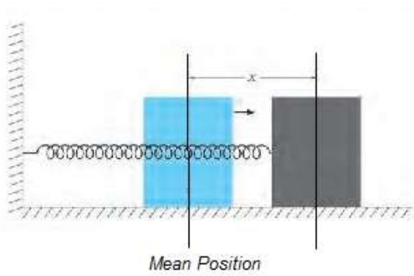
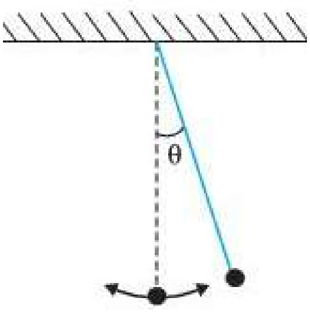
Chapter- 14
Oscillation

PERIODIC MOTION

- A motion that repeats itself after a regular interval of time is called periodic motion.
- Examples:
  (i) The motion of all planets, comets, etc. around the sun.
  (ii) The motion of an electron around the nucleus.
  (iii) Vibrations of a simple pendulum.
  (iv) Vibrations of the prongs of a tuning fork. etc.
- The smallest interval of time after which the motion is repeated is called it's period. It is denoted as T .
- The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the **frequency** of the periodic motion. It is represented by the symbol ν
- The relation between ν and T is
  $$\nu = \frac{1}{T}$$
- The unit of ν is thus s^{-1}. It is called hertz (abbreviated as Hz).
  1 hertz = 1 Hz = 1 oscillation per second $=1s^{-1}$.
- **Displacement in periodic motion :**
  - It refers to change with time of any physical property under consideration.
  - For example, in the case of the rectilinear motion of a steel ball on a surface, the distance from the starting point as a function of time is its **position displacement.**
  - Consider a block attached to a spring, the other end of the spring is fixed to a rigid wall. Generally, it is convenient to measure the displacement of the body from its equilibrium position.



  - For an oscillating simple pendulum, the angle from the vertical as a function of time may be regarded as a displacement variable.



- Any displacement variable is said to be periodic if it is a function $f(t)$ of time satisfying the condition ;

  $f(t+T) = f(t)$

  Where $T =$ the time period of the function
- Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients. This representation is given by the French mathematician, Jean Baptiste Joseph Fourier (1768–1830) and called a Fourier series. It is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T} \right]$$

Oscillatory Motion (Vibratory motion) :

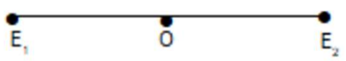
The motion of a body is said to be oscillatory if it moves back & forth about a fixed point after regular intervals of time.

The fixed point about which the body oscillates is called mean position or equilibrium position.

Note:

Every oscillatory motion is a periodic motion, but every periodic motion is not oscillatory.

Oscillation (Vibration):

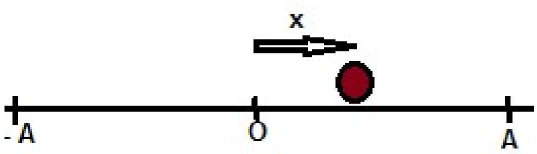


Let 'O' be the equilibrium position & E_1& E_2 are the extreme position of the oscillatory motion. The motion of the body from O to E_2& then to E_1& back to 'O' form one oscillation or vibration.

Simple Harmonic Motion (S.H.M.):

- ***A particle is said to be in S.H.M. if it is oscillating back and forth about the origin of an x-axis between the limits +A and –A and its position displacement at any instant is given by ;*** $x(t) = A\cos(\omega t + \theta_0)$

  where A, ω, and θ_0 are constants.



- Thus, simple harmonic motion (SHM) is not any periodic motion but one in which displacement is a sinusoidal function of time.
- **The terms in S.H.M. :**

  x (*t*) = displacement *x* as a function of time *t*

  A = amplitude = magnitude of maximum displacement

ω = angular frequency
$\omega t + \theta_0$ = phase angle θ (time-dependent)
θ_0 = phase constant or initial phase or phase

- **Time period (T)and frequency (ν) in S.H.M. :**
  By definition of time period ;
  $x(t) = x(t+T)$
  $\Rightarrow A\cos(\omega t + \theta_0) = A\cos(\omega(t+T) + \theta_0)$
  $\Rightarrow A\cos(\omega t + \theta_0) = A\cos(\omega t + \omega T + \theta_0)$
  As cosine function repeats after every 2π, hence
  $$\omega T = 2\pi$$
  $$\Rightarrow T = \frac{2\pi}{\omega}$$
  So frequency ; $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$

The relation between S.H.M. & Uniform Circular Motion:

Let a particle p moves with uniform angular velocity ω in a circle of radius 'A'. Then the particle 'p' is called the reference point. The circle is called the circle of reference.

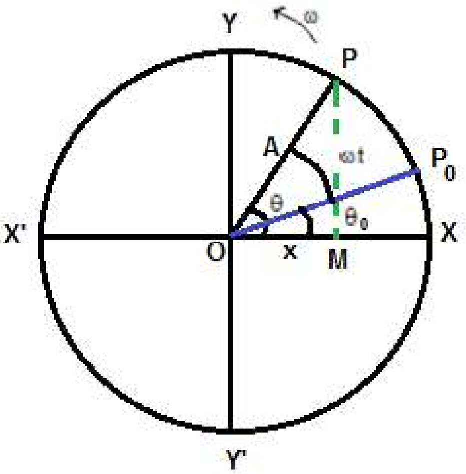
Let XX' be the diameter of the circle of reference & PM be the perpendicular on it. Then M is called the projection of P on XX'.

As the particle moves from X to X' through Y, its projection moves from X to X' through O. As the particle moves from X' to X through Y' its projection moves from X' to X through O. Thus, the projection M is said to execute S.H.M. along with XX'.

Thus S.H.M. is regarded as the projection of uniform circular motion upon the diameter of the circle of reference.

CHARACTERISTICS OF S.H.M.:

Displacement:



It is the distance of the particle executing S.H.M. from its mean position and is directed towards the point from the mean position.

Let OM = x = displacement of the particle at any instant.

From the figure, in triangle OMP,

$$\cos\theta = \frac{OM}{OP} = \frac{x}{A}$$

$\Rightarrow x = A\cos\theta$

Let t = time taken by the particle to reach the point P from P_0.

Then $\theta = \omega t + \theta_0$ [where θ_0 = initial angular position i.e. angle corresponding to initial position P_0]

$\Rightarrow x = A\cos(\omega t + \theta_0)$ ....(i)

Amplitude :

It is the maximum displacement of the particle from its mean position.

At the extreme position,

$$\cos(\omega t + \theta_0) = \pm 1$$
$$\Rightarrow x = \pm A$$

Where 'A' is called the amplitude of vibration.

Velocity :

It is defined as the time rate of change of displacement.

$$v = \frac{dx}{dt} = \frac{d}{dt}\{A\cos(\omega t + \theta_0)\}$$
$$\Rightarrow v = -\omega A\sin(\omega t + \theta_0)$$
$$\Rightarrow v = \omega A\sqrt{1 - \cos^2(\omega t + \theta_0)} = \omega\sqrt{A^2 - A^2\cos^2(\omega t + \theta_0)}$$
$$\Rightarrow v = \omega\sqrt{A^2 - x^2}$$

- When $x = \pm A$ i.e. at extreme ends $\Rightarrow$ $v = 0$ =minimum velocity
- When $x = 0$ i.e. at mean position $\Rightarrow$ $v = \pm\omega A$ =maximum velocity
- **Thus, the velocity is maximum at the mean position & zero at extreme positions.**

Acceleration:

It is defined as the time rate of change of velocity.

$$a = \frac{dv}{dt} = \frac{d}{dt}\{-\omega A\sin(\omega t + \theta_0)\}$$
$$\Rightarrow a = -\omega^2 A\cos(\omega t + \theta_0)$$
$$\Rightarrow a = -\omega^2 x$$

- When $x = \pm A$ i.e. at extreme ends $\Rightarrow$ $a = \mp\omega A$ =maximum velocity
- When $x = 0$ i.e. at mean position $\Rightarrow$ $a = 0$ =minimum velocity

Thus, acceleration is zero at mean position & maximum at extreme positions.

Graphical Representation of displacement, velocity, and acceleration in S.H.M:

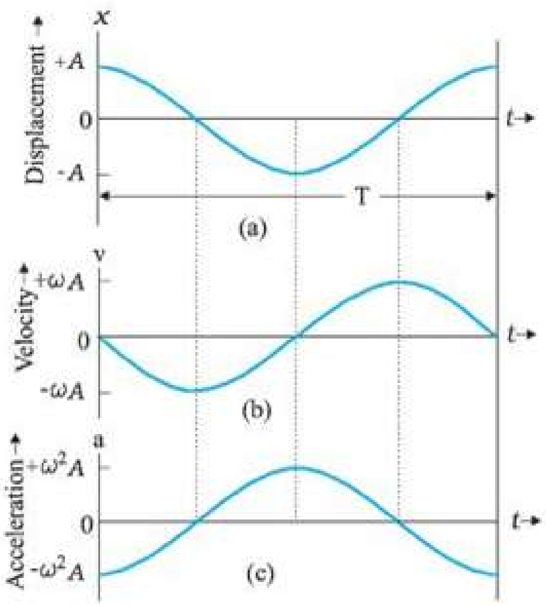
1. **Particle starting from extreme position :**
   i.e. at $t = 0$, $x = A \Rightarrow A\cos\theta_0 = A \Rightarrow \theta_0 = 0$
   So the equations representing displacement, velocity and acceleration are ;

$x = A\cos\omega t$

$v = -\omega A\sin\omega t$

$a = -\omega^2 A\cos\omega t$

So graphs $x \sim t, v \sim t$ and $a \sim t$ are as shown below.



2. Particle starting from the mean position in a positive direction :

i.e. at $t = 0$, $x = 0$ and $v = +ve \Rightarrow \cos\theta_0 = 0$ and $\sin\theta_0 = -1 \Rightarrow \theta_0 = \dfrac{3\pi}{2}$

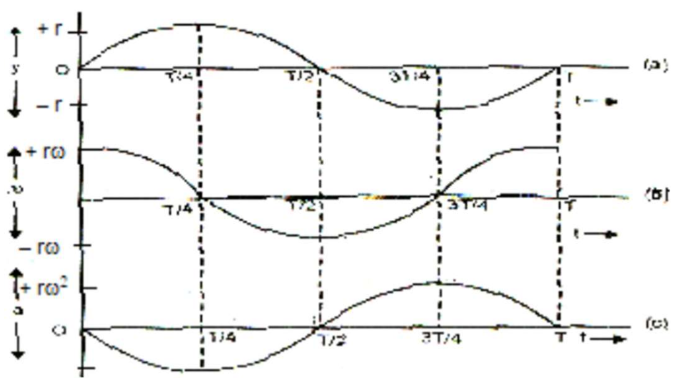
So the equations representing displacement, velocity and acceleration are ;

$x = A\sin\omega t$

$v = \omega A\cos\omega t$

$a = -\omega^2 A\sin\omega t$

So graphs $x \sim t, v \sim t$ and $a \sim t$ are as shown below.



3. Graphs showing the variation of velocity and acceleration with displacement :

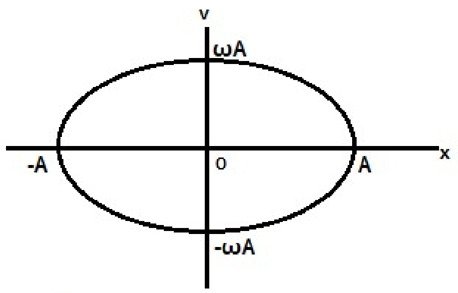
(a) $v \sim x$ graph :

As $v = \omega\sqrt{A^2 - x^2}$

$\Rightarrow v^2 = \omega^2 A^2 - \omega^2 x^2$

$\Rightarrow v^2 + \omega^2 x^2 = \omega^2 A^2$

$\Rightarrow \frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1$

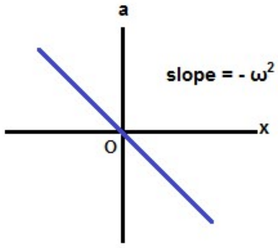


Hence $v \sim x$ graph is an **ellipse**.

(b) $a \sim x$ graph :

As $a = -\omega^2 x$

hence $a \sim x$ graph is a straight line passing through origin with slope = $-\omega^2$



Note:

(i) The phase difference between displacement - time curve & velocity-time curve is π/2 rad.

(ii) The phase difference between the displacement-time curve & acceleration - time curve is π rad.

(iii) The phase difference between the velocity-time curve & acceleration - time curve is π/2 rad.

Force law for the simple harmonic motion :

By Newton's 2nd law, resultant of all forces acting on a particle is given by $F = ma$

For S.H.M., we have $a = -\omega^2 x$

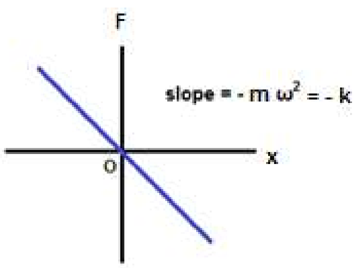
$\Rightarrow F = ma = -m\omega^2 x$

$\Rightarrow F = -kx$

Where, $k = m\omega^2$ = force constant or spring constant in S.H.M.

-ve sign indicates that force is directed opposite to the displacement i.e. always directed towards the mean position. So this force is also called as **restoring force**.

Graphically;



The expression for angular frequency, time period, and frequency in S.H.M. :

As the force constant is defined as, $k = m\omega^2$

$$\Rightarrow \omega^2 = \frac{k}{m}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\text{force constant}}{\text{Inertia factor}}}$$

Again $a = -\omega^2 x$, or in magnitude $|a| = \omega^2 |x|$

$$\Rightarrow \omega = \sqrt{\frac{|a|}{|x|}} = \sqrt{\frac{\text{acceleration}}{\text{displcement}}}$$

These are the two expressions for angular frequency.

Now time period ;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{|x|}{|a|}}$$

The frequency ;

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{|a|}{|x|}}$$

The energy in simple harmonic motion :

A particle executing simple harmonic motion possesses two types of energies.

(i) Potential Energy (E_p) :

The restoring force in simple harmonic motion is defined as; $F = -kx$

Also, the restoring force is conservative.

So potential energy of the system for any displacement x is ;

$$E_p = -\int_{x_{ref.}}^{x} F dx = -\int_{x_{ref.}}^{x} -kx dx = \int_{x_{ref.}}^{x} kx dx$$

$$\Rightarrow E_p = \left[\frac{kx^2}{2}\right]_{x_{ref.}}^{x} = \frac{1}{2}k\left[x^2 - x_{ref.}^2\right]$$

Where $x_{ref.}$ = displacement for reference point i.e. a point where the particle has 0 potential energy.

Usually mean position (i.e. x = 0) is taken as zero potential energy position .

So potential energy is ;

$$E_p = \frac{1}{2}kx^2$$

(ii) Kinetic Energy (E_k) :

The K.E. is due to the motion of the particle.

It is given by;

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\omega\sqrt{A^2 - x^2}\right)^2 = \frac{1}{2}m\omega^2\left(A^2 - x^2\right)$$

$$\Rightarrow E_k = \frac{1}{2}k\left(A^2 - x^2\right)$$

Here we have used the relations; $v = \omega\sqrt{A^2 - x^2}$ and $k = m\omega^2$

Total mechanical energy (E) :

The total mechanical energy of the particle at any instant is given by

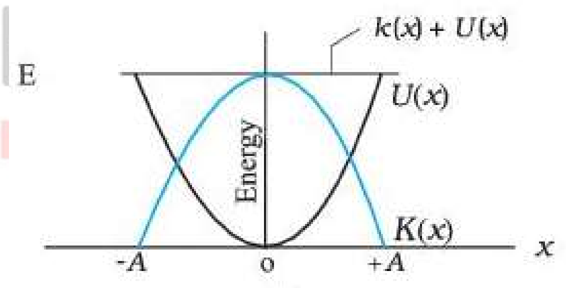
$$E = E_p + E_k = \frac{1}{2}kx^2 + \frac{1}{2}k\left(A^2 - x^2\right)$$

$$\Rightarrow E = \frac{1}{2}kA^2$$

This shows that the total mechanical energy is independent of position or time, i.e. total mechanical energy is conserved in simple harmonic motion.

Graphical representation of energy in S.H.M. :

1. **Variation with displacement :**



2. **Variation with time when starting from an extreme end :**

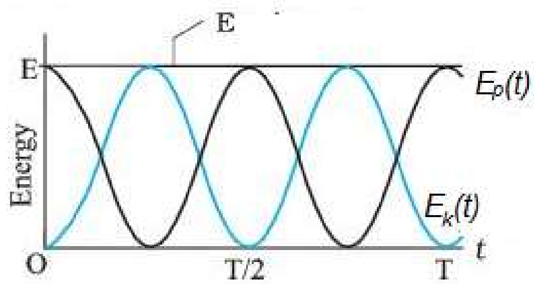
In this case; $x = A\cos\omega t$ and $v = -\omega A\sin\omega t$

$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2\omega t$$

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2\omega t == \frac{1}{2}kA^2\sin^2\omega t$$

$$[\because k = m\omega^2]$$

$$E = \frac{1}{2}kA^2\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}kA^2$$



3. **Variation with time when starting from mean position :**

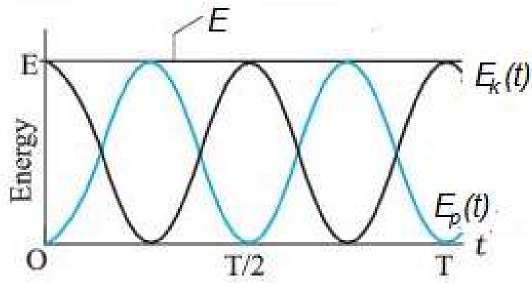
In this case; $x = A\sin\omega t$ and $v = \omega A\cos\omega t$

$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\omega t$$

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2\omega t == \frac{1}{2}kA^2\cos^2\omega t$$

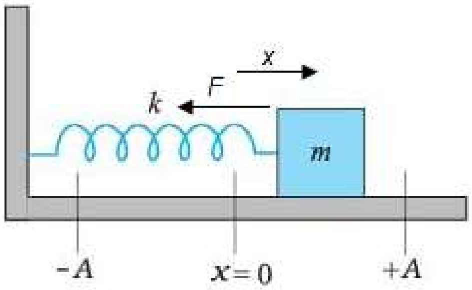
$$[\because k = m\omega^2]$$

$$E = \frac{1}{2}kA^2\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}kA^2$$



Some examples of S.H.M:

Spring-Block system oscillating horizontally on a frictionless surface :



In this case, the net force on the block for any displacement x is

$$F = -kx$$

Hence it satisfies the condition for S.H.M. and hence the block is in S.H.M.

As $k = m\omega^2$

$$\Rightarrow \omega^2 = \frac{k}{m}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

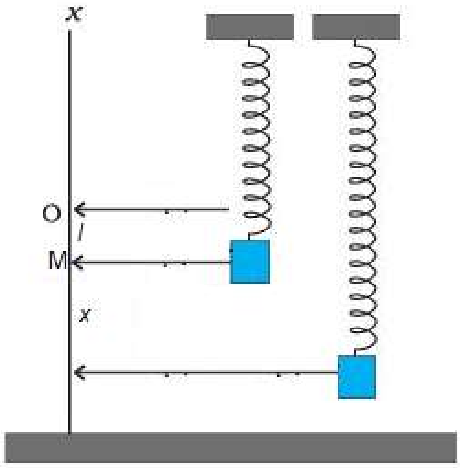
Now time period ;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency ;

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Spring-Block system oscillating vertically from a rigid surface :



Let, O: The unstretched free end of the spring.

M: Equilibrium position as the block is connected to the free end.

So the force equation is; $kl = mg$ ....(i)

For further displacement x, the net force on the block is

$$F = mg - k(l + x) = mg - kl - kx$$

$$\Rightarrow F = -kx \qquad \text{[Using equation (i)]}$$

Hence it satisfies the condition for S.H.M. and hence the block is in S.H.M.

As $k = m\omega^2$

$$\Rightarrow \omega^2 = \frac{k}{m}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

Now time period ;

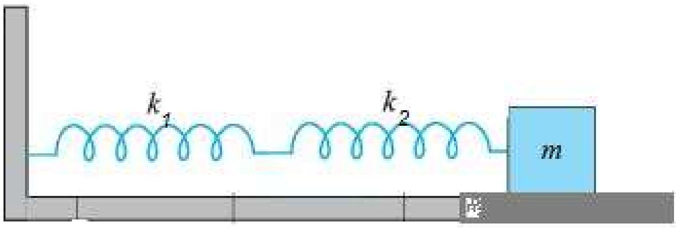
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency ;

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Multiple Spring-Block systems:

(a) Springs in series :



Force on each spring is the same .

Total elongation is ; $x = x_1 + x_2 + ...$

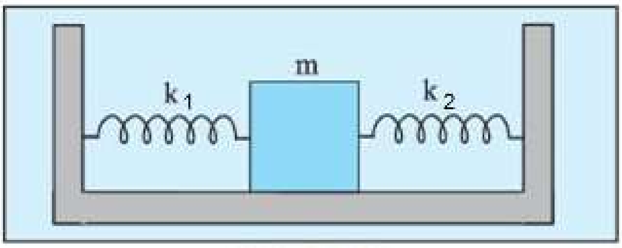
$$\Rightarrow \frac{F}{k_s} = \frac{F}{k_1} + \frac{F}{k_2} + ...$$

$$\Rightarrow \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + ...$$

Now time period ;

$$T = 2\pi\sqrt{\frac{m}{k_s}}$$

(b) Springs in parallel:

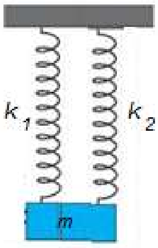


The deformation of each spring is the same .

The total force is ; $F = F_1 + F_2 + ...$

$\Rightarrow k_P x = k_1 x + k_2 x + ...$

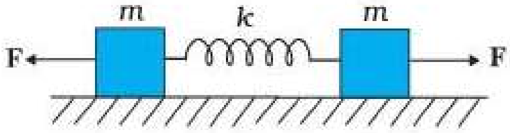
$\Rightarrow k_P = k_1 + k_2 + ...$



Now time period ;

$$T = 2\pi\sqrt{\frac{m}{k_P}}$$

(c) Two masses connected to the two ends of a spring :



In this case, the system can be reduced to one mass system of reduced mass given as;

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Now time period ;

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

The expression for spring constant of a spring :

Let a spring of length L, area of cross-section A, and Young's modulus Y be stretched by a length x.

Let the restoring force be F.

By definition ; $Y = \frac{F/A}{x/L}$

$$\Rightarrow F = \frac{YAx}{L}$$

This is in form ; $F = kx$

So the spring constant is;

$$k = \frac{YA}{L}$$

$$\Rightarrow k \alpha A \text{ and } k \alpha \frac{1}{L}$$

Angular S.H.M. :

In this oscillation net torque on the system is directly proportional to the angular displacement and directed opposite to the angular displacement. This is called as restoring torque.

So restoring torque is given by; $\tau = -k\theta$

Where k = torque constant.

Here at any instant;

angular displacement is $\theta = \theta_0 \cos(\omega t + \phi_0)$

angular velocity is $\Omega = -\Omega_0 \sin(\omega t + \phi_0)$

angular acceleration is $\alpha = \alpha_0 \cos(\omega t + \phi_0)$

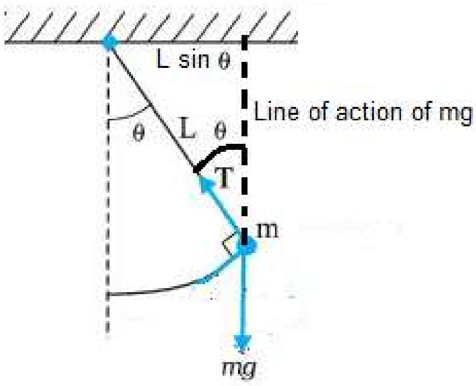
Where; $\omega = \text{Angular frequency} = \sqrt{\frac{\text{k}}{\text{I}}} = \sqrt{\frac{\text{torque constant}}{\text{inertia factor}}}$

So time period of oscillation is; $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}}$

Frequency of oscillation is; $\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$

Simple Pendulum:

- A simple pendulum is a point mass suspended by weightless & inextensible flexible string fixed rigidly to support.
- Since the point mass is a theoretical concept, in actual practice a heavy metallic bob suspended with a strong thread from rigid support is taken as the simple pendulum.
- The string is fixed rigidly at one end 'S' called the point of suspension. When the bob is displaced from the mean position 'O' to the extreme point 'E_1' & released, it starts, oscillating about 'O' & the motion is S.H.M. as explained below.
- As the bob gets displaced through 'O' to the point E_1, the various forces acting on the pendulum are –
  (a) Weight 'mg' of the bob in the vertically downward direction.
  (b) Tension (F_T) along the length of the string & towards the point of suspension.



- As tension is along the string and hence always intersects the axis of rotation, its torque is 0.
  So the net torque on the pendulum is;
  $\tau = (mg).r_\perp = mgl\sin\theta$
  For very small angular displacement; $\theta \to 0$
  $\Rightarrow \sin\theta \to \theta$
  So the net torque on the pendulum is;
  $\tau = mgl\theta$
  Vectorially; $\tau = -mgl\theta$ [$\because \tau$ is directed opposite to θ]

Hence motion is angular S.H.M. with; $k = mgl$

So time period of oscillation is; $T = 2\pi\sqrt{\frac{I}{k}}$

Here I = moment of inertia of the bob $= ml^2$

So the time period is; $T = 2\pi\sqrt{\frac{mgl}{ml^2}}$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$

Discussion :

(a) The time period does not depend upon the mass of the bob i.e. pendulum of equal length but different masses will have the same period 'T'.

(b) By moving up from earth surface :
g decreases $\Rightarrow$ T increases

(c) By moving down into earth surface :
g decreases $\Rightarrow$ T increases

(d) At the center of the earth; $g = 0 \Rightarrow T = \infty$

(e) By moving from equator to pole ; g increases $\Rightarrow$ T decreases

(f) Pendulum in an elevator :

(i) The elevator at rest or uniform motion :

Effective weight; $F_T = mg$

$\Rightarrow g_{eff} = g$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g}}$ i.e. no change

(ii) Elevator accelerating up :

Effective weight; $F_T = m(g+a)$

$\Rightarrow g_{eff} = g + a$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g+a}}$ i.e. less than the actual time period

(iii) Elevator accelerating down :

Effective weight; $F_T = m(g-a)$

$\Rightarrow g_{eff} = g - a$

$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g-a}}$ i.e. more than the actual time period

(iv) Elevator falling freely :

Effective weight; $F_T = m(g-g) = 0$

$\Rightarrow g_{eff} = 0$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{0}} = \infty$$

(v) Elevator accelerating horizontally :

In this case, the equilibrium position makes angle θ with vertical in the opposite direction of motion.

So; $F_T \cos\theta = mg$

And $F_T \sin\theta = ma$

So the effective weight is; $F_T = \sqrt{(F_T \cos\theta)^2 + (F_T \sin\theta)^2}$

$$\Rightarrow F_T = \sqrt{(mg)^2 + (ma)^2} = m\sqrt{g^2 + a^2}$$

$$\Rightarrow g_{eff} = {F_T}/{m} = \sqrt{g^2 + a^2}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}} = 2\pi\left(\frac{l}{\sqrt{g^2 + a^2}}\right)^{1/2}$$

(vi) Elevator moving in a uniform circular motion:

In this case, the equilibrium position makes angle θ with vertical in the radially outward direction.

So; $F_T \cos\theta = mg$

And $F_T \sin\theta = \frac{mv^2}{r}$

So the effective weight is; $F_T = \sqrt{(F_T \cos\theta)^2 + (F_T \sin\theta)^2}$

$$\Rightarrow F_T = \sqrt{(mg)^2 + \left(\frac{mv^2}{r}\right)^2} = m\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2}$$

$$\Rightarrow g_{eff} = {F_T}/{m} = \sqrt{g^2 + \left(\frac{v^2}{r}\right)^2}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + (v^2/r)^2}}} = 2\pi\left(\frac{l}{\sqrt{g^2 + v^2/r}}\right)^{1/2}$$

(g) **By the rise in temperature;** the length of string increases. Hence time period also increases.

(h) **By fall in temperature;** the length of string decreases. Hence time period also decreases.

(i) **When the time period of the pendulum increases the clock becomes slower and when the time period of the pendulum decreases the clock becomes faster.**

Second, 's pendulum :

A pendulum is said to be second's pendulum if its time period is 2 s.

As $T = 2\pi\sqrt{\frac{l}{g}}$

$$\Rightarrow T^2 = 4\pi^2 \frac{l}{g}$$

$$\Rightarrow l = \frac{gT^2}{4\pi^2} = \frac{9.8 \times 4}{4\pi^2} \approx 1m$$

So the second's pendulum is also called as meter's pendulum.

A. Free Vibrations :

- When a body vibrates with its natural frequency, It is said to execute free vibrations.
- In this case, the resultant force is;

$$F = -kx$$

$$\Rightarrow ma = -kx$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is the differential equation of free vibration

Example

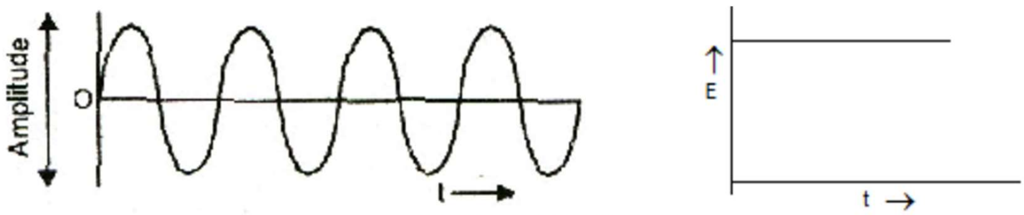
(a) When a tuning fork is struck against a rubber pad, the prongs begin to execute free vibrations.

(b) When a stretched string is plucked, it executes free vibrations.

(c) When the bob of a simple pendulum is displaced from its mean position and released, it executes free oscillations.

Note:

Free oscillations are also called ***undamped oscillations****. The amplitude and energy remain constant with time because of the absence of a resistive medium.*



Undamped Oscillations

B. Damped Oscillations:

- In actual practice, most of the oscillations occur in resistive media like air, water, etc. Thus a fraction of energy of the oscillating system is dissipated in the form of heat overcoming resistive forces. Thus, the amplitude of oscillation gradually decreases with time, and finally, the oscillating system stops.
- Such oscillations in which amplitude decreases continuously are called damped oscillations.
- In this case, a damping force is proportional to the speed and directed opposite to the direction of motion.

  This force is given by; $F_d = -bv$

  Where b = damping constant which depends upon the medium only
- So resulting force in damped oscillation is;

  $$F = -bv - kx$$

  $$\Rightarrow ma = -bv - kx$$

  $$\Rightarrow ma + bv + kx = 0$$

  $$\Rightarrow a + \frac{b}{m}v + \frac{k}{m}x = 0$$

  $$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = 0$$

  This is the differential equation of the damped oscillation

  Here $\omega = \sqrt{\frac{k}{m}}$ = Angular frequency of undamped oscillation
- The solution of the differential equation represents the displacement.

  $$x = Ae^{\frac{-bt}{2m}}\cos(\omega' t + \theta_0)$$

  Where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ = Angular frequency of the damped oscillation

  Here amplitude of the damped oscillation;

  $$A' = Ae^{\frac{-bt}{2m}}$$

  This shows that the amplitude decreases exponentially with time.

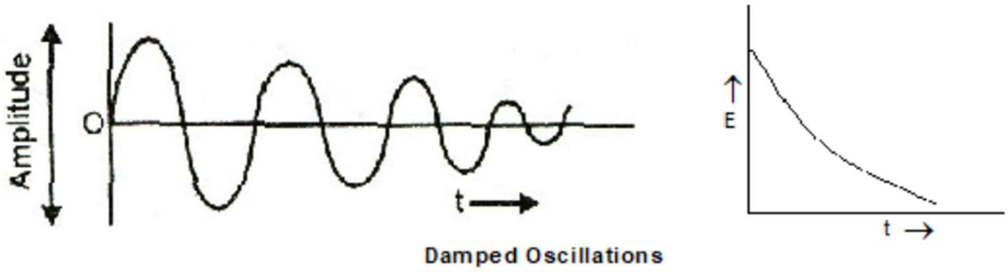
  The mechanical energy of damped oscillation is;

  $$E' = \frac{1}{2}kA'^2 = \frac{1}{2}kA^2\left(e^{\frac{-bt}{2m}}\right)^2 = Ee^{\frac{-bt}{m}}$$

  Where $E = (1/2)kA^2$ = Energy at (t = 0) or mechanical energy of undamped oscillation

Examples:(a) The oscillations of a pendulum in air.

(b)oscillations of a mass attached to a spring and placed on the table.



C. Forced Vibrations:

- When a body is maintained in a state of vibration by a strong periodic force of frequency other than the natural frequency of the body, the vibrations are called forced vibrations.
- In this oscillation; after some time of starting the particle oscillates with the frequency of the periodic force.
- Let the periodic force acting be $F = F_0 \cos\omega_d t$

  So the net force on the particle; $F_{net} = -kx - bv - F_0 \cos\omega_d t$

  By Newton's 2nd law ; $F_{net} = ma$

  $\therefore ma = -bv - kx + F_0 \cos\omega_d t$

  $\Rightarrow ma + bv + kx = F_0 \cos\omega_d t$

  $$\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega_d t$$

  $$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega_d t$$

  $$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\cos\omega_d t$$

  This is the differential equation of forced vibration.

  Let the solution be;

  $x = A\cos(\omega_d t + \theta_0)$

  $$\Rightarrow \frac{dx}{dt} = -\omega_d A\sin(\omega_d t + \theta_0)$$

  $$\Rightarrow \frac{d^2x}{dt^2} = -\omega_d^2 A\cos(\omega_d t + \theta_0)$$

  Substituting these in the differential equation we get;

  $$-\omega_d^2 A\cos(\omega_d t + \theta_0) - \omega_d A\frac{b}{m}\sin(\omega_d t + \theta_0) + \omega^2 A\cos(\omega_d t + \theta_0) = \frac{F_0}{m}\cos(\omega_d t + \theta_0 - \theta_0)$$

  $$(\omega^2 - \omega_d^2)A\cos(\omega_d t + \theta_0) - \omega_d A\frac{b}{m}\sin(\omega_d t + \theta_0) = \frac{F_0}{m}\cos\theta_0\cos(\omega_d t + \theta_0) + \frac{F_0}{m}\sin\theta_0\sin(\omega_d t + \theta_0)$$

Comparing the coefficients we have;

$$\left(\omega^2-\omega_d^2\right)A=\frac{F_0}{m}\cos\theta_0$$

$$-\omega_d A\frac{b}{m}=\frac{F_0}{m}\sin\theta_0$$

Squaring and adding both

$$\frac{F_0}{m}=A\sqrt{\left(\omega^2-\omega_d^2\right)^2+\frac{\omega_d^2b^2}{m^2}}$$

$$\Rightarrow F_0=A\sqrt{m^2\left(\omega^2-\omega_d^2\right)^2+\omega_d^2b^2}$$

$$\Rightarrow A=\frac{F_0}{\sqrt{m^2\left(\omega^2-\omega_d^2\right)^2+\omega_d^2b^2}}$$ .... (i) [Expression for amplitude]

Again dividing both

$$\tan\theta_0=\frac{-\omega_d b/m}{\omega^2-\omega_d^2}=\frac{-\omega_d b}{m\left(\omega^2-\omega_d^2\right)}$$

$$\Rightarrow \theta_0=\tan^{-1}\left[\frac{-\omega_d b}{m\left(\omega^2-\omega_d^2\right)}\right]$$ ....(ii).

[Phase difference between displacement and force]

Examples:

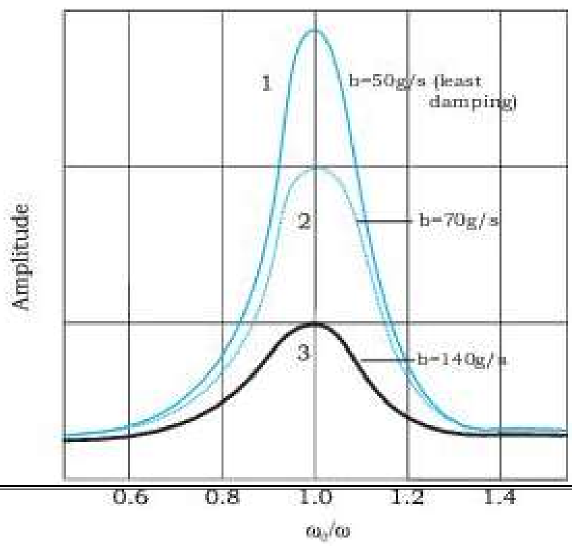
(a) When the stem of a vibrating tuning fork is pressed against the top of a table, the table starts vibrating with the frequency of the tuning fork which is different from the natural frequency of vibration of the table.

(b) The soundboards of all stringed musical instruments like sitar, violin, etc. execute forced oscillations and the frequency of oscillations is equal to the natural frequency of vibrating string.

D. Resonant Vibrations:

- When a body is maintained in a state of vibration by periodic force having the same natural frequency as that of the body, the vibrations are called resonant vibrations.
- In this stage; $\omega_d=\omega$
  i.e. frequency of applied periodic force = natural frequency of vibration
- As in forced vibration;

$$A=\frac{F_0}{\sqrt{m^2\left(\omega^2-\omega_d^2\right)^2+\omega_d^2b^2}}$$



At resonance amplitude is maximum and is given by;

$$A_{\max} = \frac{F_0}{\sqrt{\omega_d^2 b^2}} = \frac{F_0}{\omega_d b}$$

- The resonant amplitude depends upon damping constant b as; $A_{\max} \alpha \frac{1}{b}$.

  Greater damping less is the resonant amplitude and vice-versa.
- When the applied frequency is far away from the natural frequency and damping is less, then;

$$A = \frac{F_0}{\sqrt{m^2\left(\omega^2 - \omega_d^2\right)^2}} = \frac{F_0}{m\left(\omega^2 - \omega_d^2\right)}$$

- **The resonance curve is shown in the figure.**

Examples

(a) Resonance can cause disasters during earthquakes. If the natural frequency of a building becomes equal to the frequency of periodic oscillations present within the earth, then the building will start oscillating with a large amplitude thereby damaging itself.

(b) Soldiers while crossing a suspension bridge are always asked to break their steps. If they march in steps while crossing the bridge, the frequency of their steps may match the natural frequency of the bridge. In that case, due to resonance the bridge may start oscillating violently and may collapse ultimately

E. Maintained Oscillations :

It the energy fed to the oscillating system continuously in such a way that the rate of feeding back the energy is equal to the rate of dissipation energy due to friction, then the system will go on oscillating with constant amplitude. such oscillations are called maintained oscillations.