Chapter- 7

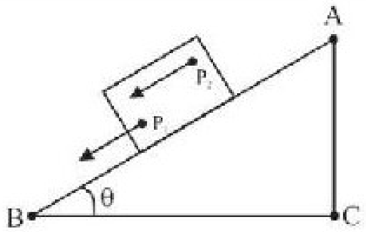
System of Particles and Rotational Motion

Rigid Body :

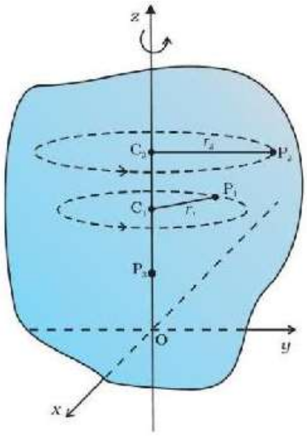
Ideally, a rigid body is a body with a perfectly definite and unchanging shape. The distance between all pairs of particles of such a body do not change.

Kinds of Motion of a rigid body :

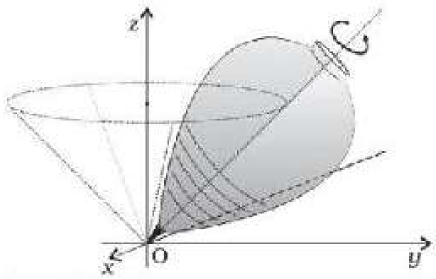
- **Pure Translation :**
  - In pure translational motion, at any instant of time, all particles of the body have the same velocity.
  - E.g. : Sliding of a block
  - Pure translational motion of a rigid body or a system of particles is considered as a translation of a point called as the *centre of mass* of the system.



- **Pure rotation :**
  - In pure rotation, the body rotates about a fixed axis called an *axis of rotation*.
  - At any instant; all particles of the body have same angular velocity (linear velocity may be different as $v = r\omega$ and all particles don't have same r i.e. distance from the axis of rotation)
  - E.g. : Rotation of a ceiling fan, rotation of a potter's wheel etc.



- **Combination of translation and rotation :**
  - Rolling Motion: Body covers a linear distance on rotating around an axis.
  - Precession: Here one point of the body is fixed. Axis of rotation passing through the point rotate around a vertical line through the fixed point. e.g.: A Spinning top.
  - Motion of planets around sun or Motion of oscillating table fan



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- **Vibratory Motion :**
  - A different particle of a system may vibrate about their mean position.
  - In vibratory motion, a particle moves to and fro about a mean position along a line.
  - E.g. : Vibration of molecules in a substance.

Centre of mass :

- Mass of centre of mass (m_{CM}) = total mass of the system = $m_1 + m_2 + m_3 + \ldots\ldots + m_N$
- Translational motion of the system is equivalent to the translation of the centre of mass. i.e. Linear momentum of C.M. ($\vec{p}_{CM}$) = $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots\ldots + \vec{p}_N$
- Position of C.M. is ; $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots.. + m_N\vec{r}_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \dfrac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i}$

  So co-ordinates of C.M.; $x_{CM} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots.. + m_Nx_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \dfrac{\sum_{i=1}^{N} m_ix_i}{\sum_{i=1}^{N} m_i}$

  $$y_{CM} = \dfrac{m_1y_1 + m_2y_2 + m_3y_3 + \ldots.. + m_Ny_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \dfrac{\sum_{i=1}^{N} m_iy_i}{\sum_{i=1}^{N} m_i}$$

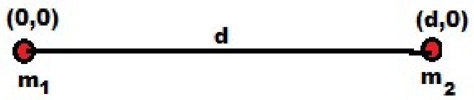
  $$z_{CM} = \dfrac{m_1z_1 + m_2z_2 + m_3z_3 + \ldots.. + m_Nz_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \dfrac{\sum_{i=1}^{N} m_iz_i}{\sum_{i=1}^{N} m_i}$$

- If C.M. of a system is taken as origin, then $\sum_{i=1}^{N} m_i\vec{r}_i = 0$

Centre of a mass of two-particle system :

Let two particles of masses m_1 and m_2 lie at a separation d.

Assume they lie on x-axis and m_1 is at the origin



So position of m_2 = (d,0)

So the centre of mass must lie on the x-axis with co-ordinates ;

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1.0 + m_2d}{m_1 + m_2} = \frac{m_2d}{m_1 + m_2}$$

So C.M. is at a distance $\frac{m_2d}{m_1 + m_2}$ from m_1 and hence at a distance $\frac{m_1d}{m_1 + m_2}$ from m_2

So the ratio of distances of C.M. from m_1 and m_2 = m_2: m_1

If both the particles have equal masses then the position of C.M. is (d/2) i.e. midpoint of the line joining the two masses.

C.M. of a continuous or rigid body :

- For a rigid body, such as a metre stick or a flywheel, is a system of closely packed particles; the number of particles (atoms or molecules) in such a body is so large that it is impossible to carry out the summations over individual particles to find C.M.
- Since the spacing of the particles is small, we can treat the body as a continuous distribution of mass.
- For such cases divide the whole body into a large number of small elementary parts each of mass dm and one such element is chosen at (x, y, z).
- Now coordinates of the centre of mass is given by ;

$$x_{CM} = \frac{\int xdm}{M}$$

$$y_{CM} = \frac{\int ydm}{M}$$

$$z_{CM} = \frac{\int zdm}{M}$$

Or $$\vec{r}_{CM} = \frac{\int \vec{r}dm}{M}$$

Where M = mass of the rigid body.

Centre of a mass of a uniform thin rod :

Let M = mass of the rod

L = length of the rod.

As mass is distributed throughout the length of the rod uniformly so mass per unit length of the rod is same throughout and equal to $\frac{M}{L}$.

Now let's divide the whole rod into elementary lengths.

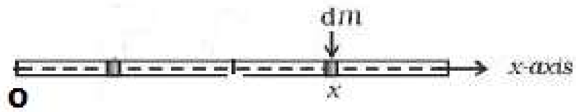
Let one such element of length *dx* at a position *x* w.r.t. one end of the rod.

So the mass of the element is

$$dm = \frac{M}{L}dx$$

As we are considering the rod along the x-axis, so the C.M. will lie on X-axis also. Hence we need to find the x-co-ordinate of C.M. only.

$$x_{CM} = \frac{1}{M}\int_0^M x\,dm = \frac{1}{M}\int_0^L x\frac{M}{L}dx = \frac{1}{M}\frac{M}{L}\int_0^L x\,dx$$



$$\Rightarrow x_{CM} = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$\Rightarrow x_{CM} = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

So C.M. of the uniform rod lies at the mid-point of the rod.

Note :

The centre of mass of homogeneous bodies of regular shapes like ***rings, discs, spheres, rods*** etc. (By a homogeneous body we mean a body with uniformly distributed mass.), by using symmetry consideration, can easily be shown to lie at their geometric centres.

Question :

Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

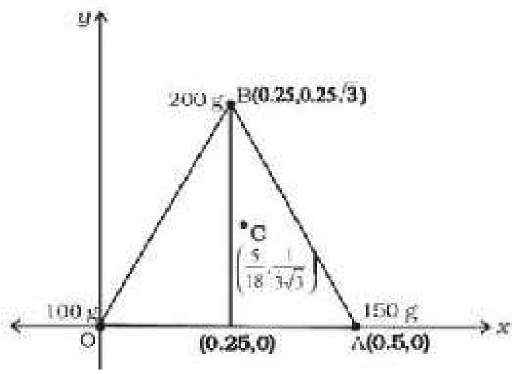
Ans :

Let co-ordinates of the particles 100 g , 150 g and 200 g be respectively as (0,0) , (0.5,0) and (0.25 , 0.25$\sqrt{3}$) respectively .

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{100(0) + 150(0.5) + 200(0.25)}{100 + 150 + 200}$$

$$= \frac{75 + 50}{450} = \frac{5}{18}m$$



$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{100(0) + 150(0) + 200(0.25\sqrt{3})}{100 + 150 + 200}$$

$$= \frac{50\sqrt{3}}{450} = \frac{\sqrt{3}}{9}m$$

Question :

Find the centre of mass of a triangular lamina.

Ans :

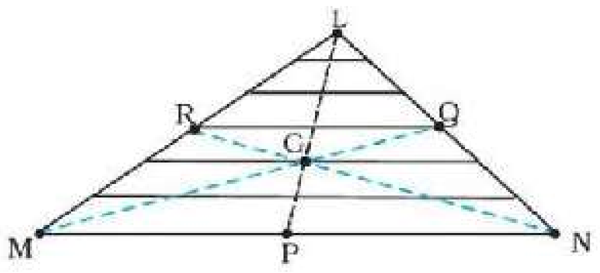
The lamina (D*LMN*) may be subdivided into narrow strips each parallel to the base *(MN).*

By symmetry, each strip has its centre of mass at its midpoint.

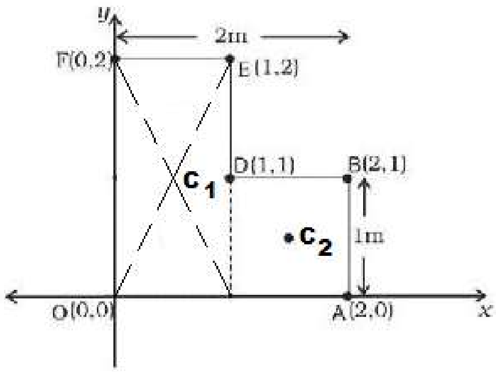
If we join the midpoint of all the strips we get the median LP.

The centre of mass of the triangle as a whole, therefore, has to lie on the median LP.

Similarly, we can argue that it lies on the median MQ and NR.
This means the centre of mass lies on the point of concurrence of the medians, i.e. on the centroid G of the triangle.



Question :
Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.
Ans :
Let thickness of the lamina be "t" i.e. uniform and its density be ρ.
Now let's consider the lamina as a combination of two rectangular plates with centres of mass C_1 and C_2.
Now due to symmetry ;



$$C_1(x_1, y_1) = \left(\frac{0+1}{2}, \frac{0+2}{2}\right) = \left(\frac{1}{2}, 1\right)$$

$$C_2(x_2, y_2) = \left(\frac{1+2}{2}, \frac{1+0}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

Now let the areas be A_1 and A_2 respectively.
So;

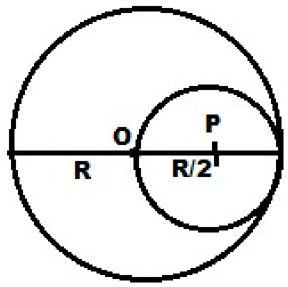
$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\rho A_1 t x_1 + \rho A_2 t x_2}{\rho A_1 t + \rho A_2 t} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2}$$

$$= \frac{2(1/2) + 1(3/2)}{2+1} = \frac{5}{6}m$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{\rho A_1 t y_1 + \rho A_2 t y_2}{\rho A_1 t + \rho A_2 t} = \frac{A_1y_1 + A_2y_2}{A_1 + A_2}$$

$$= \frac{2(1) + 1(1/2)}{2+1} = \frac{5}{6}m$$

Question :
From a uniform disk of radius *R*, a circular hole of radius *R*/2 is cut out. The centre of the hole is at *R*/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.
Ans :



Let the centre of mass of the whole disc be origin O.
So the co-ordinate of the centre of mass of the cut portion = P(R/2,0)
Thickness is uniform equal to *t*.
Let the density of the material be ρ.

Now the co-ordinates of the C.M. of the remaining portion be ;

$$x_{CM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2} = \frac{\rho A_1 t x_1 - \rho A_2 t x_2}{\rho A_1 t - \rho A_2 t} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{\pi R^2(0) - (\pi R^2/4)(R/2)}{\pi R^2 - (\pi R^2/4)} = \frac{-R}{6}$$

$$y_{CM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2} = \frac{\rho A_1 t y_1 - \rho A_2 t y_2}{\rho A_1 t - \rho A_2 t} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$= \frac{\pi R^2(0) - (\pi R^2/4)(0)}{\pi R^2 - (\pi R^2/4)} = 0$$

So C.M. becomes (-R/6, 0) i.e. shifts by R/6 towards the material side of the disc.

motion of the centre of mass :

- **Velocity and acceleration of C.M.:** As

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots.. + m_N\vec{r}_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i}$$

- Taking derivatives w.r.t. t in both the sides we have a velocity of C.M. as ;

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots.. + m_N\vec{v}_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{v}_i}{\sum_{i=1}^{N} m_i}$$

Taking further derivatives w.r.t. t we have an acceleration of C.M. as ;

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots.. + m_N\vec{a}_N}{m_1 + m_2 + m_3 + \ldots. + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{a}_i}{\sum_{i=1}^{N} m_i}$$

- **Net force on C.M. :**

As the momentum of C.M. is ;

$$\vec{p}_{CM} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots\ldots + \vec{p}_N$$

Taking derivatives w.r.t. t and using Newton's 2nd law we have

$$\vec{F}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots + \vec{F}_N$$
$$\Rightarrow m_{CM}\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots + \vec{F}_N$$

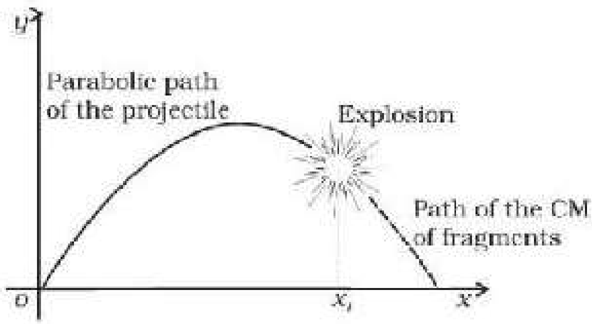
Thus, the total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.
Here the force **F_1** on the first particle means the vector sum of all the forces on the first particle; likewise for the second particle etc.
Among these forces on each particle, there will be external forces exerted by bodies outside the system and also internal forces exerted by the particles on one another.
We know from Newton's third law that these internal forces occur in equal and opposite pairs and the sum of forces, their contribution is zero.
So only the external forces contribute to the acceleration of the C.M.
Hence

$$m_{CM}\vec{a}_{CM} = \sum \vec{F}_{ext}$$
$$\Rightarrow \vec{F}_{CM} = \sum \vec{F}_{ext}$$
$$\Rightarrow \frac{d\vec{p}_{CM}}{dt} = \sum \vec{F}_{ext}$$

This is Newton's 2nd law for a system of particles.
Conclusion: The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.

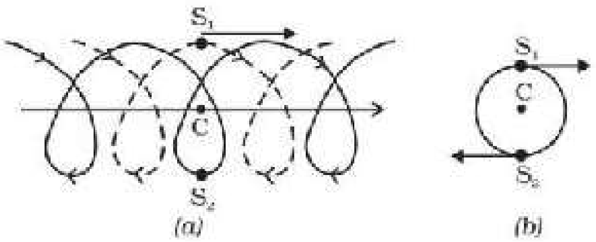
- **If** $\sum \vec{F}_{ext} = \vec{0}$
  Then the momentum of the C.M. will be conserved.
- **If during motion of a system any interaction occurs among particles, still C.M. will move as of its original path.**
  e.g. :

1.



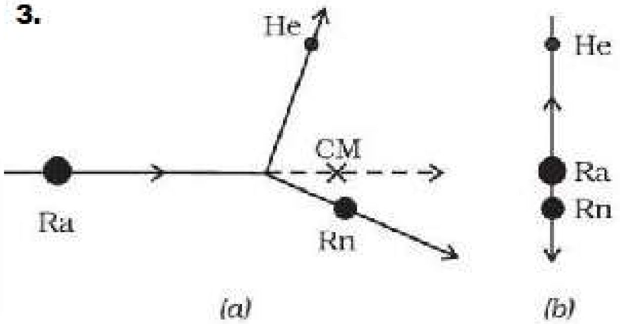
[*The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.*]

2.



(a) Trajectories of two stars, S_1 (dotted line) and S_2 (solid line) forming a binary system with their centre of mass C in uniform motion.
(b) The same binary system, with the centre of mass C at rest.

3.



(a) A heavy nucleus radium (Ra) splits into a lighter nucleus radon (Rn) and an alpha particle (nucleus of helium atom). The CM of the system is in uniform motion.
(b) The same splitting of the heavy nucleus radium (Ra) with the centre of mass at rest. The two product particles fly back to back.

Relation between Angular velocity and linear velocity in pure rotation :

If a rigid body or system of a particle is rotating about an axis then all particle have the same angular velocity ω directed long axis of rotation.

The linear velocity of a particle at a perpendicular distance r from the axis of rotation is v directed along a tangent.

So vectorially,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

In magnitude ;

$$v = \omega r$$

Angular acceleration: At any instant ;

$$\alpha = \frac{d\omega}{dt}$$

Linear acceleration :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{a}_T + \vec{a}_r$$

$\Rightarrow \vec{a}_T = \vec{\alpha}\times\vec{r} =$ tangential component of acceleration

and $\vec{a}_r = \vec{\omega}\times\vec{v} =$ Radial component of acceleration or centripetal acceleration

In magnitude ; $a_T = \alpha r$

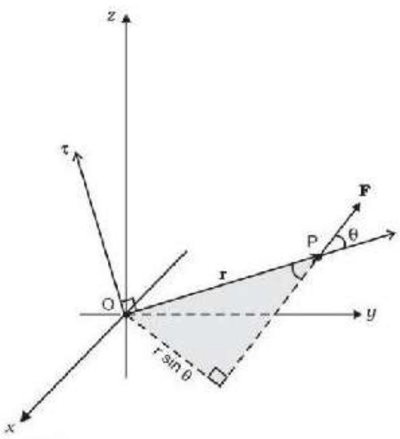
$a_r = \omega v = v^2/r = \omega^2 r$

$a = \sqrt{a_T^2 + a_r^2}$

Moment of a force (Torque) :

- The rotational analogue of force is the ***moment of force.*** It is also referred to as ***torque***.
- If a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector r, the moment of the force acting on the particle with respect to the origin O is defined as the vector product

$$\vec{\tau} = \vec{r}\times\vec{F}$$



- In magnitude ; $\tau = rF\sin\theta = (r\sin\theta)F = r_\perp F$
  Where $r_\perp$ = perpendicular distance of the line of force from axis of rotation called a **moment arm**.
- Also ; $\tau = rF\sin\theta = r(F\sin\theta) = rF_\perp$
  Where $F_\perp$ = Perpendicular component of force to **r.**
- If line of action of the force is in such direction that if produced it can meet the axis of rotation then torque of the force = 0
  Because in this case, $\theta = 0^0$ or 180^0
- If line of action is parallel to the axis of rotation then also torque of the force is 0.
- S.I. unit of torque is N m .
- Dimensional formula for torque is $[M^1L^2T^{-2}]$

Question :

Find the torque of a force 7iˆ + 3j ˆ – 5k ˆ about the origin. The force acts on a particle whose position vector is iˆ – j ˆ + k ˆ...

Answer :

$$\vec{\tau} = \vec{r}\times\vec{F} = \hat{i}(yF_z - zF_y) + \hat{j}(zF_x - xF_z) + \hat{k}(xF_y - yF_x)$$

$$\Rightarrow \vec{\tau} = \hat{i}(5-3) + \hat{j}(7+5) + \hat{k}(3+7)$$

$$\Rightarrow \vec{\tau} = 2\hat{i} + 12\hat{j} + 10\hat{k}$$

Angular Momentum (Moment of momentum) of a particle :

- If a particle of mass *m* and linear momentum $\vec{p}$ at a position $\vec{r}$ relative to the origin O, then the angular momentum $\vec{l}$ of the particle with respect to the origin O is defined to be

  $$\vec{l} = \vec{r} \times \vec{p}$$

- In magnitude ; l = r p sinθ = (r sinθ) p = $r_{\perp}p$

  Where $r_{\perp}$ = perpendicular distance of direction of motion of the particle from the axis of rotation .
- Also ; l = rpsinθ = r (psinθ) = $r\,p_{\perp}$

  Where $p_{\perp}$ = Perpendicular component of force to **r.**
- If line of motion of a particle is in such direction that if produced it can meet the axis of rotation then its angular momentum = 0

  Because in this case, θ = 0^0 or 180^0
- If a line of motion is parallel to the axis of rotation then also angular momentum of the particle is 0.
- S.I. unit of torque is kg m^2s^{-1} .
- Dimensional formula for torque is $[M^1L^2T^{-1}]$

Question :

Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Answer :

Let the particle with velocity **v** be at point P at some instant *t.*

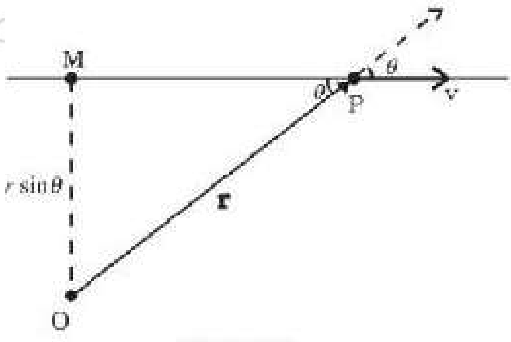
We want to calculate the angular momentum of the particle about arbitrary point O.

The angular momentum is

l = **r** × *m***v**.

Its magnitude is *l = mvr* sinθ,

where θ is the angle between **r** and **v**.



Although the particle changes position with time, the line of direction of **v** remains the same and hence OM = *r sin* θ. is a constant.

Further, the direction of ***l*** is perpendicular to the plane of **r** and **v**.

This direction does not change with time.

Thus, ***l*** remains the same in magnitude and direction and is therefore conserved.

Relation for angular momentum and torque for particle :

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\Rightarrow \frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}(\vec{r}) \times \vec{p} + \vec{r} \times \frac{d}{dt}(\vec{p})$$

$$\Rightarrow \frac{d\vec{l}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F}_{net}$$ [As $\frac{d\vec{r}}{dt} = \vec{v}$ and by Newton's 2nd law $\frac{d\vec{p}}{dt} = \vec{F}_{net}$]

Since $\vec{p} = m\vec{v} \Rightarrow \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = \vec{0}$ and $\vec{r} \times \vec{F}_{net} = \vec{\tau}_{net}$ = Net torque on the particle.

$$\therefore \frac{d\vec{l}}{dt} = \vec{\tau}_{net}$$

Thus, the time rate of change of the angular momentum of a particle is equal to the net torque acting on it.
This is the rotational analogue of the equation **F** = d**p**/d*t*, which expresses Newton's second law for the translational motion of a single particle.

Torque and angular momentum for a system of particle or rigid body :

- Angular momentum of a system of particles is

$$\vec{L} = \sum_{j=1}^{N} \vec{l}_j = \sum_{j=1}^{N} \vec{r}_j \times \vec{p}_j$$

- Now taking the time derivative of both sides we have ;

$$\frac{d}{dt}\vec{L} = \sum_{j=1}^{N} \frac{d}{dt}\vec{l}_j = \sum_{j=1}^{N} \frac{d}{dt}(\vec{r}_j \times \vec{p}_j)$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \sum_{j=1}^{N}\left[\left\{\frac{d}{dt}(\vec{r}_j)\right\} \times \vec{p}_j + \vec{r}_j \times \frac{d}{dt}(\vec{p}_j)\right]$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \sum_{j=1}^{N}\left[\vec{v}_j \times \vec{p}_j + \vec{r}_j \times \left(\vec{F}_j\right)_{net}\right]$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \sum_{j=1}^{N}\left[\vec{0} + \vec{r}_j \times \left\{\left(\vec{F}_j\right)_{\text{internal}} + \left(\vec{F}_j\right)_{\text{external}}\right\}\right]$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \sum_{j=1}^{N}\left[\vec{r}_j \times \left\{\left(\vec{F}_j\right)_{\text{internal}}\right\}\right] + \sum_{j=1}^{N}\left[\vec{r}_j \times \left\{\left(\vec{F}_j\right)_{\text{external}}\right\}\right]$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \vec{0} + \sum_{j=1}^{N}\left[\vec{r}_j \times \left\{\left(\vec{F}_j\right)_{\text{external}}\right\}\right] = \sum_{j=1}^{N}\left[\vec{r}_j \times \left\{\left(\vec{F}_j\right)_{\text{external}}\right\}\right]$$

$$\Rightarrow \frac{d}{dt}\vec{L} = \sum \vec{\tau}_{\text{external}}$$

Here We shall assume not only Newton's third law, i.e. the forces between any two particles of the system are equal and opposite, but also that these forces are directed along the line joining the two particles. In this case, the contribution of the internal forces to the total torque on the system is zero, since the torque resulting from each action-reaction pair of forces is zero.

- **Thus, the time rate of the total angular momentum of a system of particles about a point (taken as the origin of our frame of reference) is equal to the sum of the external torques (i.e. the torques due to external forces) acting on the system taken about the same point.**

Law of conservation of angular momentum :

- **Statement:** if the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved, i.e. remains constant.

- **Proof :**
  Since for a particle or a system; total torque acting is equal to time derivatives of its angular momentum i.e.
  $$\frac{d}{dt}\vec{L} = \sum \vec{\tau}_{\text{external}}$$
  So if $\sum \vec{\tau}_{\text{external}} = 0$
  $$\Rightarrow \frac{d}{dt}\vec{L} = \vec{0}$$
  $$\Rightarrow \vec{L} = \text{conserved}$$

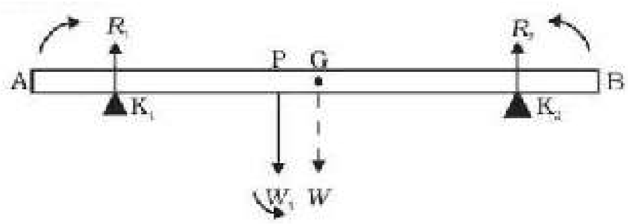
Equilibrium of a rigid body :

1. **Translational equilibrium:** If the resultant of all external forces acting on the system is 0 then it is in translational equilibrium.
   i.e. $\sum \vec{F} = \vec{0}$
   $$\sum F_X = 0$$
   $$\sum F_Y = 0$$
   $$\sum F_Z = 0$$
2. **Rotational equilibrium:** If the resultant of all external torques acting on the system is 0 then it is in rotational equilibrium.
   i.e. $\sum \vec{\tau} = \vec{0}$
   $$\sum \tau_X = 0$$
   $$\sum \tau_Y = 0$$
   $$\sum \tau_Z = 0$$

Question :

A metal bar 70 cm long and 4.00 kg in mass supported on two knife-edges placed 10 cm from each end. A 6.00 kg load is suspended at 30 cm from one end. Find the reactions at the knife-edges. (Assume the bar to be of uniform cross-section and homogeneous.)

Ans :



The weight of the rod W acts at its centre of gravity G.
As the rod is uniform in cross-section and homogeneous; hence G is at the centre of the rod.
AB = 70 cm.
AG = 35 cm
AP = 30 cm
PG = 5 cm
$AK_1 = BK_2$ = 10 cm and
$K_1G = K_2G$ = 25 cm.
Also, W= weight of the rod = 4.00 kg and W_1= suspended load = 6.00 kg;
R_1 and R_2 are the normal reactions of the support at the knife edges.

For translational equilibrium of the rod,
$R_1+R_2-W_1-W=0$
$\Rightarrow R_1+R_2 = W_1+W = 60\ N + 40\ N = 100\ N$ ...(i) (Taking $g = 10\ ms^{-2}$)

For considering rotational equilibrium, we take moments of the forces. A convenient point to take moments about is G. The moments of R_2 and W_1 are anticlockwise (+ve), whereas the moment of R_1 is clockwise (-ve).

For rotational equilibrium,
$-R_1\ (K_1G) + W_1\ (PG) + R_2\ (K_2G) = 0$
$\Rightarrow -R_1$ (25 cm) + 60N (5cm) + R_2 (25cm) = 0
$\Rightarrow 25\ (R_1-R_2) = 300$ N
$\Rightarrow R_1-R_2 = 12$ ....(ii)

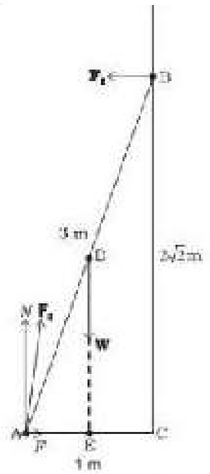
Solving equations (i) and (ii) we get ;
$R_1 = 56$ N
$R_2 = 44$ N

Question :

A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in the figure. Find the reaction forces of the wall and the floor.

Answer :

The ladder AB is 3 m long, its foot A is at distance AC = 1 m from the wall.
From Pythagoras theorem, BC = $2\sqrt{2}$ m.
The forces acting on the ladder t different points are shown in the figure.

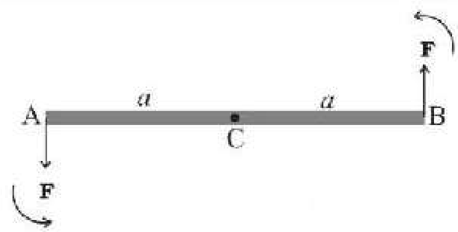


For translational equilibrium, taking the forces in the vertical direction,
$N - W = 0$
$\Rightarrow N = W = 200\ N$ ......(i) (Taking $g = 10\ m\ s^{-2}$)
Taking the forces in the horizontal direction,
$F - F_1 = 0$
$\Rightarrow F = F_1$ ....(ii)
For rotational equilibrium, taking the moments of the forces about A,
$(2\sqrt{2}\ m)F_1 - (1/2m)\ W = 0$
$\Rightarrow (2\sqrt{2}\ m)F_1 = (1/2m)\ W = (1/2m)\ (200N)$
$\Rightarrow F_1 = \frac{100N}{2\sqrt{2}} = 25\sqrt{2}N$
$\therefore F = F_1 = 25\sqrt{2}N$

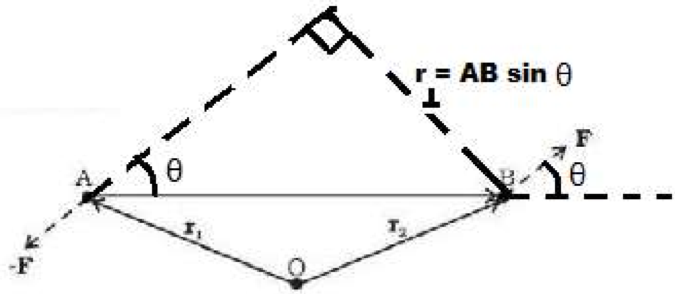
Now net contact force on the ladder due to the ground has magnitude F_2 making angle α with horizontal.

$$\therefore F_2 = \sqrt{N^2+F^2} = \sqrt{(200)^2+(25\sqrt{2})^2} = 203.1N$$
$$\therefore \alpha = \tan^{-1}(N/F) = \tan^{-1}\left(200/25\sqrt{2}\right) = \tan^{-1}\left(4\sqrt{2}\right)$$

Couple :



- A pair of equal and opposite forces with different lines of action is known as a couple.
- A couple produces rotation without translation. Because net force due to a couple is 0 and it produces torque only.
- E.g. : Forces applied to open the lid of a bottle, forces applied to open a water tap, forces on the steering of a car while driving etc.
- **Torque due to a couple :**



The moment of the couple = sum of the moments of the two forces making the couple

$\Rightarrow \boldsymbol{\tau} = \mathbf{r_1} \times (-\mathbf{F}) + \mathbf{r_2} \times \mathbf{F}$

$\Rightarrow \boldsymbol{\tau} = \mathbf{r_2} \times \mathbf{F} - \mathbf{r_1} \times \mathbf{F}$

$\Rightarrow \boldsymbol{\tau} = (\mathbf{r_2} - \mathbf{r_1}) \times \mathbf{F}$

But $\mathbf{r_1} + \mathbf{AB} = \mathbf{r_2}$, and hence $\mathbf{AB} = \mathbf{r_2} - \mathbf{r_1}$.

The moment of the couple, therefore, is

$\boldsymbol{\tau} = \mathbf{AB} \times \mathbf{F}$.

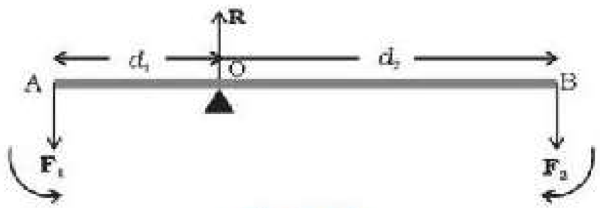
This is independent of the origin, the point about which we took the moments of the forces.

Now in magnitude ;

$\tau = (AB) F \sin\theta = (AB \sin\theta) F = r_{\perp} F$

i.e. Torque due to a couple = (Either force magnitude) (Perpendicular distance between their lines of action)

Principle of moments :



The figure shows a simple lever AB in equilibrium.

O = its fulcrum

F_1 = load to be lifted

F_2 = effort applied to lift .

OA = d_1 = load Arm

OB = d_2 = effort arm

Now by considering rotational equilibrium about the axis through O we have ;

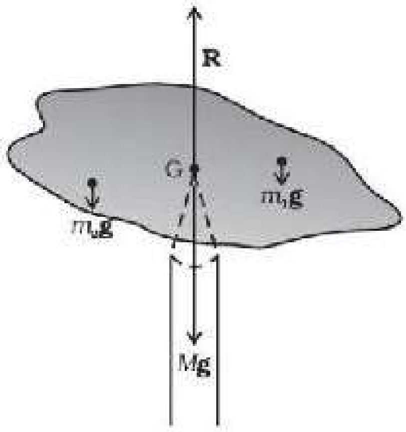
$d_1F_1 - d_2F_2 = 0$

$\Rightarrow \quad d_1F_1 = d_2F_2$

$\Rightarrow$ *(load arm)(load) = (effort arm) (effect)*

Now mechanical advantages (M.A.) of the lever = $\dfrac{load(F_1)}{effort(F_2)} = \dfrac{\text{effort arm}}{\text{load arm}}$

Centre of gravity :



- The CG of a body is defined as that point where the total gravitational torque on the body is zero.

  i.e. $\vec{\tau}_g = \sum \vec{\tau}_i = \sum \vec{r}_i \times m_i \vec{g} = \vec{0}$ ....(i)

  Here positions $\vec{r}_i$ of different particles are taken w.r.t. CG i.e. CG is taken as origin.

  Now as g = acceleration due to gravity which is non – zero, hence

  $\sum m_i \vec{r}_i = \vec{0}$

  This defines C.M.

  So C.M. coincides with CG
- If acceleration due to gravity varies from point to point on the body then equation (i) can't give $\sum m_i \vec{r}_i = \vec{0}$, hence C.M. differs from CG.

Moment of Inertia:

- Moment of inertia of a system about an axis is defined as

  $I = \sum_{i=1}^{N} m_i r_i^2$

  Where r_i = the radius of i^{th} particle about an axis of rotation or perpendicular distance of the i^{th} particle from the axis of rotation .
- M.I. is the rotational analogue mass in translational motion.
- S.I. unit of M.I. = kg m^2
- Dimensional formula of M.I. = M L^2
- Radius of gyration (k) :

  Radius of gyration of a system about an axis is defined as a distance k given by ;

  $k = \sqrt{\dfrac{I}{M}}$

  Where *I* = M.I. of the system of mass M about the axis
- **Moment of inertia and radius of gyration of some homogeneous bodies about some fixed axes are given below in a table**.

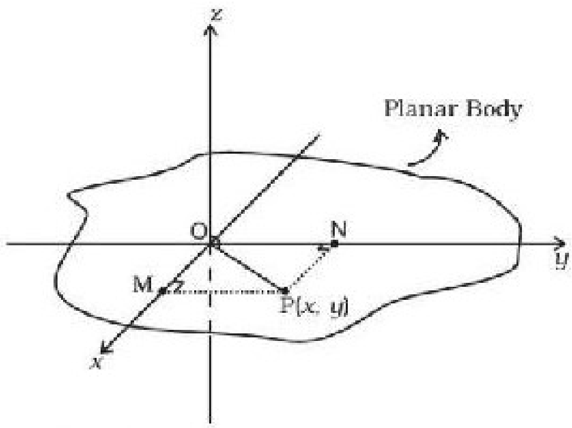
Shape	Axis	I	k
Thin uniform rod	1. Perpendicular bisector 2. Perpendicular through an end	1. $ML^2/12$ 2. $ML^2/3$	1. $L/2\sqrt{3}$ 2. $L/\sqrt{3}$
Rectangular lamina of length a and breadth b	1. Normal to the plane passing through the centre 2. Bisector of length 3. Bisector of breadth	1. $M(a^2+b^2)/12$ 2. $Ma^2/12$ 3. $Mb^2/12$	1. $\sqrt{(a^2+b^2)/12}$ 2. $a/2\sqrt{3}$ 3. $b/2\sqrt{3}$
Uniform thin ring	1. Perpendicular to the plane passing through the centre 2. Any diametre	1. MR^2 2. $(MR^2)/2$	1. R 2. $R/\sqrt{2}$
Uniform solid disc	1. Perpendicular to the plane passing through the centre 2. Any diametre	1. $(MR^2)/2$ 2. $(MR^2)/4$	1. $R/\sqrt{2}$ 2. $R/2$
Annular disc	Perpendicular to the plane passing through the centre	$M(r^2+R^2)/2$	$\sqrt{(r^2+R^2)/2}$
Solid sphere	Any diametre	$(2/5)MR^2$	$R\sqrt{2/5}$
Solid Cylinder	Its axis	$MR^2/5$	$R/\sqrt{2}$

Theorem of perpendicular axis :

The moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

i.e. $I_z = I_x + I_y$

Where I_x, I_y = M.I. of the lamina w.r.t. x and y axes respectively.



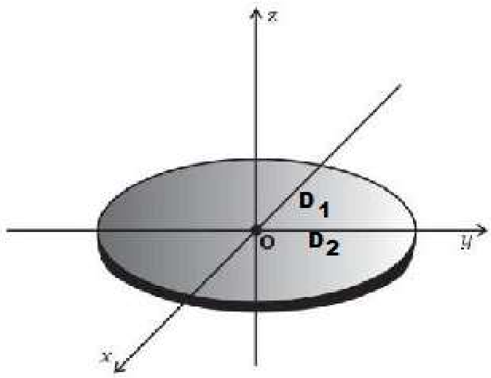
Question :

What is the moment of inertia of a uniform disc about one of its diameters?

Ans:

We know that M.I. of a disc about an axis passing through its centre and perpendicular to its plane is ;

$I = \frac{1}{2}MR^2$



Now if two perpendicular diameters D_1 and D_2 are imagined then by perpendicular axis theorem ;

$I = I_{D1} + I_{D2}$

Since the disc is symmetrical to all diameters, hence

$I_{D1} = I_{D2} = I_D$ (say)

$\therefore I = I_{D1} + I_{D2} = 2I_D$

$\Rightarrow I_D = \frac{I}{2} = \frac{(1/2)MR^2}{2} = \frac{MR^2}{4}$

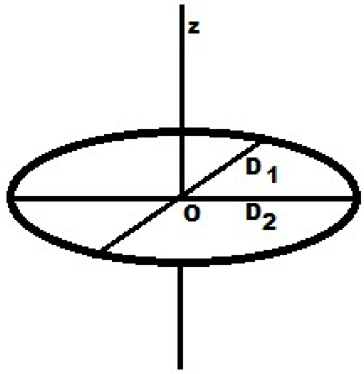
Question :

What is the moment of inertia of a uniform thin ring about one of its diameters?

Ans:

We know that M.I. of a ring about an axis passing through its centre and perpendicular to its plane is ;

$I = MR^2$



Now if two perpendicular diameters D_1 and D_2 are imagined then, by perpendicular axis theorem ;

$I = I_{D1} + I_{D2}$

Since the ring is symmetrical to all diameters, hence

$I_{D1} = I_{D2} = I_D$ (say)

$\therefore I = I_{D1} + I_{D2} = 2I_D$

$\Rightarrow I_D = \frac{I}{2} = \frac{MR^2}{2} = \frac{MR^2}{2}$

Theorem of parallel axis :

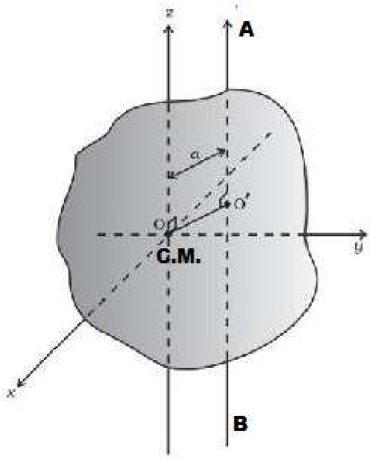
The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

i.e. $I_{AB} = I_{C.M.} + Ma^2$

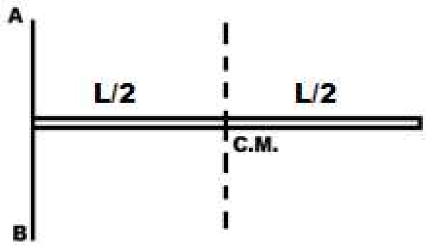
Where I_{AB} = M.I. of the body about an axis AB .

$I_{C.M.}$ = M.I. of the body about an axis passing through the C.M. and parallel to AB .

a = distance between the two parallel axes.



Question :
What is the moment of inertia of a rod of mass *M,* length *l* about an axis perpendicular to it through one end?
Ans :



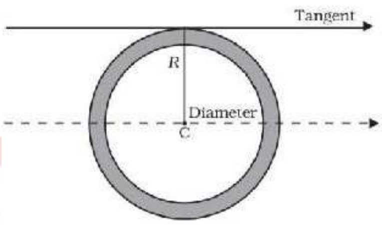
We know that M.I. of a rod about an axis passing through its centre and perpendicular to its length is ;

$$I_{C.M.} = ML^2/12$$

a = L/2

$$\therefore I_{AB} = I_{C.M.} + Ma^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

Question: What is the moment of inertia of a ring about a tangent to the circle of the ring?
Ans :



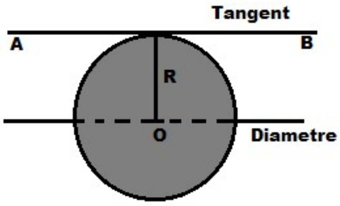
We know that M.I. of a ring about a diametre is ;

$$I_{C.M.} = MR^2/2$$

a = R

$$\therefore I_{AB} = I_{C.M.} + Ma^2 = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}$$

Question: What is the moment of inertia of a disc about a tangent to the circle of the disc?
Ans :



We know that M.I. of a ring about a diametre is ;

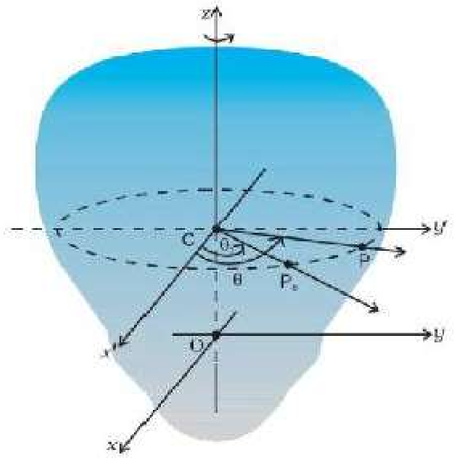
$$I_{C.M.} = MR^2/4$$

a = R

$$\therefore I_{AB} = I_{C.M.} + Ma^2 = \frac{MR^2}{4} + MR^2 = \frac{5MR^2}{4}$$

Kinematics of rotational motion about a fixed axis :

- As, in case of rotational motion about a fixed axis, angular variables like angular displacement (θ), angular velocity (ω) and angular acceleration (α) are same for all particles at an instant therefore kinematic equation in rotational motion is the relations among them with time.



- **For uniform rotation :**
  ω = constant
  In any time t, angular displacement is ; θ = ωt
- For uniformly accelerated rotation ;
  α = constant

(i) Angular velocity at any instant :

$$\alpha = \frac{d\omega}{dt}$$
$$\Rightarrow d\omega = \alpha dt$$
$$\Rightarrow \int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt$$
$$\Rightarrow \omega - \omega_0 = \alpha(t-0)$$
$$\Rightarrow \omega = \omega_0 + \alpha t$$

(ii) Angular displacement in a time t :

$$\omega = \frac{d\theta}{dt}$$
$$\Rightarrow d\theta = \omega dt = (\omega_0 + \alpha t)dt$$
$$\Rightarrow \int_0^{\theta} d\theta = \omega_0 \int_0^t dt + \alpha \int_0^t tdt$$
$$\Rightarrow \theta - 0 = \omega_0(t-0) + \alpha\left(\frac{t^2}{2} - \frac{0^2}{2}\right)$$
$$\Rightarrow \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

(iii) Angular displacement in term of angular velocity :

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{dt}\frac{d\theta}{d\theta} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

$$\Rightarrow \alpha d\theta = \omega d\omega$$

$$\Rightarrow \alpha\int_0^\theta d\theta = \int_{\omega_0}^{\omega}\omega d\omega$$

$$\Rightarrow \alpha(\theta - 0) = \frac{\omega^2 - \omega_0^2}{2}$$

$$\Rightarrow \omega^2 - \omega_0^2 = 2\alpha\theta$$

- Number of revolutions in a time t is ; $n = \frac{\theta}{2\pi}$

Questions :

The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds.

(i) What is its angular acceleration, assuming the acceleration to be uniform?

(ii) How many revolutions does the engine make during this time?

Ans :

$$\omega_0 = 1200rpm = \frac{1200}{60}rps = \frac{1200\times 2\pi}{60}rad/s = 40\pi\ rad/s$$

$$\omega = 3120rpm = \frac{3120}{60}rps = \frac{3120\times 2\pi}{60}rad/s = 104\pi\ rad/s$$

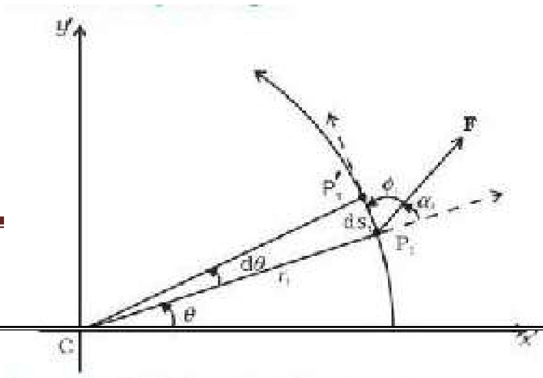
(i) $\alpha = \frac{\omega - \omega_0}{t} = \frac{104\pi - 40\pi}{16}rad/s^2 = 4\pi\ rad/s^2$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 40\pi\times 16 + \frac{1}{2}\times 4\pi\times 16^2 = 1152\pi$ rad

$n = \frac{\theta}{2\pi} = \frac{1152\pi}{2\pi} = 576$ revolutions

__Dynamics of rotation about a fixed axis :__

- In this case, only the components of torque along the axis are considered. So-net torque is along the axis of rotation.
- In this case the component of position along the perpendicular to the axis i.e. the radius of the circular path of the particle is taken as its position.
- In this case components of forces that lie in planes perpendicular to the axis are considered. Components of forces which are parallel to the axis will give torques perpendicular to the axis and need not be taken into account.
- **Work done by a torque :**
  During rotation about an axis, any elementary displacement **ds** is along the tangent i.e. perpendicular to the radius (r).
  ds = r dθ



Let force **F**, at this instant, makes angle ϕ with **ds** and α with **r.**
So, $\phi + \alpha = 90^0$
So torque of the force is ;
$\tau = r F \sin\alpha$
Torque is along the axis of rotation along which all angular variables are directed.
Now work done for this elementary displacement is;

$$dW = Fds\cos\phi$$
$$\Rightarrow dW = F(rd\theta)\cos(90^0 - \alpha)$$
$$\Rightarrow dW = F(r\sin\alpha)d\theta$$
$$\Rightarrow dW = \tau d\theta$$

So work done by a torque in rotating from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta$$

- **Instantaneous power delivered by a torque :**
  As work done by a torque for an elementary displacement is ;
  $$dW = \tau d\theta$$
  Hence power delivered by the torque at any instant is
  $$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt}$$
  $$\Rightarrow P = \tau\frac{d\theta}{dt}$$
  $$\Rightarrow P = \tau\omega$$
- **Kinetic energy of a system rotating about an axis :**
  $$K = \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2 = \sum_{i=1}^{N}\frac{1}{2}m_i(r_i\omega)^2$$
  $$\Rightarrow K = \sum_{i=1}^{N}\frac{1}{2}m_i r_i^2\omega^2$$
  $$\Rightarrow K = \frac{1}{2}\left(\sum_{i=1}^{N}m_i r_i^2\right)\omega^2$$
  $$\Rightarrow K = \frac{1}{2}I\omega^2$$
- **Expression for net torque on a system rotating about an axis :**
  By work-energy theorem; for an elementary displacement ;
  $$dW = dK \Rightarrow \frac{dW}{d\omega} = \frac{dK}{d\omega}$$
  $$\Rightarrow \frac{\tau d\theta}{d\omega} = \frac{d}{d\omega}\left(\frac{1}{2}I\omega^2\right) = \frac{1}{2}I.2\omega = I\omega$$
  $$\Rightarrow \tau d\theta = I\omega d\omega$$
  Dividing dt in both sides we have ;

$$\tau \frac{d\theta}{dt} = I\omega \frac{d\omega}{dt} \Rightarrow \tau\omega = I\omega\alpha$$

$$\Rightarrow \tau = I\alpha$$

- **Expression for the angular momentum of a system rotating about an axis :**

$$L = \sum_{i=1}^{N} m_i r_i v_i \sin 90^0 = \sum_{i=1}^{N} m_i r_i v_i$$

(Since $\mathbf{v_i}$ is along the tangent and $\mathbf{r_i}$ is along the radius, so the angle between them is 90^0)

$$\Rightarrow L = \sum_{i=1}^{N} m_i r_i (r_i \omega) = \sum_{i=1}^{N} m_i r_i^2 \omega$$

$$\Rightarrow L = \left(\sum_{i=1}^{N} m_i r_i^2 \right) \omega$$

$$\Rightarrow L = I\omega$$

$\vec{L}$ is directed along the axis of rotation . So vectorially ;

$$\vec{L} = I\vec{\omega}$$

- **Angular impulse :**

$$I_R = \Delta L = \Delta(I\omega) = I\Delta\omega$$

- **Conservation of angular momentum for a system rotating about an axis :**

Statement :

If net torque acting on a system rotating about an axis is 0, then total angular momentum is conserved.

Proof :

Since $\vec{L} = I\vec{\omega}$

$$\Rightarrow \frac{d}{dt}\vec{L} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d}{dt}\vec{\omega}$$

$$\Rightarrow \frac{d}{dt}\vec{L} = I\vec{\alpha} = \vec{\tau}$$

If $\vec{\tau} = \vec{0}$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{0}$$

$$\Rightarrow I\omega = \text{constant}$$

$$\Rightarrow I \;\alpha\; \frac{1}{\omega}$$

Comparison of Translational and Rotational Motion

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\, d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F\,v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Question :

A cord of negligible mass is wound around the rim of a flywheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. The flywheel is mounted on a horizontal axle with frictionless bearings.

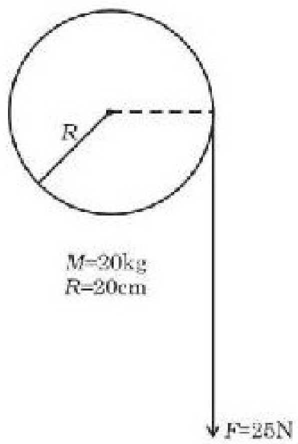
(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

Ans :



M.I. of the flywheel ; $I = \frac{MR^2}{2} = \frac{20 \times (0.2)^2}{2} = 0.4kgm^2$

(a) Torque on the flywheel is ; $\tau = FR = 25N \times (0.2m) = 5Nm$

Again as; τ = Iα

$$\Rightarrow \alpha = \frac{\tau}{I} = \frac{5}{0.4} = 12.5 rad/s^2$$

(b) Work is done by the pull unwinding 2m of the cord = 25 N × 2m = 50 J

(c) Let ω be the final angular velocity.

The initial angular velocity (ω_0) = 0

The angular displacement during 2m rope being uncovered is ;

$$\theta = \frac{s}{R} = \frac{2m}{0.2m} = 10 rad$$

As ; $\omega^2 - \omega_0^2 = 2\alpha\theta$

$$\Rightarrow \omega^2 - 0 = 2(12.5)(10)(rad/s)^2$$

$$\Rightarrow \omega^2 = 250(rad/s)^2$$

So K.E. achieved by the flywheel = $\frac{1}{2}I\omega^2 = \frac{1}{2}(0.4kgm^2)(250rad^2/s^2) = 50J$

(d) The answers are the same,
i.e. the kinetic energy gained by the wheel = work done by the force.
There is no loss of energy due to friction.

Question :

If earth shrinks and its radius becomes half of its initial value, keeping mass unchanged, then what will be the duration of a day?

Ans :

As R' = R/2

$$\Rightarrow I' = \frac{2}{5}M(R')^2 = \frac{2}{5}M(R/2)^2 = \frac{1}{4}\left(\frac{2}{5}MR^2\right) = \frac{I}{4}$$

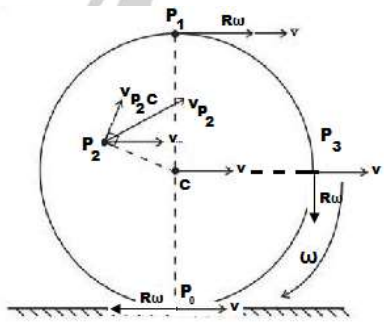
By the law of conservation of angular momentum ;

$$I\omega = I'\omega'$$

$$\Rightarrow I.\frac{2\pi}{T} = I'.\frac{2\pi}{T'}$$

$$\Rightarrow T' = \frac{I'}{I}.T = \frac{T}{4} = \frac{24h}{4} = 6h$$

Pure Rolling Motion i.e. rolling without slipping :



- The point of contact of the body with the platform is momentarily at rest w.r.t. the platform.
  i.e. $V_{P0} = 0$
- Rolling Motion is the combination of translation of C.M. and rotation about an axis through C.M.
  Let v = velocity of translation of C.M.
  Every other particle rotates about the axis through the C.M. with an angular velocity ω.
  So the velocity of any particle P_2 w.r.t. C.M. C is;
  $\vec{v}_{P_2C} = r\omega\hat{e}_t$
  Where r = distance of the point P_2 from C
  $\hat{e}_t$ = unit vector along a tangent to the circle along which P_2 is rotating about C.
  As $\vec{v}_{P_2C} = \vec{v}_{P_2} - \vec{v}_C$
  $\vec{v}_{P_2} = \vec{v}_{P_2C} + \vec{v}_C = r\omega\hat{e}_t + \vec{v}$
- As for the lowermost point i.e. point of contact P_0 ;
  r = R = radius of the body
  $\hat{e}_t$ is opposite to $\vec{v}$ i.e. velocity of C.M.
  As; $\vec{v}_{P_0} = 0$
  $\Rightarrow R\omega - v = 0$

$\Rightarrow \omega = \frac{v}{R}$ ……… (i)

Differentiating both sides with time we have

$\alpha = \frac{a}{R}$ ………..(ii)

Equations (i) and (ii) represent the conditions for pure rolling.

- For topmost point (P_1) or point exactly opposite to the point of contact; $\hat{e}_t$ is in the direction of $\vec{v}$ i.e. velocity of C.M.
  Hence $v_{P_1} = R\omega + v$
- For horizontal point (P_3) or point in a position parallel to contact platform; $\hat{e}_t$ is in the direction perpendicular to $\vec{v}$ i.e. velocity of C.M.
  Hence $v_{P_3} = \sqrt{\left(R^2\omega^2 + v^2\right)}$
- **Kinetic energy in pure rolling motion :**
  K_R = K.E. of the body due to its rotation about an axis through C.M.

  $\Rightarrow K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}(Mk^2)\left(\frac{v}{R}\right)^2$ [k = radius of gyration]

  $\Rightarrow K_R = \frac{1}{2}Mv^2\left(\frac{k^2}{R^2}\right)$

  K_T = K.E. of the body due to its translation along with C.M.

  $\Rightarrow K_T = \frac{1}{2}Mv^2$

  So total K.E. of the body in pure rolling motion is ;

  $K = K_R + K_T$

  $\Rightarrow K = \frac{1}{2}Mv^2\left(\frac{k^2}{R^2}\right) + \frac{1}{2}Mv^2$

  $\Rightarrow K = \frac{1}{2}Mv^2\left(\frac{k^2}{R^2} + 1\right)$

Impure Rolling Motion i.e. rolling wit slipping :

In this case, the point of contact of the body with the platform is not at rest w.r.t. the platform.

i.e. $v_{P_0} \neq 0$

Question :

Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. (i) Which of the bodies reaches the ground with maximum velocity? (ii) Which one reaches earliest?

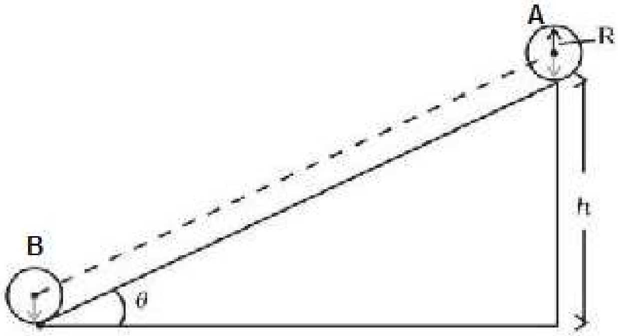
Answer :

As this is the case of pure rolling i.e. rolling without slipping, hence no work is done against friction or no energy is lost due to friction.

Hence; Mechanical energy at the starting point A = Mechanical energy at the ground.

$$Mgh = \frac{1}{2}Mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$\Rightarrow v = \sqrt{\frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}}$$



For a ring ; $k^2 = R^2$
For a solid cylinder ; $k^2 = R^2/2$
For a solid sphere; $k^2 = 2R^2/5$

$$v_{Ring} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$$

$$v_{Cylinder} = \sqrt{\frac{2gh}{1+1/2}} = \sqrt{\frac{4gh}{3}}$$

$$v_{Sphere} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10gh}{7}}$$

This shows $v_{Sphere} > v_{Cylinder} > v_{Ring}$

i.e. Sphere will reach with maximum velocity.
So the sphere will reach the earliest, then the cylinder and finally ring.

Question :

A body rolls down an inclined plane without slipping from rest. Find the expression for the acceleration of the body and expression for static friction.

Answer :

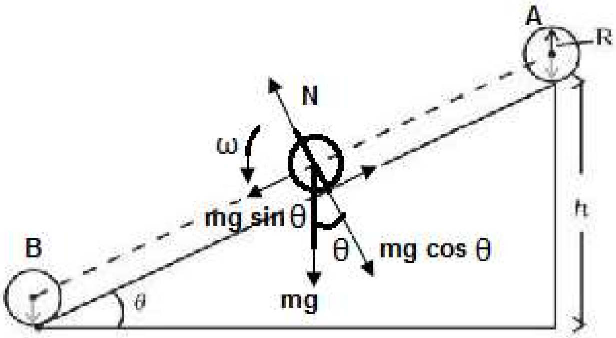
Force equation along the axis parallel to the incline;

$Mg \sin\theta - F_s = Ma$ ....(i)

Force equation along the axis perpendicular to the incline

$N - Mg\cos\theta = 0$

$\Rightarrow$ $N = Mg\cos\theta$ ....(ii)



Torque equation ;

$F_sR = I\alpha$ .... (iii)

For pure rolling; $a = \alpha R \Rightarrow \alpha = \frac{a}{R}$ ......(iv)

Using equation (iv) in (iii);

$$\Rightarrow F_sR = \frac{Ia}{R}$$

$$\Rightarrow F_s = \frac{Ia}{R^2} = \frac{Mk^2a}{R^2} = Ma\left(\frac{k^2}{R^2}\right) \text{.......(v)}$$ [k = Radius of gyration]

Using equation (v) in equation (i) we get

$$Mg\sin\theta - Ma\left(\frac{k^2}{R^2}\right) = Ma$$

$$\Rightarrow Ma\left(1+\frac{k^2}{R^2}\right) = Mg\sin\theta$$

$$\Rightarrow a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$$ ....(vi) [Expression for acceleration]

Using equation (vi) in equation (v) we get ;

$$F_s = M\left(\frac{k^2}{R^2}\right).\frac{g\sin\theta}{1+\frac{k^2}{R^2}}$$

$$\Rightarrow F_s = \frac{Mg\sin\theta}{\frac{R^2}{k^2}+1}$$ ....(vii) [Expression for static friction in Pure rolling]