body at the surface of the earth will be just sufficient to perform work W.

$$\therefore \quad \frac{1}{2}mv_e^2 = \frac{GMm}{R} \quad \text{or} \quad v_e^2 = \frac{2GM}{R}$$

Escape velocity $v_e = \sqrt{\frac{2GM}{R}}$ ....(i)

As $g = \frac{GM}{R^2}$ or $GM = gR^2$

$$\therefore \quad v_e = \sqrt{\frac{2gR^2}{R}} \quad \text{or} \quad v_e = \sqrt{2gR}$$ .... (ii)

It ρ is the mean density of the earth, then

$$M = \frac{4}{3}\pi R^3 \rho$$

$\therefore \quad v_e = \sqrt{\frac{2G}{R} \times \frac{4}{3}\pi R^3 \rho} = \sqrt{\frac{8\pi\rho G R^2}{3}}$ .... (iii)

Equations (i), (ii) and (iii) give different expressions for the escape velocity of a body. The escape velocity does not depend on the mass of the body projected.

Example:- Find the velocity of escapee at the earth given that its radius is 6.4×10^6m and the value of g at its surface is $9.8 ms^{-2}$.

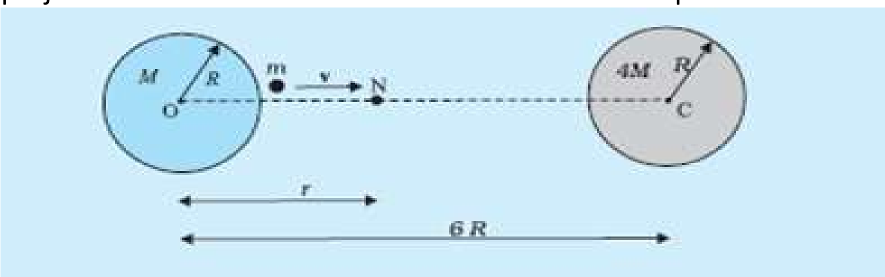
Solution:- Here $R = 6.4 \times 10^6 m$, $g = 9.8 ms^{-2}$

$v_e = \sqrt{2gR} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$

$= 11.2 \times 10^3 ms^{-1} = 11.2 Kms^{-1}$

NCERT EXAMPLE=

Two uniform solid spheres of equal radii R, but mass M and 4M have a centre to centre separation 6R, as shown in the figure. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere?



Solution-The projectile is acted upon by two mutually opposite gravitational forces of the two spheres. At neutral Point, N two forces cancel each other exactly. If ON =r, We have

$$\frac{GMm}{r^2} = \frac{4GMm}{(6R-r)^2}$$

$$(6R - r)^2 = 4r^2$$

$$6R - r = \pm 2r$$

$$r = 2R \quad \text{or} - 6R.$$

The neutral point r =-6R does not concern us in the example. Thus ON=r=2R. It is sufficient to project the particle with the speed which would enable to reach N.Thereafter the gravitational pull of 4M would suffice. The mechanical energy at the surface of M is

$$E_i = \frac{1}{2} m v^2 - \frac{GMm}{R} - \frac{4GMm}{5R}.$$

At the neutral Point N, the speed approaches Zero. Here the Mechanical energy is purely potential.

$$E_N = -\frac{GMm}{2R} - \frac{4GMm}{4R}.$$

From the principle of conservation of mechanical energy,

$$\frac{1}{2}\upsilon^2 - \frac{GM}{R} - \frac{4GM}{5R} = -\frac{GM}{2R} - \frac{GM}{R}$$

or

$$\upsilon^2 = \frac{2GM}{R}\left(\frac{4}{5} - \frac{1}{2}\right)$$

$$\upsilon = \left(\frac{3GM}{5R}\right)^{1/2}$$

Example:- Determine the escape velocity of a body from the moon. Take the moon to be a uniform sphere of radius 1.76×10^6m and mass 7.36×10^{22}kg. Given 6.67×10^{-11}Nm^{2}kg^{-2}.

Solution:- Here $R = 1.76 \times 10^6 m, M = 7.36 \times 10^{22} kg$

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{1.76 \times 10^6}} = 2375 ms^{-1} = 2.375 kms^{-1}$$

Example:- Calculate the escape velocity for an atmospheric particle 1600 km above the earth's surface, given that the radius of the earth is 6400 km and acceleration due to gravity on the surface of the earth is $9.8 ms^{-2}$.

Solution:- At a height h above the earth's surface, we have

$$v_e = \sqrt{2g_h(R+h)},\ g_h = \frac{gR^2}{(R+h)^2}$$

$$\therefore \quad v_e = \sqrt{\frac{2 \times gR^2}{(R+h)^2} \times (R+h)} = \sqrt{\frac{2gR^2}{R+h}}$$

But $g = 9.8\ ms^{-2}, R = 6.4 \times 10^6 m, h = 1600 km = 1.6 \times 10^6 m$

$$R + h = (6.4 + 1.6) \times 10^6 = 8 \times 10^6 m$$

$$\therefore \quad v_e = \sqrt{\frac{2 \times 9.8 \times (6.4 \times 10^6)^2}{8 \times 10^6}} = 10.02 \times 10^3 ms^{-1} = 10.02 kms^{-1}$$

Problems for Practice:-

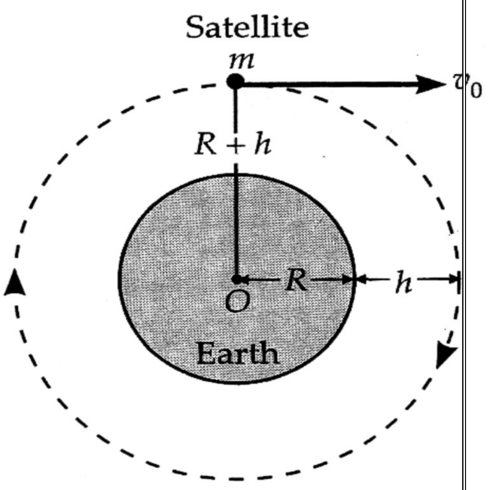
01. Find the velocity of escape at the moon. Given that its radius is $1.7\times10^{6}m$ and the value of g is $1.63ms^{-2}$.

02. The mass of Jupiter is $1.90\times10^{36}kg$ and its diameter is $13.1\times10^{7}m$. Calculate the escape velocity on the surface of Jupiter.

03. If the earth has a mass 9 times and radius twice that of a planet mars, calculate the minimum velocity required by a rocket to pull out of gravitational force of Mars. Take the escape velocity on the surface of the earth to be $11.2kms^{-1}$.

04. Find the velocity of escape from the sun, if its mass is $1.89\times10^{30}kg$ and its distance from the earth is $1.58\times10^{8}km$. Take $G=6.67\times10^{-11}Nm^{2}kg^{-2}$.

05. A body of mass 100 kg falls on the earth from infinity. What will be its velocity on reaching the earth? What will be its K.E? Radius of the earth is 6400 km and $g=9.8ms^{-2}$. Air friction is negligible.



Lecture – 6

Orbital Velocity:- The velocity required to put the satellite into its orbit around the earth is called Orbital Velocity.

Let M = Mass of the earth, R = Radius of the earth, m = Mass of the satellite, v_0 = orbital velocity of the satellite, h= Height of the satellite above the earth's surface, R + h = Orbital radius of the satellite.

From the law of gravitation, the force of gravity on the satellite is

$$F = \frac{GMm}{(R+h)^2}$$

The centripetal force required by the satellite to keep it in its orbit is $F = \frac{mv_0^2}{R+h}$

In equilibrium, the centripetal force is just provided by the gravitational pull of the earth, so

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2}$$

Or $$v_0^2 = \frac{GM}{R+h}$$

$\therefore$ Orbital velocity, $v_0 = \sqrt{\frac{GM}{R+h}}$

If g is the acceleration due to gravity on the earth's surface, then $g = \frac{GM}{R^2}$

Or $GM = gR^2$

Hence $v_0 = \sqrt{\frac{gR^2}{R+h}} = R\sqrt{\frac{g}{R+h}}$

When the satellite revolves close to the surface of the earth, h = 0 and the orbital velocity will become.

$$v_0 = \sqrt{gR}$$

As $g = 9.8 ms^{-2}$ and $R = 6.4 \times 10^6 m$

So, $v_0 == \sqrt{9.8 \times 6.4 \times 10^6} = 7.92 \times 10^3 ms^{-1} = 7.92 kms^{-1}$

The relation between orbital velocity and escape velocity:- The escape velocity of the body from the earth's surface is

$$v_e = \sqrt{2gR}$$

The orbital velocity of a satellite revolving close to the earth's surface is

$$v_0 = \sqrt{gR}$$

$$\therefore \frac{v_e}{v_0} = \frac{\sqrt{2gR}}{\sqrt{gR}} = \sqrt{2}$$

Or $v_e = \sqrt{2}v_0$

The escape velocity of a body from the earth's surface is timing *its velocity in a circular orbit just above the earth's surface.*

Geostationary Satellites:- The satellite which revolves around the earth in its equatorial plane with the same angular speed and the same direction as the earth rotates about its axis is called a geostationary or synchronous satellite.

Characteristics:=

1. **It should revolve at a height of nearly 36000 km above the earth surface.**
2. **Its period of revolution around the earth should be exactly as that of the earth about its axis, i,e 24 hours.**
3. **Its rotation from west to east as that of the earth with same angular speed as that of earth about its axis.**
4. **This appears stationary w.r.t an observer on the surface of the earth since its relative velocity w.r.t earth is zero.**

USES Of Geostationary Satellites:-

(a).In communication purpose,T.V, Telephone signal etc
(b).In studying the upper region of the atmosphere.
(c).In forecasting weather.
(d) .In determining the exact shape and dimension of earth.

Height of a geostationary satellite:- The height of a satellite above the earth's surface is given by

$$h = \left[\frac{T^2R^2g}{4\pi}\right]^{1/3} - R$$

But T = 24h = 86400 s

R = radius of the earth = 6400 km

$g = 9.8ms^2 = 0.0098kms^{-2}$

$$\therefore\ h = \left[\frac{(86400)^2 \times (6400)^2 \times 0.0098}{4 \times 9.87}\right]^{1/3} - 6400 = 42330 - 6400 = 35930km$$

Total Energy and Binding Energy of a Satellite:-

Let a satellite of mass m moving around the earth with velocity v_0 in an orbit of radius r. Due to the gravitational pull of the earth, the satellite has potential energy which is given by

$$U = -\frac{GMm}{r}$$

The kinetic energy of a satellite due to its orbital motion is

$$K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$$

The total energy of the satellite is $E = U + K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$

Or $E = -\frac{GMm}{2r}$

The total energy of the satellite is negative. It indicates that the satellite is bound to the earth. At infinity $(r = \infty)$, the potential energy is zero and also the kinetic energy is zero. Hence total energy at infinity is zero. Thus negative total energy means that to send the satellite to infinity, it needs to be given extra energy (to make total energy zero), otherwise, it will continue revolving in a closed orbit. **Thus satellite is bound to the earth.**

The binding energy of a satellite:-

The energy required by a satellite to leave its orbit around the earth and escape to infinity is called its binding energy.

The total energy of a satellite is $-\frac{GMm}{2r}$. To escape to infinity, it must be supplied extra energy equal to $+\frac{GMm}{2r}$ so that its total energy E becomes equal to zero.

Hence Binding energy of a satellite = $\frac{GMm}{2r}$

Example(NCERT) A 400 kg satellite is in a circular orbit of radius 2 R_E about the earth. How much energy is required to transfer it to a circular orbit of radius $4R_E$? What are the changes in the kinetic and potential energies?

Solution:- Total energy in orbit of radius $2R_E$, $E_i = \frac{GM_E m}{4R_E}$

The total energy in orbit of radius 4 R_E $\quad E_f = \frac{GM_E m}{8R_E}$

The energy required to transfer the satellite to orbit of radius $4R_E$.

$$\Delta E = E_f - E_i = \frac{GM_E m}{8R_E} + \frac{GM_E m}{4R_E}$$

$$= \frac{GM_E m}{8R_E} = \left(\frac{GM_E}{R_E^2}\right)\frac{mR_E}{8} = \frac{gmR_E}{8}$$

But $g = 9.8 ms^{-2}$, $m = 400 kg$, $R_E = 6.37 \times 10^6 m$

$$\therefore \quad \Delta E = \frac{9.8 \times 400 \times 6.37 \times 10^6}{8} = 3.13 \times 10^9 J$$

Change in K.E, $\Delta K = -\Delta E = -3.13 \times 10^9 J$

Change in P.E, $\Delta U = 2\Delta E = -6.26 \times 10^9 J$

<u>Example</u>:- A satellite orbits the earth at a height of 500 km from its surface. Compute its (i) kinetic energy, (ii) potential energy, and (iii) total energy. Mass of the satellite = 300kg. Mass of the earth = $6.0 \times 10^{24} kg$, a radius of the earth = $6.4 \times 10^6 m$, $G = 6.67 \times 10^{-11} Nm^2 kg^{-2}$. Will your answer alter if the earth were to shrink suddenly to half its size?

Solution:- Here

$h = 500 km = 500 \times 10^3 m$

$m = 300 kg, M = 6.0 \times 10^{24} kg$

$R = 6.4 \times 10^6 m, G = 6.67 \times 10^{-11} Nm^2 kg^{-2}$

(i) Kinetic Energy

$$= \frac{1}{2}mv^2 = \frac{1}{2}m.\frac{GM}{R+h}$$

$$= \frac{1}{2} \times \frac{300 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6 + 500 \times 10^3} = 8.7 \times 10^9 J$$

(ii) Potential Energy

$$= -\frac{GMm}{(R+h)} = -\frac{6.67\times10^{-11}\times6.0\times10^{24}\times300}{6.4\times10^{6}+500\times10^{3}} = -17.4\times10^{9}\,J$$

(iii) Total Energy $= K.E + P.E = 8.7\times10^{9} - 17.4\times10^{9} = -8.7\times10^{9}\,J$

Problems for Practice:-

01. A rocket is launched vertically from the surface of the earth with an initial velocity of $10\,kms^{-1}$. How far above the surface of the earth would it go? The radius of the earth = 6400 km and $g = 9.8ms^{-2}$.

02. A satellite of mass 250 kg is orbiting the earth at a height of 500 km above the surface of the earth. How much energy must be expended to rocket the satellite out of the gravitational influence of the earth? Given the mass of the earth = $6.0\times10^{24}kg$, $G = 6.67\times10^{-11}Nm^{2}kg^{-2}$

03. A body is to be projected vertically upwards from the earth's surface to reach a height of 9R, where R is the radius of the earth. What is the velocity required to do so? Given $g = 10ms^{-2}$ and radius of the earth $= 6.4\times10^{6}m$.

04. Calculate the energy required to move an earth satellite of mass 10^{3}kg from a circular orbit of radius 2R to that of radius 3R. Given the mass of the earth, $M = 5.98\times10^{24}kg$ and the radius of the earth, $R = 6.37\times10^{6}m$.

5. Find the energy required to move an earth satellite of mass 10^{3} kg from a circular orbit of radius 2R to that of radius 3R.Given Mass of earth M= 5.98×10^{24} kg and R= 6.37×10^{6} m.