Chapter- 15

WAVES

Transverse and longitudinal waves

We have seen that motion of mechanical waves involves oscillations of constituents of the medium. If the constituents of the medium oscillate perpendicular to the direction of wave propagation, we call the wave a transverse wave. If they oscillate along the direction of wave propagation, we call the wave a longitudinal wave.

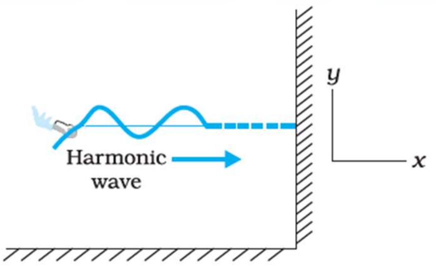


Fig. 15.3 *A harmonic (sinusoidal) wave travelling along a stretched string is an example of a transverse wave. An element of the string in the region of the wave oscillates about its equilibrium position perpendicular to the direction of wave propagation.*

Fig. 15.4 *Longitudinal waves (sound) generated in a pipe filled with air by moving the piston up and down. A volume element of air oscillates in the direction parallel to the direction of wave propagation.*

NCERTExample 15.1

Given below are some examples of wave motion. State in each case if the wave motion is transverse, longitudinal or a combination of both: (a) Motion of a kink in a longitudinal spring produced by displacing one end of the spring sideways. (b) Waves produced in a cylinder containing a liquid by moving its piston back and forth. (c) Waves produced by a motorboat sailing in water. (d) Ultrasonic waves in air produced by a vibrating quartz crystal.

ANSWER

(a) Transverse and longitudinal

(b) Longitudinal

(c) Transverse and longitudinal

(d) Longitudinal

THE SPEED OF A TRAVELLING WAVE

To determine the speed of propagation of a travelling wave, we can fix our attention on any particular point on the wave (characterized by some value of the phase) and see how that point moves in time. It is convenient to look at the motion of the crest of the wave. Fig. 15.8 gives the shape of the wave at two instants of time which differ by a small time interval Δt. The entire wave pattern is seen to shift to the right(positive direction of *x*-axis) by a distance Δx. In particular, the crest shown by a dot$(\bullet)$ moves a distance Δx in time Δt. The speed of the wave is then $\Delta x / \Delta t$.

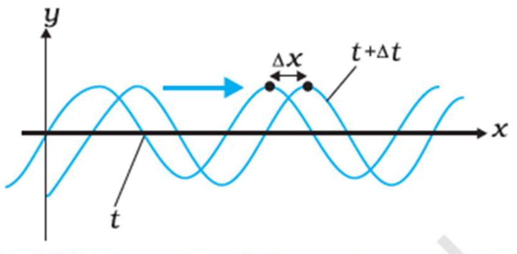


Fig. 15.8 *Progression of a harmonic wave from time t to t + Δt. where Δt is a small interval. The wave pattern as a whole shifts to the right. The crest of the wave (or a point with any fixed phase) moves right by the distance Δx in time Δt.*

The motion of a fixed phase point on the wave is given by

$kx - \omega t =$ constant

Thus, as time *t* changes, the position *x* of the fixed phase point must change so that the phase remains constant. Thus,

$$kx - \omega t = k\left(x + \Delta x\right) - \omega\left(t + \Delta t\right)$$

Taking Δx, Δt vanishingly small, this gives

$$\frac{dx}{dt} = \frac{\omega}{k} = v$$

Relating ω to *T* and *k* to λ, we get

$$v = \frac{2\pi \nu}{2\pi / \lambda} = \nu\lambda = \frac{\lambda}{T}$$

Thus for all progressive waves, in the time required for one full oscillation by any constituent of the medium, the wave pattern travels a distance equal to the wavelength of the wave.

It should be noted that the speed of a mechanical wave is determined by the inertial (linear mass density for strings, mass density in general) and elastic properties (Young's modulus for linear media/ shear modulus, bulk modulus) of the medium.

DISPLACEMENT RELATION IN A PROGRESSIVE WAVE

If the position of the constituents of the medium is denoted by x, the displacement from the equilibrium position may be denoted by y. A sinusoidal travelling wave is then described by:

$y\left(x,t\right) = a\sin\left(kx - \omega t + \phi\right)$ ---eq.(1)

Let

$$y(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$

Then,

$a = \sqrt{A^2 + B^2}$,

$$\phi = \tan^{-1}\left(\frac{B}{A}\right)$$

In eq.(1), y(x,t) is displacement as a function of position x and time t, a is the amplitude of a wave, ω is the angular frequency of the wave k is the angular wavenumber, $(kx - \omega t + \phi)$

Is phase, ϕ is the initial phase angle.

On the other hand, a function

$y(x,t) = a\sin(kx + \omega t + \phi)$ represents a wave travelling in the negative direction of the x-axis.

Amplitude and Phase

In equation

$y(x,t) = a\sin(kx - \omega t + \phi)$,

since the sine function varies between 1 and –1, the displacement y (x,t) varies between a and –a. We can take a to be a positive constant, without any loss of generality. Then, a represents the maximum displacement of the constituents of the medium from their equilibrium position. Note that the displacement y may be positive or negative, but a is positive. It is called the amplitude of the wave. The quantity $(kx - \omega t + \phi)$ appearing as the argument of the sine function in Eq. (15.2) is called the phase of the wave.

Wavelength and Angular WaveNumber

The minimum distance between two points having the same phase is called the wavelength of the wave, usually denoted by λ. The wavelength is the distance between two consecutive crests or troughs in a wave. Taking $\phi = 0$ in equation

$$y(x,t) = a\sin(kx - \omega t + \phi)$$

the displacement at t = 0 is given by

y(x,0) = a sin kx

Since the sine function repeats its value after every 2π change in angle,

$$\sin(kx) = \sin(kx + 2n\pi) = \sin k\left(x + \frac{2n\pi}{k}\right)$$

That is the displacements at points x and at $\frac{2n\pi}{k}$ are the same, where n=1,2,3,... The least distance between points with the same displacement (at any given instant of time) is obtained by taking n = 1. λ is then given by

$$\lambda = \frac{2n\pi}{k}$$

k is the angular wavenumber or propagation constant; its SI unit is radian per metre.

Period, Angular Frequency and Frequency

We may for, simplicity, take equation

$$y(x,t) = a\sin(kx - \omega t + \phi)$$

with $\phi = 0$ and monitor the motion of the element say at *x* = 0. We then get

$$y(0,t) = a\sin(-\omega t) = -a\sin(\omega t)$$

Now, the period of oscillation of the wave is the time it takes for an element to complete one full oscillation. That is

$a\sin(-\omega t) = -a\sin\omega(t+T) = -a\sin(\omega t + \omega T)$

Since sine function repeats after every 2π , $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$

ω is called the angular frequency of the wave. Its SI unit is rad/s. The frequency ν is the number of oscillations per second. Therefore,

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi}$$

NCERT Example 15.2

A wave travelling along a string is described by, y(x, t) = 0.005 sin (80.0 x – 3.0 t),

in which the numerical constants are in SI units (0.005 m, 80.0 rad/m, and 3.0 rad/s). Calculate (a) the amplitude, (b) the wavelength, and (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance x = 30.0 cm and time t = 20 s?

SOLUTION

(a) the amplitude of the wave is 0.005 m = 5 mm.

(b) the angular wavenumber *k* and angular frequency ω are

k = 80.0 m^{-1} and ω = 3.0 s^{-1}, so

$$\lambda = \frac{2\pi}{k} = 7.85 \text{ cm}$$

(c) Now, we relate *T* to ω by the relation

$$\lambda = \frac{2\pi}{\omega} = 2.09\text{s}$$

and frequency, $\nu = \frac{1}{T} =$ 0.48 Hz

The displacement *y* at *x* = 30.0 cm and time *t* = 20 s is given by

y = (0.005 m) sin (80.0 × 0.3 – 3.0 × 20)

= (0.005 m) sin $\left(-36+12\pi\right)$

= (0.005 m) sin (1.699)

= (0.005 m) sin $\left(97^{\circ}\right)\approx 5mm$

Speed of a Transverse Wave on Stretched String

The speed of a mechanical wave is determined by the restoring force set up in the medium when it is disturbed and the inertial properties (mass density) of the medium. The speed is expected to be directly related to the former and inversely to the latter. For waves on a string, the restoring force is provided by the tension T in the string. The inertial property will, in this case, be linear mass density μ, which is mass m of the string divided by its length L. The dimension of μ is $\left[ML^{-1}\right]$ and that of T is like force, namely $\left[MLT^{-2}\right]$. We need to combine these dimensions to get the dimension of speed v$\left[LT^{-1}\right]$. Simple inspection shows that the the quantity T/μ has the relevant dimension $\frac{\left[MLT^{-2}\right]}{\left[ML^{-1}\right]}=\left[L^2T^{-2}\right]$

Thus

$$v=C\sqrt{\frac{T}{\mu}}$$

where C is the undetermined constant of dimensional analysis. In the exact formula, it turns out, C=1. The speed of transverse waves on a stretched string is given by

$$v=\sqrt{\frac{T}{\mu}}$$

Speed of a Longitudinal Wave(Speed of Sound)

In a longitudinal wave, the constituents of the medium oscillate forward and backwards in the direction of propagation of the wave. We have already seen that the sound waves travel in the form of compressions and rarefactions of small volume elements of air. The elastic property that determines the stress under compressional strain is the bulk modulus of the medium defined by $B = -\frac{\Delta P}{\Delta V / V}$ Here, the change in pressure ΔP produces a volumetric strain $\Delta V / V$.

B has the dimension $\left[ML^{-1}T^{-2}\right]$. The inertial property relevant for the propagation of the wave is the mass density ρ, with dimensions $\left[ML^{-3}\right]$. Simple inspection reveals that quantity B/ρ has the relevant dimension: $\frac{\left[ML^{-1}T^{-2}\right]}{\left[ML^{-3}\right]} = \left[L^2L^{-2}\right]$

Thus, if B and ρ are considered to be the only relevant physical quantities,

$$v = C\sqrt{\frac{B}{\rho}}$$

where *C* is the undetermined constant from dimensional analysis. The exact derivation shows that *C*=1. Thus, the general formula for longitudinal waves in a medium is:

$$v = \sqrt{\frac{B}{\rho}} \text{ ---(1)}$$

The speed of longitudinal waves in a solid bar is given by

$$v = \sqrt{\frac{Y}{\rho}}$$

where Y is Young's modulus of the material of the bar.

We can estimate the speed of sound in a gas in the ideal gas approximation. For an ideal gas, the pressure P, volume V and temperature T are related by

$PV = Nk_BT$ ---(2)

where N is the number of molecules in volume V, k_B is the Boltzmann constant and T the temperature of the gas (in Kelvin). Therefore, for an isothermal change, it follows from eq.(2) that

$P\Delta V + V\Delta P = 0$

Hence,

B = P

Therefore, from eq. (1) the speed of a longitudinal wave in an ideal gas is given by,

$$v = \sqrt{\frac{P}{\rho}}$$

This relation was first given by Newton and is known as Newton's formula.

If we examine the basic assumption made by Newton that the pressure variations in a medium during propagation of sound are isothermal, we find that this is not correct. It was pointed out by Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain a constant temperature. These variations, therefore, are adiabatic and not isothermal. For adiabatic processes, the ideal gas satisfies the relation

$PV^{\gamma} =$ constant

$\Rightarrow \Delta\left(PV^{\gamma}\right) = 0$

$\Rightarrow V^{\gamma}\Delta P + P\gamma V^{\gamma-1}\Delta V = 0$

where γ is the ratio of two specific heats, C_P / C_V .

Thus, for an ideal gas, the adiabatic bulk modulus is given by,

$$B_{ad} = -\frac{\Delta P}{\Delta V / V} = \gamma P$$

The speed of sound is, therefore

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad \text{---(3)}$$

This modification of Newton's formula is referred to as the Laplace correction. For air γ = 7/5. Now using eq. (3) to estimate the speed of sound in air at STP, we get a value 331.3 m/s, which agrees with the measured speed.

NCERT Example 15.3

A steel wire 0.72 m long has a mass of 5.0 ×10–3 kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire?

SOLUTION

$$\mu = \frac{5.0\times10^{-3}kg}{0.72m} = 6.9\times10^{-3}kg/m$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{60N}{6.9\times10^{-3}kg/m}} = 93m/s$$

NCERT Example 15.4

Estimate the speed of sound by Newton's formula in the air at standard temperature and pressure. The mass of 1 mole of air is $29.0\times10^{-3}kg$.

SOLUTION

$$\rho = \frac{29.0\times10^{-3}kg}{22.4l} = 1.29kg/m^3$$

$$v = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{1.01\times10^5 N/m^2}{1.29kg/m^3}} = 280m/s$$

THE PRINCIPLE OF SUPERPOSITION OF WAVES

When the pulses overlap, the resultant displacement is the algebraic sum of the displacement due to each pulse. This is known as the principle of superposition of waves.

To put the principle of superposition mathematically, let $y_1(x,t)$ and $y_2(x,t)$ be the displacements due to two wave disturbances in the medium. If the waves arrive in a region simultaneously, and therefore, overlap, the net displacement *y* (*x*,*t*) is given by

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

For simplicity, consider two harmonic travelling waves on a stretched string, both with the same ω (angular frequency) and *k* (wavenumber), and, therefore, the same wavelength λ. Their wave speed will be identical. Let us further assume that their amplitudes are equal and they are both travelling in the positive direction of the *x*-axis. The waves only differ in their initial phase.

$$y_1(x,t) = a\sin(kx - \omega t) \text{---eq. (1)}$$

$$y_2(x,t) = a\sin(kx - \omega t + \phi) \text{---eq. (2)}$$

The net displacement is then, by the principle of superposition, given by

$$y(x,t) = a\left[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)\right] \Rightarrow$$

$$y(x,t) = a\left[2\sin\frac{(kx - \omega t) + (kx - \omega t + \phi)}{2}\cos\frac{-(kx - \omega t) + (kx - \omega t + \phi)}{2}\right]$$

$$\Rightarrow y(x,t) = a\left[2\sin\left(kx - \omega t + \frac{\phi}{2}\right)\cos\frac{\phi}{2}\right] \text{---eq. (3)}$$

Eq. (3) is also a harmonic travelling wave in the positive direction of the *x*-axis, with the same frequency and wavelength. However, its initial phase angle is $\frac{\phi}{2}$. The significant thing is that its amplitude is a function of the phase difference ϕ between the constituent two waves:

$$A(\phi) = a\cos\frac{\phi}{2} \text{---(4)}$$

For ϕ = 0, when the waves are in phase,

$$y(x,t) = 2a\sin(kx - \omega t) \text{---(5)}$$

i.e., the resultant wave has amplitude 2a, the largest possible value for *A*. For $\phi = \pi$, the waves are completely, out of phase and the resultant wave has zero displacements everywhere at all times

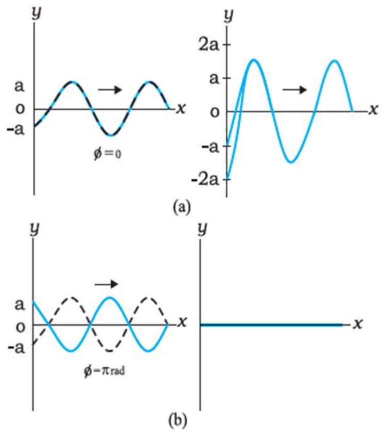


Fig. 15.10 The resultant of two harmonic waves of equal amplitude and wavelength according to the principle of superposition. The amplitude of the resultant wave depends on the phase difference ϕ, which is zero for (a) and π for (b)

$$y(x,t) = 0 \text{---(6)}$$

Eq. (5) refers to the so-called constructive interference of the two waves where the amplitudes add up in the resultant wave. Eq. (6) is the case of destructive interference where the amplitudes subtract out in the resultant wave.

REFLECTION OF WAVES

Fig. 15.11 shows a pulse travelling along a stretched string and being reflected by the boundary. Assuming there is no absorption of energy by the boundary, the reflected wave has the same shape as the incident pulse but it suffers a phase change of π or 180° on reflection. This is because the boundary is rigid and the disturbance must have zero displacements at all times at the boundary. By the principle of superposition, this is possible only if the reflected and incident waves differ by a phase of π so that the resultant displacement is zero. This reasoning is based on boundary condition on a rigid wall. We can arrive at the same conclusion dynamically also. As the pulse arrives at the wall, it exerts a force on the wall. By Newton's Third Law, the wall exerts an equal and opposite force on the string generating a reflected pulse that differs by a phase of π .

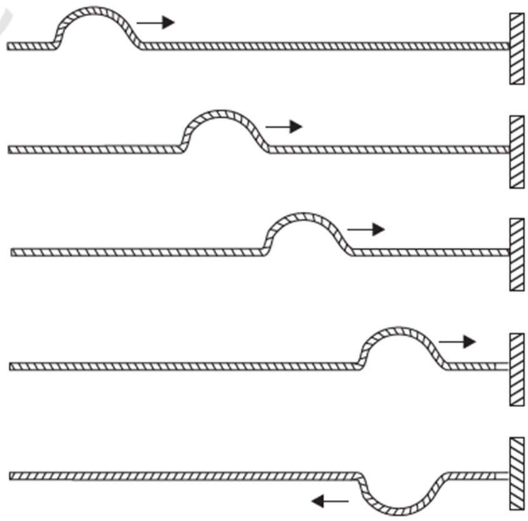


Fig. 15.11 *Reflection of a pulse meeting a rigid boundary.*

If on the other hand, the boundary point is not rigid but completely free to move (such as in the case of a string tied to a freely moving ring on a rod), the reflected pulse has the same phase and amplitude (assuming no energy dissipation) as the incident pulse. The net maximum displacement at the boundary is then twice the amplitude of each pulse. An example of a non-rigid boundary is the open end of an organ pipe.

To summarise, a travelling wave or pulse suffers a phase change of π on reflection at a rigid boundary and no phase change on reflection at an open boundary. To put this mathematically, let the incident travelling wave be

$$y_2(x,t) = a\sin(kx - \omega t)$$

At a rigid boundary, the reflected wave is given by

$y_r(x,t) = a\sin(kx - \omega t + \pi) = -a\sin(kx - \omega t)$

At an open boundary, the reflected wave is given by

$y_r(x,t) = a\sin(kx - \omega t + 0) = a\sin(kx - \omega t)$

Clearly, at the rigid boundary, $y = y_r + y_r$ at all times.

Standing Waves and Normal Modes

In a string, a wave travelling in one direction will get reflected at one end, which in turn will travel and get reflected from the other end. This will go on until there is a steady wave pattern set up on the string. Such wave patterns are called standing waves or stationary waves. To see this mathematically, consider a wave travelling along the positive direction of the x-axis and a reflected wave of the same amplitude and wavelength in the negative direction of the x-axis.

$y_1(x,t) = a\sin(kx - \omega t)$ ---eq. (1)

$y_2(x,t) = a\sin(kx + \omega t)$ ---eq. (2)

The resultant wave on the string is, according to the principle of superposition:

$y(x,t) = a\left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$

$= 2a\sin kx \cos \omega t$ ---eq. (1)

The amplitude of this wave is 2a sin kx. Thus, in this wave pattern, the amplitude varies from point-to-point, but each element of the string oscillates with the same angular frequency w or time period. There is no phase difference between the oscillations of different elements of the wave. The string as a whole vibrates in phase with differing amplitudes at different points. The wave pattern is neither moving to the right nor the left. Hence, they are called standing or stationary waves. The amplitude is fixed at a given location but, as remarked earlier, it is different at different locations. The points at which the amplitude is zero (i.e., where there is no motion at all) are nodes; the points at which the amplitude is the largest are called antinodes. Fig. 15.12 shows a stationary wave pattern resulting from the superposition of two travelling waves in opposite directions.

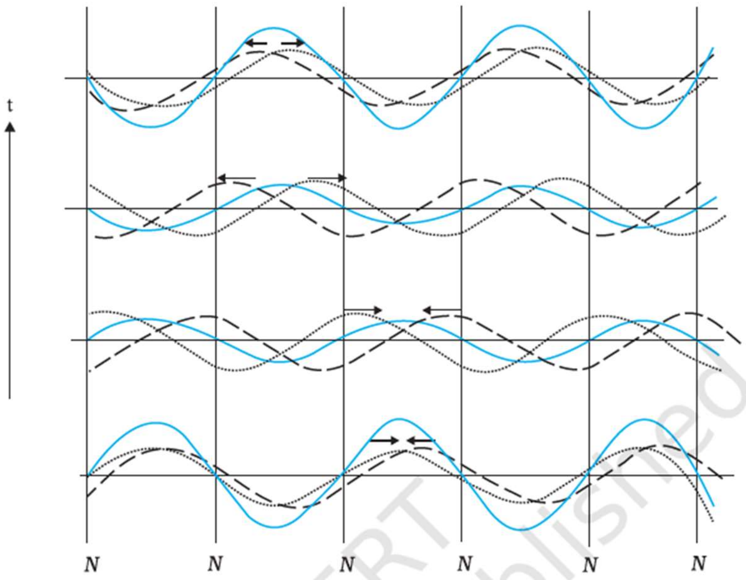


Fig. 15.12 *Stationary waves arising from superposition of two harmonic waves travelling in opposite directions. Note that the positions of zero displacement (nodes) remain fixed at all times.*

The most significant feature of stationary waves is that the boundary conditions constrain the possible wavelengths or frequencies of vibration of the system. The system can not oscillate with any arbitrary frequency (contrast this with a harmonic travelling wave) but is characterised by a set of natural frequencies or normal modes of oscillation. Let us determine these normal modes for a stretched string fixed at both ends.

First, from Eq. (1), the positions of nodes (where the amplitude is zero) are given by sin kx = 0, which implies

$kx = n\pi, n = 0,1,2,3,...$

Since $k = \frac{2\pi}{\lambda}$

$x = n\frac{\lambda}{2}, n = 0,1,2,3,...$ ---eq. (2)

Clearly, the distance between any two successive nodes is $\frac{\lambda}{2}$. In the same way, the positions of antinodes (where the amplitude is the largest) are given by the largest value of sin *kx* :

$|\sin kx| = 1$ which implies

$kx = \left(n + \frac{1}{2}\right)\pi, n = 0,1,2,3,...$

with $k = \frac{2\pi}{\lambda}$

$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, n = 0,1,2,3,...$ ---eq. (3)

Again, the distance between any two consecutive antinodes is $\frac{\lambda}{2}$. The above equations can be applied to the case of a stretched string of length *L* fixed at both ends. Taking one end to be at *x* = 0, the boundary conditions are that *x* = 0 and *x* = *L*are positions of nodes. The *x* = 0 condition is already satisfied. The *x* = *L* node condition requires that the length *L* is related to λ by

$L = n\frac{\lambda}{2}, n = 0,1,2,3,...$

Thus, the possible wavelengths of stationary waves are constrained by the relation

$\lambda = \frac{2L}{n}, n = 0,1,2,3,...$ ---eq. (4)

with corresponding frequencies

$\nu = n\frac{v}{2L}, n = 0,1,2,3,...$ ---eq. (5)

We have thus obtained the natural frequencies - the normal modes of oscillation of the system. The lowest possible natural frequency of a system is called its fundamental mode or the first harmonic. For the stretched string fixed at either end, it is given by $\nu = \frac{v}{2L}$, corresponding to *n* = 1 of Eq. (5). Here v is the speed of wave determined by the properties of the medium. The *n* = 2 frequency is called the second harmonic; *n* = 3 is the third harmonic and so on.

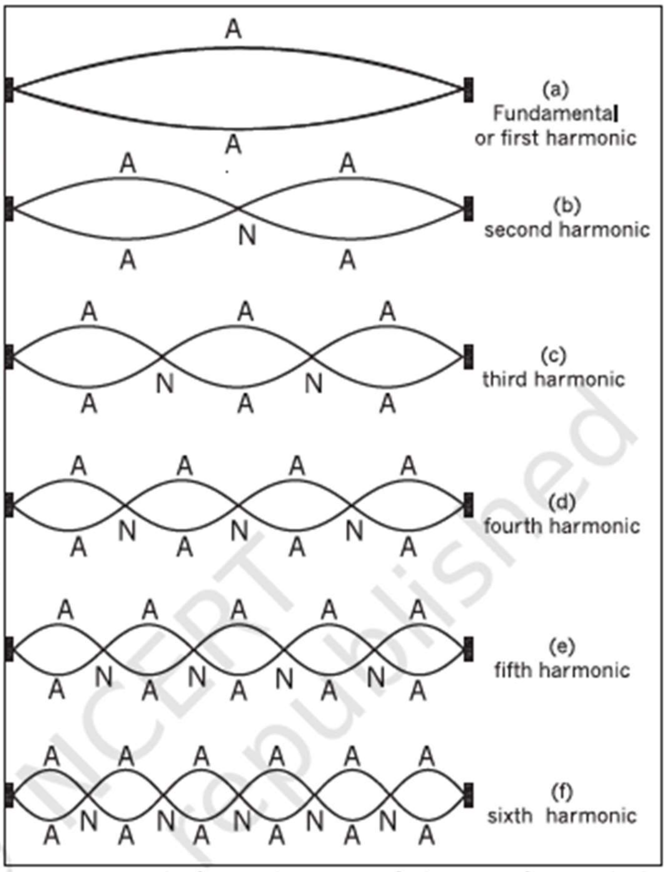


Fig. 15.13 *The first six harmonics of vibrations of a stretched string fixed at both ends.*

standing waves in organ pipes, fundamental mode and harmonics

Consider normal modes of oscillation of an air column with one end closed and the other open. A glass tube partially filled with water illustrates this system. The end in contact with water is a node, while the open end is an antinode. At the node, the pressure changes are the largest, while the displacement is minimum (zero). At the open end - the antinode, it is just the other way - least pressure change and maximum amplitude of displacement. Taking the end in contact with water to be *x* = 0, the node condition (Eq. 2) is already satisfied. If the other end *x* = *L* is an antinode, Eq. (3) gives

$$L = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, n = 0,1,2,3,...$$

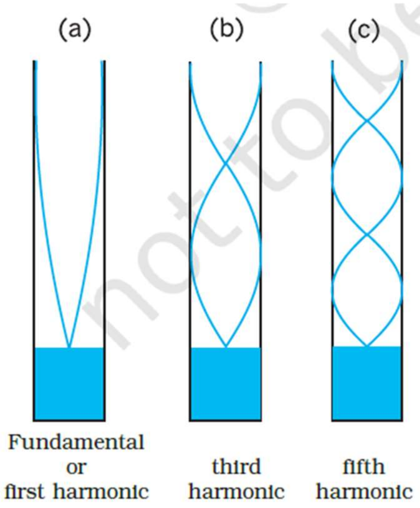
The possible wavelengths are then restricted by the relation :

$$\lambda = \frac{2L}{n + \frac{1}{2}}, n = 0,1,2,3,...$$

The normal modes – the natural frequencies – of the system are

$$\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L}, n = 0,1,2,3,...$$

The fundamental frequency corresponds to *n* = 0 and is given by $\frac{v}{4L}$. The higher frequencies are odd harmonics, i.e., odd multiples of the fundamental frequency: $3\frac{v}{4L}, 5\frac{v}{4L}, 7\frac{v}{4L}$,..., etc.



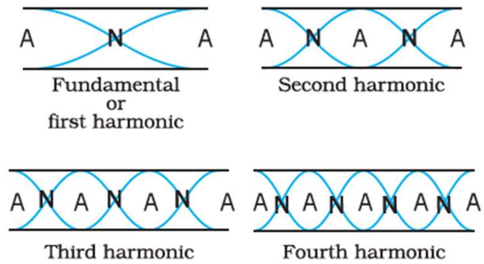


Fig. 15.15 *Standing waves in an open pipe, first four harmonics are depicted.*

The systems above, strings and air columns, can also undergo forced oscillations. If the external frequency is close to one of the natural frequencies, the system shows resonance.

NCERT Example 15.5

A pipe, 30.0 cm long, is open at both ends. Which harmonic mode of the pipe resonates a 1.1 kHz source? Will resonance with the same source be observed if one end of the pipe is closed? Take the speed of sound in air as 330 m/s.

SOLUTION

For *L* = 30.0 cm, *v* = 330 m/s,

$$\nu_n = n\frac{v}{2L} = n550s^{-1}$$

Clearly, a source of frequency 1.1 kHz will resonate at ν_2, i.e., the second harmonic. Now if one end of the pipe is closed, then

$$\nu_n = n\frac{v}{4L} = n275s^{-1}, n = 1,3,5,7,...$$

So, no resonance will be observed with the source, the moment one end is closed.

BEATS

'Beats' is an interesting phenomenon arising from the interference of waves. When two harmonic sound waves of close (but not equal) frequencies are heard at the same time, we hear a sound of similar frequency (the average of two close frequencies), but we hear something else also. We hear audibly distinct waxing and waning of the intensity of the sound, with a frequency equal to the difference in the two close frequencies. Artists use this phenomenon often while tuning their instruments with each other. They go on tuning until their sensitive ears do not detect any beats.

To see this mathematically, let us consider two harmonic sound waves of nearly equal angular frequency ω_1 and ω_2 fix the location to be x = 0 for convenience and, assuming equal amplitudes and ω_1 slightly greater than ω_2

$$s_1(x,t) = a\cos\omega_1 t$$

$$s_2(x,t) = a\cos\omega_2 t$$

The resultant displacement is, by the principle of superposition, is

$$s = s_1 + s_2 = a\cos\omega_1 t + a\cos\omega_2 t = a\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$= 2a\cos\frac{(\omega_1+\omega_2)}{2}t\cos\frac{(\omega_1-\omega_2)}{2}t = (2a\cos\omega_b t)\cos\omega_a t$$

where, $\omega_a = \dfrac{(\omega_1+\omega_2)}{2}$, $\omega_b = \dfrac{(\omega_1-\omega_2)}{2}$

The resultant wave is oscillating with the average angular frequency ω_a; however, its amplitude is not constant in time, unlike a pure harmonic wave. The amplitude is the largest when the term $\cos\omega_b t$ takes its limit +1 or –1. In other words, the intensity of the resultant wave waxes and wanes with a frequency which is $2\omega_b = \omega_1 - \omega_2$. The beat frequency is given by $\nu_{\text{beat}} = \nu_1 - \nu_2$

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Example 15.6 Two sitar strings A and B playing the note '*Dha*' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the original frequency of B if the frequency of A is 427 Hz?

SOLUTION

$\nu_A - \nu_B = 5Hz\ \nu_A = 427Hz\ \nu_B = 422Hz$

DOPPLER EFFECT

It is an everyday experience that the pitch (or frequency) of the whistle of a fast-moving train decreases as it recedes away. When we approach a stationary source of sound with high speed, the pitch of the sound heard appears to be higher than that of the source. As the observer recedes away from the source, the observed pitch (or frequency) becomes lower than that of the source. This motion-related frequency change is called the Doppler effect. We shall analyse changes in frequency under three different situations: (1) observer is stationary but the source is moving, (2) observer is moving but the source is stationary, and (3) both the observer and the source are moving.

Source Moving; Observer stationery

Let us choose the convention to take the direction from the observer to the source as the positive direction of velocity. Consider a source S moving with velocity v_s and an observer who is stationary in a frame in which the medium is also at rest. Let the speed of a wave of angular frequency ω and period T_0, both measured by an observer at rest with respect to the medium, be *v*. We assume that the observer has a detector that counts every time a wave crest reaches it. As shown in Fig. 15.17, at time *t* = 0 the source is at the point S_1, located at a distance *L* from the observer, and emits a crest. This reaches the observer at the time $t_1 = L/v$. At time *t* = T_0 the source has moved a distance $v_s T_0$ and is at the point S_2, located at a distance ($L + v_s T_0$) from the observer. At S_2, the

source emits a second crest. This reaches the observer at

$$t_2 = T_0 + \frac{l + v_s T_0}{v}$$

At the time nT_0, the source emits its (*n*+1)th crest and this reaches the observer at the time

$$t_{n+1} = nT_0 + \frac{L + nv_s T_0}{v}$$

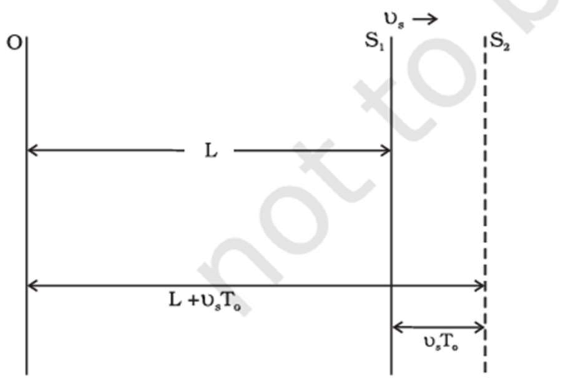

Fig. 15.17 *Doppler effect (change in frequency of wave) detected when the source is moving and the observer is at rest in the medium.*

Hence, in a time interval

$$nT_0 + \frac{L + nv_s T_0}{v} - \frac{L}{v}$$

the observer's detector counts *n* crests and the observer records the period of the wave as *T* given by

$$T = \frac{nT_0 + \frac{L + nv_s T_0}{v} - \frac{L}{v}}{n} = T_0\left(1 + \frac{v_s}{v}\right)$$

$$\Rightarrow \nu = \nu_0\left(1 + \frac{v_s}{v}\right)^{-1}$$

If v_s is small compared with the wave speed v, taking binomial expansion to terms in first order in $\frac{v_s}{v}$ and neglecting higher power,

$$\nu \approx \nu_0\left(1-\frac{v_s}{v}\right)$$

For a source approaching the observer, we replace v_s by $-v_s$ to get

$$\nu \approx \nu_0\left(1+\frac{v_s}{v}\right)$$

The observer thus measures a lower frequency when the source recedes from him than he does when it is at rest. He measures a higher frequency when the source approaches him.

Observer Moving; SourceStationary

Now to derive the Doppler shift when the observer is moving with velocity v_o towards the source and the source is at rest. We work in the reference frame of the moving observer. In this reference frame, the source and medium are approaching at speed v_o and the speed with which the wave approaches is $v_o + v$. Following a similar procedure as in the previous case, we find that the time interval between the arrival of the first and the *(n+*1*)* th crests is

$$t_{n+1} - t_1 = nT_0 - \frac{nv_oT_0}{v_o + v}$$

The observer thus measures the period of the wave to be

$$T_0\left(1-\frac{v_o}{v_o+v}\right) = T_0\left(1+\frac{v_o}{v}\right)^{-1}$$

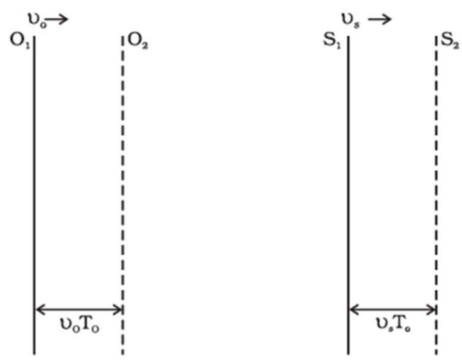

Fig. 15.18 *Doppler effect when both the source and observer are moving with different velocities.*

$$\Rightarrow \nu = \nu_0\left(1+\frac{v_o}{v}\right)$$

If $\frac{v_o}{v}$ is small, the Doppler shift is almost the same whether it is the observer or the source moving.

Both Source and Observer Moving

We will now derive a general expression for Doppler shift when both the source and the observer are moving. Let us take the direction from the observer to the source as the positive direction. Let the source and the observer be moving with velocities v_s and v_o respectively as shown in Fig.15.18. Suppose at time *t* = 0, the observer is at O_1 and the source is at S_1, O_1 being to the left of S_1. The source emits a wave of velocity v, of frequency ν and period T_0 all measured by an observer at rest with respect to the medium. Let *L* be the distance between O_1 and S_1 at *t* = 0 when the source emits the first crest. Now, since the observer is moving, the velocity of the wave relative to the observer is $v_o + v$. Therefore, the first crest reaches the observer at the time $t_1 = \frac{L}{v_o + v}$. At time *t* = T_0, both the observer and the source have moved to their new positions O_2 and S_2 respectively. The new distance between the observer and the source, O_2S_2, would be $L+(v_s - v_o)T_0$. At S_2, the source emits a second crest. This reaches the observer at the time.

$$t_2 = T_0 + \frac{L+(v_s - v_o)T_0}{v + v_o}$$

At time nT_0 the source emits its (*n*+1) th crest and this reaches the observer at the time

$$t_{n+1} = nT_0 + \frac{L + n(v_s - v_o)T_0}{v + v_o}$$

Hence, in a time interval $t_{n+1} - t_1$, i.e.,

$$nT_0 + \frac{L + n(v_s - v_o)T_0}{v + v_o} - \frac{L}{v + v_o}$$

the observer counts *n* crests and the observer records the period of the wave as equal to *T* given by

$$T = T_0 \frac{v + v_s}{v + v_o}$$

The frequency *v* observed by the observer is given by

$$v = v_0 \frac{v + v_o}{v + v_s}$$

NCERT Example 15.7

A rocket is moving at a speed of 200 m/ s towards a stationary target. While moving, it emits a wave of frequency 1000 Hz. Some of the sound reaching the target gets reflected back to the rocket as an echo. Calculate (1) the frequency of the sound as detected by the target and (2) the frequency of the echo as detected by the rocket.

SOLUTION

(i) $v = v_0 \left(1 - \frac{v_s}{v}\right)^{-1} = 2540 Hz$

(ii) $v' = v \left(\frac{v + v_o}{v}\right) = 4080 Hz$