Chapter- 4

MOTION IN A PLANE

Scalar quantities

A scalar quantity is a quantity with magnitude only. It is specified completely by a single number with a proper unit.

For example mass, length, time, temperature etc.

Vector quantities

Any physical quantities having magnitude and direction and which obey triangle law of addition or equivalently parallelogram law of addition are called vector quantities.

For example displacement, velocity, acceleration, etc.

Notation of vectors

We will denote the scalar quantities by English or Greek alphabets like A, a, Θ, θ, etc. and vector quantities by modified English or Greek alphabets like $\vec{\mathbf{A}}$, $\vec{\mathbf{a}}$, $\vec{\mathbf{\Theta}}$, $\vec{\mathbf{\theta}}$ etc.

Let $\vec{\mathbf{a}}$ is a vector then we will denote its magnitude which is a positive scalar as $|\vec{\mathbf{a}}|$ or a.

Representation of vectors

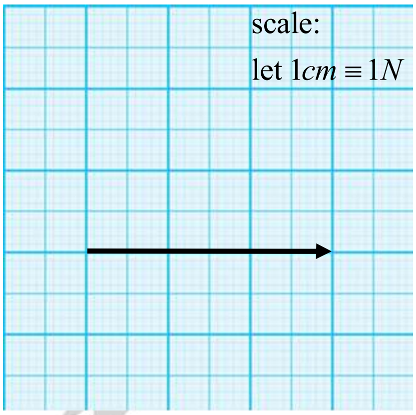
A vector is represented in a diagram by an arrow with the following conditions

The length of the arrow according to our established scale is equal to the magnitude of the vector.

The direction of the arrow, given by the arrowhead, is parallel to the direction of the vector.

For example:

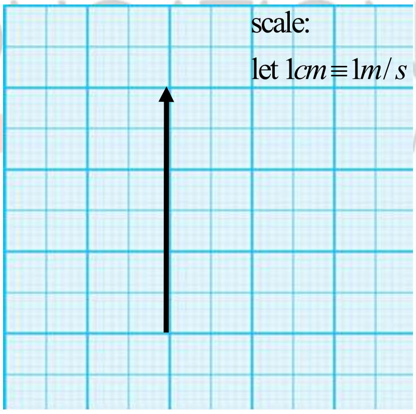
Let us represent a vector 3N due east. To draw the arrow let choose the scale " 1cm represents 1N". We shall draw an arrow of length 3 cm due east. This arrow represents a vector 5N due east.

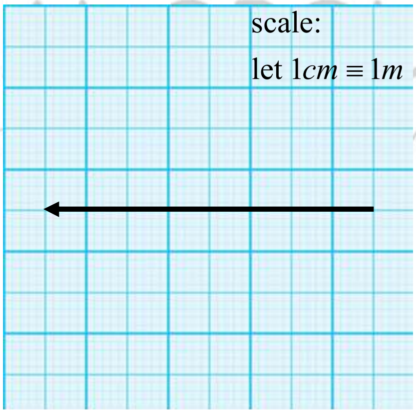


QUESTION

Represent the vector (a) 3 m/s due north, (b) 4 m due west, on graph paper.

ANSWER





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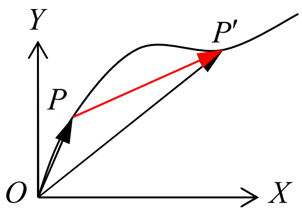
Position and displacement vector

Let P and P' are positions of the particle at the times t and t' respectively. $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the position vectors



of the particle at the times t and t' respectively. The displacement vector of the particle in this interval is $\overrightarrow{PP'}$.

The magnitude of the displacement of a particle is either less than or equal to the distance travelled (length of the path). These are equal if the particle moves in one direction.

QUESTION

If a particle moves along a circle of radius r from one point to a diametrically opposite point what is the distance travelled and the magnitude of displacement?

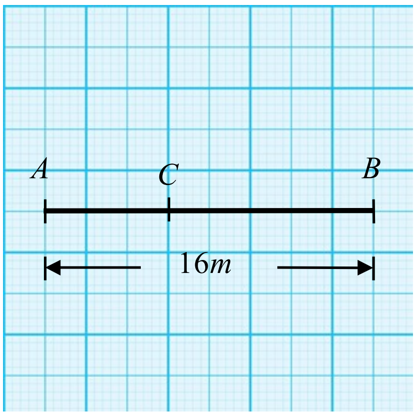
ANSWER

$\text{distance} = \pi r$

$\text{magnitude of displacement} = 2r$

QUESTION

If a particle moves along a straight line from the point A to B and then to the point C what are the distance travelled and the magnitude of displacement?

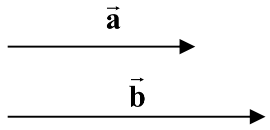


ANSWER

Distance=AB+BC=26m

Magnitude of displacement=AC=6m

Parallel vectors



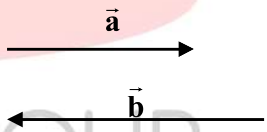
The vectors having the same directions are called parallel vectors.

In the figure $\vec{a}, \vec{b}$ are parallel vectors.

Example 1: the vectors 2m due west and 4N due west are parallel.

Example 2: the vectors 3m/s due north and 5N due north are parallel.

Antiparallel vectors



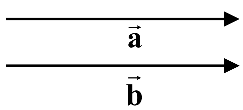
Two vectors having opposite directions are called antiparallel vectors.

In the figure $\vec{a}, \vec{b}$ are antiparallel vectors.

Example 1: the vectors 2m due west and 4N due east are antiparallel.

Example 2: the vectors 4m/s due north and 7N due south are antiparallel.

Equality of vectors

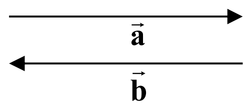


Two vectors having the same magnitudes and directions are called equal vectors.

In the figure $\vec{a} = \vec{b}$.

Negative vector

In the figure $|\vec{\mathbf{a}}| = |\vec{\mathbf{b}}|$ and $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ are antiparallel vectors. So $\vec{\mathbf{a}} = -\vec{\mathbf{b}}$ or $-\vec{\mathbf{a}} = \vec{\mathbf{b}}$, i.e., one is the negative of the other.



Example 1: 2m due east and 2m due west are negative of each other.

Example 2: $2m/s$ due north and $-2m/s$ due north are negative of each other.

Multiplication of a vector by a real number

If $\vec{a}$ is a vector and λ is a positive number then $\lambda\vec{a}$ is the vector in the direction of $\vec{a}$ and its magnitude is λ times that of $\vec{a}$. $|\lambda\vec{a}| = \lambda|\vec{a}|$

If λ is a negative number then $\lambda\vec{a}$ is the vector in the opposite direction of $\vec{a}$ and its magnitude is $-\lambda$ times that of $\vec{a}$. $|\lambda\vec{a}| = -\lambda|\vec{a}|$

Example 1: if $\vec{a} = 4\text{m due west}$ and $\lambda = 3(\text{a positive number})$, then $|\vec{a}| = 4\text{m}$,

$\lambda\vec{a} = \lambda(4\text{m due west}) = 3(4\text{m due west}) = 12\text{m due west}$, $|\lambda\vec{a}| = 12\text{m}$, $\lambda|\vec{a}| = 3\times 4\text{m} = 12\text{m}$, so

$|\lambda\vec{a}| = \lambda|\vec{a}|$.

Example 2: if $\vec{a} = 4\text{m due west}$ and $\lambda = -3(\text{a negative number})$, then $|\vec{a}| = 4\text{m}$,

$\lambda\vec{a} = \lambda(4\text{m due west}) = -3(4\text{m due west}) = -12\text{m due west} = 12\text{m due east}$, $|\lambda\vec{a}| = 12\text{m}$,

$-\lambda|\vec{a}| = -(-3)\times 4\text{m} = 12\text{m}$, so $|\lambda\vec{a}| = -\lambda|\vec{a}|$.

QUESTION

Choose the parallel and antiparallel vectors from the group: 2m due east, 3N due east, 4m/s due west, 5N due west, 6m due east, 3N due south.

ANSWER

2m due east, 3N due east, 6m due east, are parallel vectors.

4m/s due west, 5N due west are parallel vectors.

2m due east, 4m/s due west, are antiparallel vectors.

3N due east, 4m/s due west, are antiparallel vectors.

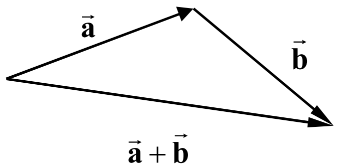
6m due east, 4m/s due west, are antiparallel vectors.

2m due east, 5N due west, are antiparallel vectors.

3N due east, 5N due west, are antiparallel vectors.

6m due east, 5N due west, are antiparallel vectors.

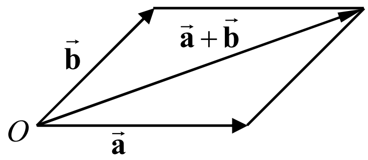
Triangle method of vector addition



To find the sum $\vec{a}+\vec{b}$, we place vector $\vec{b}$ so that its tail is at the head of $\vec{a}$. Then we join the tail of $\vec{a}$ to the head of $\vec{b}$. This line represents $\vec{a}+\vec{b}$.

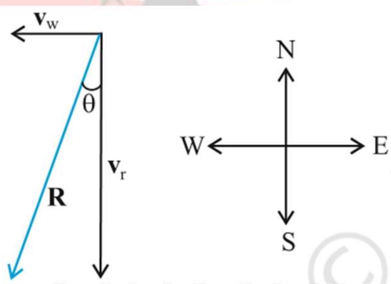
Parallelogram method of vector addition

To find the sum $\vec{a}+\vec{b}$, we bring the tails of $\vec{a}$ and $\vec{b}$ to a common origin O. Then we draw a line from the head of $\vec{a}$ parallel to $\vec{b}$ and another line from the head of $\vec{b}$ parallel to $\vec{a}$ complete a parallelogram. Now we join the point of intersection of these two lines to the origin O. The resultant vector $\vec{a}+\vec{b}$ is directed from the common origin O along the diagonal of the parallelogram.



NCERT Example 4.1

Rain is falling vertically with a speed of 35 m/s. The wind starts blowing after some time with a speed of 12 m/ s in the east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella?



SOLUTION

Resultant of $\vec{v}_r$ and $\vec{v}_w$ is $\vec{R}$. The magnitude of $\vec{R}$ is

$$R=\sqrt{{v_r}^2+{v_w}^2}=\sqrt{35^2+12^2}\,m/s=37m/s$$

The direction θ that R makes with the vertical is given by

$$\tan\theta=\frac{v_w}{v_r}=0.343 \Rightarrow \theta=\frac{v_w}{v_r}=\tan^{-1}(0.343)=19^\circ$$

Therefore, the boy should hold his umbrella in the vertical plane at an angle of about 19° with the vertical towards the east.

Subtraction of vectors

We define the difference of two vectors $\vec{a}$ and $\vec{b}$ as the sum of two vectors $\vec{a}$ and $-\vec{b}$.

$$\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$$

NCERT Example 4.2

Find the magnitude and direction of the resultant of two vectors $\vec{A}$ and $\vec{B}$ in terms of their magnitudes and angle θ between them.

SOLUTION

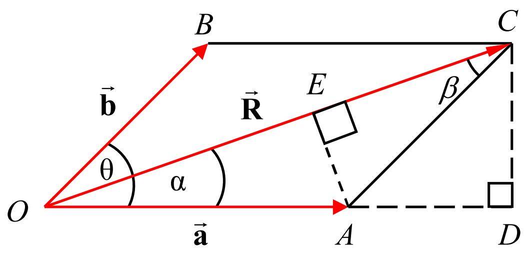
Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the two vectors $\vec{a}$ and $\vec{b}$ making an angle θ. Then, using the parallelogram method of vector addition $\overrightarrow{OC}$ represents the resultant vector $\vec{R}=\vec{a}+\vec{b}$

$$OA = a$$

$$OB = AC = b$$

$$R = OC = \left|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right|$$

CD is normal to OA and AE is normal to OC. From the geometry of the figure,



$$OC = \sqrt{(OD)^2+(DC)^2}$$

$$= \sqrt{(OA+AD)^2+(DC)^2}$$

$= \sqrt{(OA)^2 + (AD)^2 + 2OA \times AD + (DC)^2}$

$= \sqrt{(a)^2 + (b\cos\theta)^2 + 2a \times b\cos\theta + (b\sin\theta)^2}$

$\Rightarrow R = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ ---eq(1)

In the right angled triangle OCD

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA + AD} = \frac{b\sin\theta}{a + b\cos\theta}$$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{b\sin\theta}{a + b\cos\theta}\right)$ ---eq(2)

$$\sin\alpha = \frac{AE}{OA} = \frac{AE}{a}$$

$$\sin\beta = \frac{AE}{AC} = \frac{AE}{b}$$

So

$$a\sin\alpha = b\sin\beta \Rightarrow \frac{\sin\alpha}{b} = \frac{\sin\beta}{a}$$

Again

$$\sin\alpha = \frac{CD}{OC} = \frac{CD}{R}$$

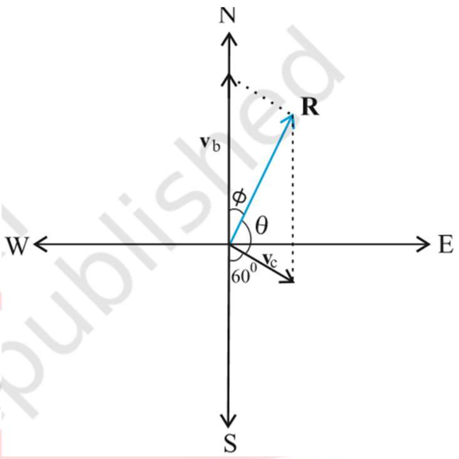
$$\sin\theta = \frac{CD}{AC} = \frac{CD}{b}$$

$$R\sin\alpha = b\sin\theta \Rightarrow \frac{\sin\alpha}{b} = \frac{\sin\theta}{R}$$

Hence

$$\frac{\sin\alpha}{b} = \frac{\sin\theta}{R} = \frac{\sin\beta}{a} \quad \text{---eq.(3)}$$

NCERT Example 4.3



A motorboat is racing towards the north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

SOLUTION

We can obtain the magnitude of $\vec{R}$ using the Law of cosine is:

$$R = \sqrt{v_b{}^2 + v_c{}^2 + 2v_b v_c \cos 120^\circ} = \sqrt{25^2 + 10^2 - \frac{2\times 250}{2}} = 22km/h$$

$$\phi = \frac{v_c\sin\theta}{R} = 0.397 \Rightarrow \phi = 23.4^\circ$$

Null vector

The vector having zero magnitude is called the null vector or zero vector.

$$\vec{a} + \vec{0} = \vec{a}$$

$\lambda\vec{0} = \vec{0}$

$0\vec{a} = \vec{0}$

$\vec{a} - \vec{a} = \vec{0}$

$|\vec{0}| = 0$

Unit vectors

A unit vector is a vector of unit magnitude and points in a particular direction. It has no dimension and unit. It is used to specify a direction only. In general, a vector $\vec{A}$ can be written as

$\vec{A} = \hat{n}|\vec{A}|$

where $\hat{n}$ is a unit vector along $\vec{A}$.

RELATIVE VELOCITY

Suppose that two objects A and B are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ (each with respect to some common frame of reference, say ground.). Then, the velocity of the object A relative to that of B is : $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$

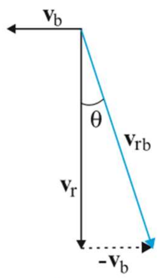
similarly, the velocity of the object B relative to that of A is: $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$ Therefore, $\vec{v}_{BA} = -\vec{v}_{AB}$ and

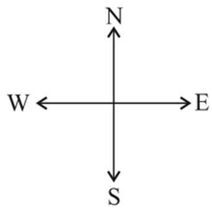
$|\vec{v}_{BA}| = |\vec{v}_{AB}|$

NCERT Example 4.6

Rain is falling vertically with a speed of 35 m/s. A woman rides a bicycle with a speed of 12 m/s in the east to west direction. What is the direction in which she should hold her umbrella?

SOLUTION





$$\tan\theta = \frac{v_b}{v_r} = \frac{12}{35} = 0.343$$

$$\theta = \tan^{-1}(0.343) = 19^\circ$$

NCERT Example 3.9

Two parallel rail tracks run north-south. Train A moves north with a speed of 54 km/h, and train B moves south with a speed of 90 km/h. What is the (a) velocity of B with respect to A ?, (b) velocity of ground with respect to B ?, and (c) velocity of a monkey running on the

the roof of the train A against its motion (with a velocity of 18 km/h with respect to the train A) as observed by a man standing on the ground?

SOLUTION

Choose the positive direction of x-axis to be from the south to north. Then,

$$v_A = +54 km/h = 15 m/s$$

$$v_B = -90 km/h = -25 m/s$$

The relative velocity of B with respect to A $= v_B - v_A = -40 m/s$, i.e. the train B appears to A to move with a speed of 40 m/s from north to south.

The relative velocity of ground with respect to B $= 0 - v_B = 25 m/s$

In (c), let the velocity of the monkey with respect to the ground be v_M . The relative velocity of the monkey with respect to A, $v_{MA} = v_M - v_A = -5 m/s$

So $v_M = 10 m/s$

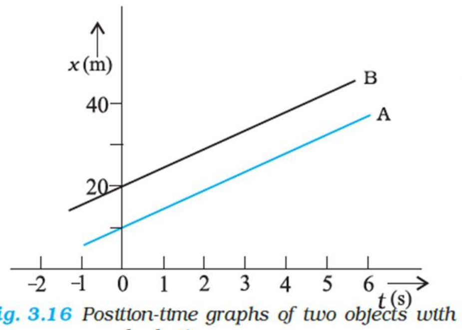


Fig. 3.16 *Position-time graphs of two objects with equal velocities.*

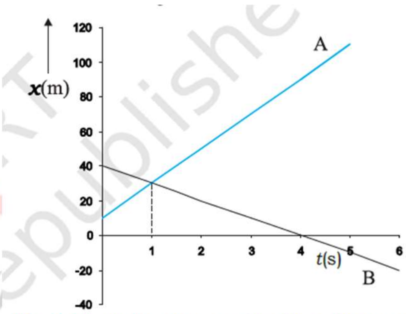


Fig. 3.18 *Position-time graphs of two objects with velocities in opposite directions, showing the time of meeting.*

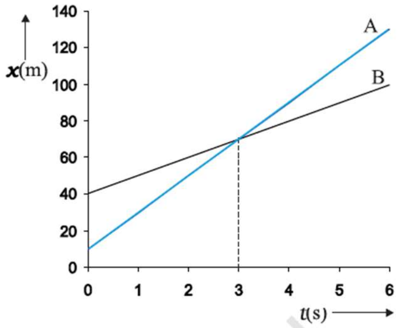
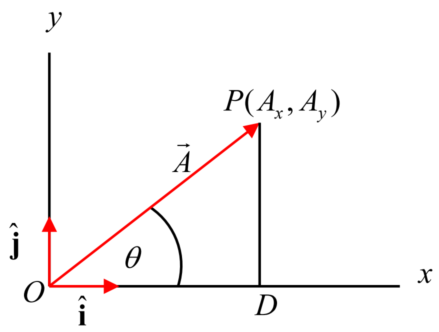


Fig. 3.17 *Position-time graphs of two objects with unequal velocities, showing the time of meeting*

Resolution of the vector into rectangular components in a two-dimensional plane



Consider a point $P(A_x, A_y)$ in xy plane as shown in fig(1). Let $\vec{A}$ is the position vector of the point P with respect to the origin O of the coordinate system. Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors along the positive x, y directions respectively.

So by the triangle law of vector addition

$$\overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OD}} + \overrightarrow{\mathbf{DP}}$$

$$\Rightarrow \vec{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad \text{---eq(1)}$$

The real numbers A_x and A_y are called x-, and y- components of the vector $\vec{A}$.

The magnitude of $\vec{A}$ is

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} \quad \text{---eq(2)}$$

Let θ is the angle that $\vec{A}$ makes with the x-axis :

$$A_y = A\sin\theta$$

$$A_x = A\cos\theta$$

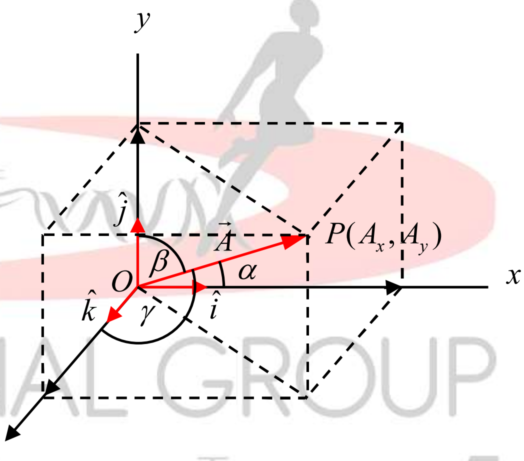
So, a component of a vector can be positive, negative or zero depending on the value of θ.

$$\tan\theta = \frac{A_y}{A_x}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \qquad \text{---eq(3)}$$

Resolution of the vector into rectangular components in three-dimensional space

Consider a point $P(A_x, A_y, A_z)$ in xyz space as shown in fig. Let $\vec{A}$ is the position vector of the point P with respect to the origin O of the coordinate system. Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the unit vectors along the positive x, y, z directions respectively.



If α, β, and γ are the angles between $\vec{A}$ and the x-, y-, and z-axes, respectively.

So by the triangle law of vector addition

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad \text{---eq(1)}$$

The magnitude of $\vec{A}$ is

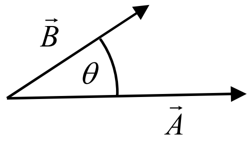
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \text{---eq(2)}$$

$A_x = A\cos\alpha$

$A_y = A\cos\beta$

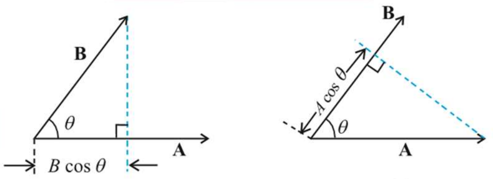
$A_z = A\cos\gamma$

The Scalar Product



The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}\cdot\vec{B}$ is defined as $\vec{A}\cdot\vec{B} = AB\cos\theta$

where θ is the angle between the two vectors.



Geometrically, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ in Fig.6.1 (b) and $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$. So, $\vec{A}\cdot\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$. Alternatively, it is the product of the magnitude of $\vec{B}$ and the component of $\vec{A}$ along $\vec{B}$.

Properties of the scalar product

$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (commutative law)

$\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ (distributive law)

$\lambda\vec{A}\cdot\left(\lambda\vec{B}\right) = \lambda\left(\vec{A}\cdot\vec{B}\right)$

For unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ we have

$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$$

Given two vectors

$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$

$$\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$$

Their scalar product is

$$\vec{A}\cdot\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\cdot\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)=A_xB_x+A_yB_y+A_zB_z$$

From the definition of the scalar product

$$\vec{A}\cdot\vec{A}=A_xA_x+A_yA_y+A_zA_z$$

$$\Rightarrow\left|\vec{A}\right|\left|\vec{A}\right|\cos 0=A^2=A_x^{\,2}+A_y^{\,2}+A_z^{\,2}$$

$\vec{A}\cdot\vec{B}=0$ if $\vec{A}$ and $\vec{B}$ are perpendicular.

NCERT Example 6.1

Find the angle between the force $\vec{F}=3\hat{i}+4\hat{j}-5\hat{k}$ unit and the displacement

$\vec{d}=5\hat{i}+4\hat{j}+3\hat{k}$ unit. Also, find the projection of $\vec{F}$ on $\vec{d}$.

SOLUTION

$$\vec{F}\cdot\vec{d} = F_x d_x + F_y d_y + F_z d_z = 3(5)+4(4)-5(3) = 16 \text{ unit}$$

$$\vec{F}\cdot\vec{F} = F_x^2 + F_y^2 + F_z^2 = 9+16+25 = 50 \text{ unit}$$

$$\vec{d}\cdot\vec{d} = d_x^2 + d_y^2 + d_z^2 = 25+16+9 = 50 \text{ unit}$$

$$\vec{F}\cdot\vec{d} = Fd\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{F}\cdot\vec{d}}{Fd} = \frac{16}{\sqrt{50}\sqrt{50}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{16}{\sqrt{50}\sqrt{50}}\right) = \cos^{-1}(0.32)$$

Definition of Vector Product

A vector product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ such that

(i)magnitude of $\vec{C} = C = AB\sin\theta$ where A and B are magnitudes of $\vec{A}$ and $\vec{B}$, θ is the angle between the two vectors.

(ii) $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

(iii) if we take a right-handed screw with its head lying in the plane of $\vec{A}$ and $\vec{B}$, and the screw perpendicular to this plane, and if we turn the head in the direction from $\vec{A}$ and $\vec{B}$, then the tip of the screw advances in the direction of $\vec{C}$.

The vector product, however, is not commutative, i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$.

$\left|\vec{A}\times\vec{B}\right| = \left|\vec{B}\times\vec{A}\right|$.

The vectors $\vec{A}\times\vec{B}$ $\vec{B}\times\vec{A}$ are in opposite directions.

We have

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\vec{A}\times\vec{B}$ does not change sign under reflection.

$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ (distributive law)

$\hat{i}\times\hat{i} = \vec{0}$ $\hat{j}\times\hat{j} = \vec{0}$ $\hat{k}\times\hat{k} = \vec{0}$

$\hat{i}\times\hat{j} = \hat{k}$ $\hat{j}\times\hat{i} = -\hat{k}$

$\hat{j}\times\hat{k} = \hat{i}$ $\hat{k}\times\hat{j} = -\hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$ $\hat{i}\times\hat{k} = -\hat{j}$

Given two vectors

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Their vector product is

$$\vec{A}\times\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$$

$$=A_x\hat{i}\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)+A_y\hat{j}\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)+A_z\hat{k}\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$$

$$=A_x\hat{i}\times B_x\hat{i}+A_x\hat{i}\times B_y\hat{j}+A_x\hat{i}\times B_z\hat{k}$$
$$+A_y\hat{j}\times B_x\hat{i}+A_y\hat{j}\times B_y\hat{j}+A_y\hat{j}\times B_z\hat{k}$$
$$+A_z\hat{k}\times B_x\hat{i}+A_z\hat{k}\times B_y\hat{j}+A_z\hat{k}\times B_z\hat{k}$$

$$=A_xB_y\hat{k}-A_xB_z\hat{j}-A_yB_x\hat{k}+A_yB_z\hat{i}+A_zB_x\hat{j}-A_zB_y\hat{i}$$

$$=\left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

We have used the elementary cross products in obtaining the above relation. The expression for $\vec{A}\times\vec{B}$ can be put in a determinant form which is easy to remember.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

NCERT Example 7.4

Find the scalar and vector products of two vectors. $\vec{A}=3\hat{i}-4\hat{j}+5\hat{k}$, $\vec{B}=-2\hat{i}+\hat{j}-3\hat{k}$

SOLUTION

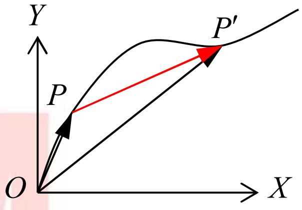
$$\vec{A}\cdot\vec{B}=\left(3\hat{i}-4\hat{j}+5\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}-3\hat{k}\right)=-6-4-15=-25$$

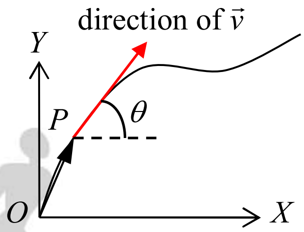
$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -4 & 5\\ -2 & 1 & -3\end{vmatrix}=7\hat{i}-\hat{j}-5\hat{k}$$

$$\vec{B}\times\vec{A}=-7\hat{i}+\hat{j}+5\hat{k}$$

Position vector and displacement

Let $P(x,y)$ and $P'(x',y')$ are positions of the particle at the times t and t' respectively. $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the position vectors of the particle at the times t and t' respectively. The displacement vector of the particle in this interval is $\overrightarrow{PP'}$.





$$\Delta\vec{r}=\overrightarrow{PP'}=\vec{r}'-\vec{r}=\hat{i}(x'-x)+\hat{j}(y'-y)=\hat{i}\Delta x+\hat{j}\Delta y$$

Velocity

The average velocity $\bar{v}$ of an object is the ratio of the displacement and the corresponding time interval

$$\bar{v}=\frac{\Delta\vec{r}}{\Delta t}=\hat{i}\frac{\Delta x}{\Delta t}+\hat{j}\frac{\Delta y}{\Delta t}=\hat{i}\bar{v}_x+\hat{j}\bar{v}_y$$

The direction of the average velocity is the same as that of $\Delta\vec{r}$.

The velocity (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero

$$\vec{v}=\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t}=\frac{d\vec{r}}{dt}=\hat{i}\frac{dx}{dt}+\hat{j}\frac{dy}{dt}=\hat{i}v_x+\hat{j}v_y$$

The direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.

The magnitude of $\vec{v}$ is then $v = \sqrt{(v_x)^2 + (v_y)^2}$ and the direction of $\vec{v}$ is given by the angle

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Acceleration

The average acceleration a of an object for a time interval Δt moving in the x-y plane is the change

in velocity divided by the time interval

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \hat{i}\frac{\Delta v_x}{\Delta t} + \hat{j}\frac{\Delta v_y}{\Delta t} = \hat{i}a_x + \hat{j}a_y$$

The acceleration (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \hat{i}\frac{dv_x}{dt} + \hat{j}\frac{dv_y}{dt} = \hat{i}a_x + \hat{j}a_y$$

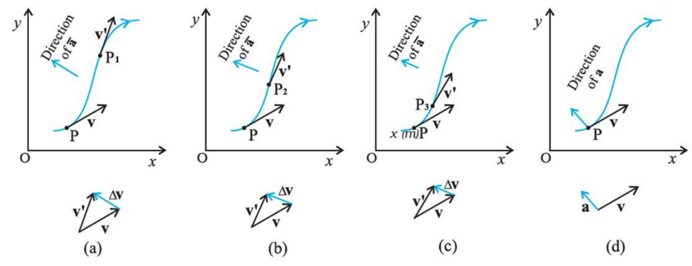


Fig. 4.15 The average acceleration for three time intervals (a) Δt_1, (b) Δt_2, and (c) Δt_3, ($\Delta t_1 > \Delta t_2 > \Delta t_3$). (d) In the limit $\Delta t \to 0$, the average acceleration becomes the acceleration.

Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line (either in the same direction or in the opposite direction). However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.

NCERT Example 4.4

The position of a particle is given by

$$\vec{r} = 3.0t\hat{i} + 2.0t^2\hat{j} + 5.0\hat{k}$$

where t is in seconds and the coefficients have the proper units for $\vec{r}$ to be in metres. (a) Find $\vec{v}(t)$ and $\vec{a}(t)$ of the particle. (b) Find the magnitude and direction of $\vec{v}(t)$ at t = 1.0 s.

SOLUTION

$$\vec{v}(t) = \frac{d}{dt}\left(3.0t\hat{i} + 2.0t^2\hat{j} + 5.0\hat{k}\right) = 3.0\hat{i} + 4.0t\hat{j}$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\left(3.0\hat{i} + 4.0t\hat{j}\right) = 4.0\hat{j}$$

$a = 4.0m/s^2$ along y-direction

At $t = 1s$, $\vec{v}(t) = 3.0\hat{i} + 4.0\hat{j}$

Its magnitude is $v = \sqrt{3.0^2 + 4.0^2} = 5m/s^2$ and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ$$ with x-axis.

MOTION IN A PLANE WITH CONSTANT ACCELERATION

Suppose that an object is moving in the x-y plane and its acceleration $\vec{a}$ is constant. Over an interval of time, the average acceleration will equal this constant value. Now, let the velocity of the object be $\vec{v}_0$ at time t = 0 and $\vec{v}$ at time t. Then, by definition

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t} \Rightarrow \vec{v} = \vec{v}_0 + \vec{a}t$$

Let us now find how the position $\vec{r}$ changes with time. Let $\vec{r}_0$ and $\vec{r}$ be the position vectors of the particle at time 0 and t and let the velocities at these instants be $\vec{v}_0$ and $\vec{v}$. Then, over this time interval t, the average velocity is $\frac{\vec{v}_0 + \vec{v}}{2}$. The displacement is the average velocity multiplied by the time interval

$$\vec{r} - \vec{r}_0 = \frac{\vec{v}_0 + \vec{v}}{2} t = \frac{\left(\vec{v}_0 + \vec{a}t\right) + \vec{v}}{2} t \Rightarrow \vec{r} - \vec{r}_0 = \frac{\vec{v}_0 + \vec{v}}{2} t = \frac{\vec{v}_0 + \left(\vec{v}_0 + \vec{a}t\right)}{2} t \Rightarrow \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

The motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.

NCERT Example 4.5

A particle starts from the origin at t = 0 with velocity $5.0\hat{i} m/s$ and moves in the x-y plane under the action of a force which produces a constant acceleration of $\left(3.0\hat{i} + 2.0\hat{j}\right) m/s^2$. (a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m? (b) What is the speed of the particle at this time?

SOLUTION

The position of the particle is given by

$$\vec{r}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 = \left(5.0\hat{i}\right)t + \frac{1}{2}\left(3.0\hat{i} + 2.0\hat{j}\right)t^2 = \left(5.0t + 1.5t^2\right)\hat{i} + 1.0t^2\hat{j}$$

$\Rightarrow x(t) = 5.0t + 1.5t^2$, $y(t) = +1.0t^2$

Given $x(t) = 5.0t + 1.5t^2 = 84m \Rightarrow t = 6s$

At $t = 6s$, $y(t) = 1.0(6^2) = 36m$

Now, the velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(5.0t + 1.5t^2)\hat{i} + 1.0t^2\hat{j} = (5.0 + 3.0t)\hat{i} + 2.0t\hat{j}$$

At $t = 6s$, $\vec{v} = 23.0\hat{i} + 12.0\hat{j}$

speed $|\vec{v}| = \sqrt{23.0^2 + 12.0^2} \cong 26m/s$

Projectile motion

An object that is in flight after being thrown or projected is called a projectile. Such a projectile might be a football, a cricket ball, a baseball or any other object.

The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity.

In our discussion, we shall assume that the air resistance has a negligible effect on the motion of the projectile. Suppose that the projectile is launched with the velocity $\vec{v}_0$ that makes an angle θ_0 with the x-axis.

After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward: $\vec{a} = -g\hat{j}$

$a_x = 0$, $a_y = -g$

The components of the initial velocity $\vec{v}_0$ are :

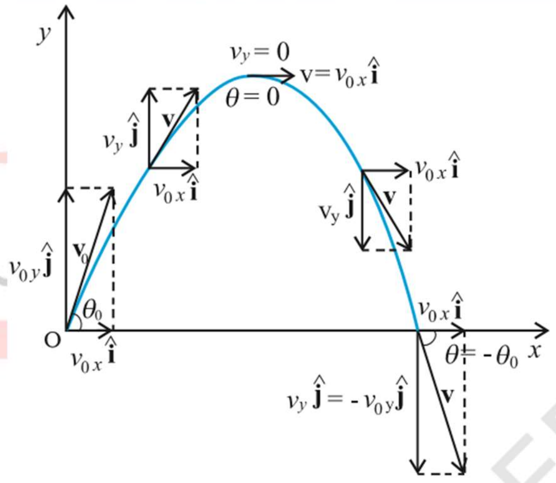
$v_{0x} = v_0 \cos\theta_0$, $v_{0y} = v_0 \sin\theta_0$ ---eq.(1)

The components of velocity at time t are

$v_x = v_0 \cos\theta_0$, $v_y = v_0 \sin\theta_0 - gt$ ---eq.(2)

The x- and y-coordinates of the position of a projectile at time t are

$x = (v_0 \cos\theta_0) t$, $y = (v_0 \sin\theta_0) t - \frac{1}{2} g t^2$ ---eq.(3)



Equation of path of a projectile

This can be seen by eliminating the time between the expressions for x and y as given in Eq. (3).

We obtain,

$y = x\tan\theta_0 - \frac{1}{2}\frac{g}{(v_0\cos\theta_0)^2}x^2$ ---eq.(4)

This is the equation of a parabola, i.e. the path of the projectile is a parabola.

Time of maximum height

Let the projectile takes time t_m to reach the maximum height. Since at this point, $v_y = 0$, we have from Eq. (1):

$v_y = v_0\sin\theta_0 - gt_m = 0$

$\Rightarrow t_m = \frac{v_0\sin\theta_0}{g}$ ---eq.(5)

time of flight

The total time T_f during which the projectile is in flight can be obtained by putting y = 0 in Eq. (3). We get :

$y = (v_0\sin\theta_0)T_f - \frac{1}{2}gT_f^{\ 2} = 0$

$\Rightarrow T_f = \frac{2v_0\sin\theta_0}{g} = 2t_m$ ---eq.(6)

T_f is known as the time of flight of the projectile.

Maximum height of a projectile

The maximum height h_m reached by the projectile can be calculated by substituting $t = t_m$ in Eq. (3), which is

$$y = (v_0 \sin\theta_0) t_m - \frac{1}{2} g t_m^{\ 2} = h_m \Rightarrow h_m = \frac{(v_0 \sin\theta_0)^2}{2g}$$ ---eq.(7)

Horizontal range of a projectile

The horizontal distance travelled by a projectile from its initial position to the position where it passes y = 0 during its fall is called the horizontal range, R. It is the distance travelled during the time of flight T_f . Therefore, the range R is

$$R = T_f v_0 \cos\theta_0 = \frac{2v_0 \sin\theta_0}{g} v_0 \cos\theta_0 = \frac{v_0^{\ 2} \sin 2\theta_0}{g}$$ ---eq.(8)

R is maximum when $\theta_0 = 45^\circ$. The maximum horizontal range is, therefore,

$$R = \frac{v_0^{\ 2}}{g}.$$

NCERT Example 4.7

Galileo, in his book Two new sciences, stated that "for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal". Prove this statement.

SOLUTION

$$R = \frac{v_0^{\ 2} \sin 2\theta_0}{g}$$

$\theta_0 = \left(45^\circ \pm \theta_0\right)$ the ranges are the same.

NCERT Example 4.8

A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of $15m/s$. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take $g = 9.8m/s^2$).

SOLUTION

The equations of motion are :

$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

Here

$x_0 = y_0 = 0$, $a_y = -g = -9.8m/s^2$

$v_{0x} = 15m/s$

The stone hits the ground when $y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = -490m$. This gives t =10 s.

The velocity components are $v_x = v_{0x} = 15m/s$ and $v_y = v_{0y} - gt = -98m/s$

Therefore, the speed of the stone is

$\sqrt{v_x^2 + v_y^2} = 99m/s$

NCERT Example 4.9

A cricket ball is thrown at a speed of $28m/s$ in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

SOLUTION

(a)The maximum height is

$$h_m = \frac{(v_0 \sin\theta_0)^2}{2g} = \frac{(28\sin 30^\circ)^2}{2\times 9.8} m = 10m$$

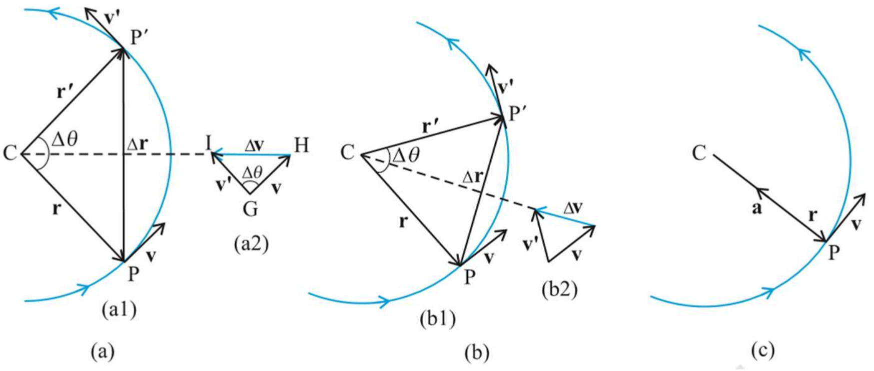
(b) $T_f = \dfrac{2v_0 \sin\theta_0}{g} = 2.9s$

(c) $R = \dfrac{v_0^2 \sin 2\theta_0}{g} = 69m$

UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called uniform circular motion.

Suppose an object is moving with uniform speed v in a circle of radius R as shown in Fig. Since the velocity of the object is changing continuously in direction, the object undergoes acceleration. Let us find the magnitude and direction of this acceleration.



Let $\vec{r}$ and $\vec{r}'$ be the position vectors, and $\vec{v}$ and $\vec{v}'$ the velocities of the object when it is at point P and P′ as shown in Fig. (a). Therefore, $\Delta\vec{v}$ is perpendicular to $\Delta\vec{r}$. Since average acceleration is along $\Delta\vec{v}$, the average acceleration $\overline{\vec{a}}$ is perpendicular to $\Delta\vec{r}$. We see that it is directed towards the centre of the circle. As $\Delta t \rightarrow 0$ and the average acceleration becomes the instantaneous acceleration. It is directed towards the centre. The magnitude of the instantaneous acceleration is,

$$|\vec{a}| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{v}|}{\Delta t}$$

Let the angle between position vectors $\vec{r}$ and $\vec{r}'$ is $\Delta\theta$. The angle between $\vec{v}$ and $\vec{v}'$ is also

$$\Delta\theta \, . \, \frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{R}$$

So

$$|\vec{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \vec{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} v \frac{|\Delta \vec{r}|}{R} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} = \frac{v}{R} v = \frac{v^2}{R}$$

Therefore, the centripetal acceleration a_c is

$$a_c = \frac{v^2}{R}$$

Thus, the acceleration of an object moving with speed v in a circle of radius R has a magnitude $\frac{v^2}{R}$ and is always directed towards the centre. This is why this acceleration is called centripetal acceleration. Since v and R are constant, the magnitude of the centripetal acceleration is also constant. However, centripetal acceleration is not a constant vector.

$\Delta\theta$ is called angular distance. We define the angular speed ω as the time rate of change of angular displacement :

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Now, if the distance travelled by the object during the time Δt is Δs, i.e. PP' is Δs, then :

$$v = \frac{\Delta s}{\Delta t}$$

$$\Delta v = R\Delta\theta$$

$$v = \frac{\Delta s}{\Delta t} = \frac{R\Delta\theta}{\Delta t} = R\omega$$

So

$$a_c = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = R\omega^2$$

The time taken by an object to make one revolution is known as its time period T and the number of revolution made in one second is called its frequency $\nu = \frac{1}{T}$. So $\omega = 2\pi\nu$.

$$a_c = R4\pi^2\nu^2$$

NCERT Example 4.10

An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (a) What is the angular speed, and the linear speed of the motion? (b) Is the acceleration vector a constant vector? What is its magnitude?

SOLUTION

This is an example of uniform circular motion. Here R = 12 cm. The angular speed ω is given by

$$\omega = \frac{2\pi}{T} = 2\pi\frac{7}{100} = 0.44 rad/s.$$

The linear speed $v = R\omega = (12cm)(0.44/s) = 5.3cm/s$

The direction of velocity $\vec{v}$ is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is not a constant vector. However, the magnitude of the acceleration is constant: $a = R\omega^2 = (12cm)(0.44/s)^2 = 2.3cm/s^2$