CHAPTER-6

WORK, ENERGY, AND POWER

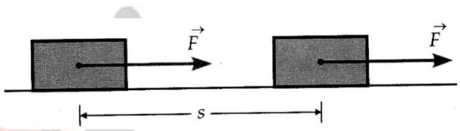
What is work?

Work is said to be done when a force acts on a body and the body moves through some distance in the direction of the force.

Example:-

Work is done when an engine pulls a train, a man moves up a hill, etc.

Work is done by a constant force:-

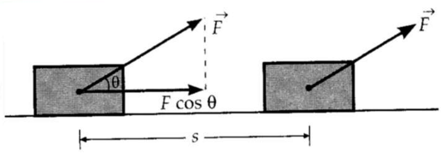


Case –I

If a force acts along the direction of motion then $w = FS$

Here $\vec{F}$ displaces a body in direction or parallel to the line of action of force)

Case – II



If force and displacement are included

W = component of force in the direction of displacement X Magnitude of displacement

$$\Rightarrow W = F\cos\theta \times S$$

$$\Rightarrow W = \vec{F}.\vec{S}$$

Work done is the dot product of force and displacement vector. Work is a **scalar** quantity

Dimension:- $W = [ML^2T^{-2}]$

Units of work:-

Absolute unit:-

(a) **Joule**:- If a force of 1N displaces a body through a distance of 1m in its direction then work done is said to be one joule. $1\text{J} = 1\text{N} \times 1\text{m}$

(b) **Erg**:- If a force of 1 dyne displaces a body through a distance of 1cm in its direction then work done is said to be one erg.

$$1\,\text{Erg} = 1\,\text{Dyne} \times 1\text{cm}$$

$$1\,joule = 10^7 erg$$

Gravitational unit:-

(a) Kilogram meter (kgm):- Work is said to be 1 kgm if a force of one-kilogram wt. displaces a body through 1 meter in its direction. $1\text{kgm} = 9.8 \times 1\text{m} = 9.8\,\text{J}$

(b) Gram – centimetre (gm cm):- Work is said to be 1gm cm if a force of one gram weight displaces a body through 1 cm in own direction. $1\text{gm cm} = 980\,\text{dyne} \times 1\,\text{cm} = 980\,\text{erg}$

Positive Work:-

If the force acting on a body has a component in the direction of displacement, then work done is called positive.

(a) When a ball falls freely under gravity.

(b) When a spring is stretched, both the stretching force and displacement act in the same direction. So work done is positive.

Negative Work:-

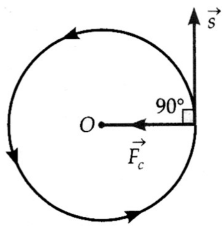
If a force acting on a body has a component in the direction of the opposite of displacement, the work done is negative.

Example:-

(a) When brakes are applied to a vehicle, the work done by the braking force is negative.

(b) When a body is lifted, work done by the gravitational force is negative.

Zero Work:-



If the body is displaced along a direction perpendicular to the direction of applied force, then work done is zero.

Example:-

(a) When a body moves in a circular path, work done by the centripetal force is zero.

(b) When a coolie walks on a horizontal road with a load on his head, the work done by the coolie on the road is zero.

(c) Work done in pushing an immovable wall is zero.

Work done in rectangular components:-

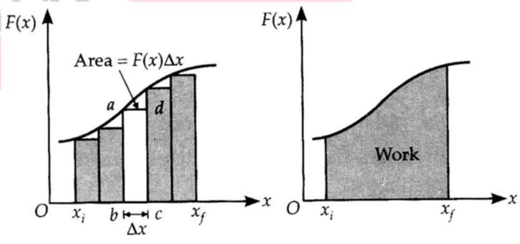
If $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$

$\vec{S} = \hat{i}S_x + \hat{J}S_y + \hat{k}S_z$

$\Rightarrow w = F_xS_x + F_yS_y + F_zS_z$

Work done by a variable force:-

Let's calculate the work done when the body moves from the initial position x_i to the final position x_j under the force F.



(a) (b)

Let for small displacement Δx, the force F can be taken as constant. So the work done is

$W = F.\Delta x =$ Area of rectangle abcd

Total work done

$W = \sum_{x_i}^{x_f} F\Delta x =$ Sum of areas of all rectangles erected overall small displacements

$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F\Delta x = \int_{x_i}^{x_f} Fdx$ = Area under the force ~ displacement curve.

QUESTION:-

A cyclist comes to a skidding stop at 10m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion.

(a) How much work done the road do on the cycle?

(b) How much work does the cycle do on the road?

Solution:-

(a) The stopping force and displacement make an angle of 180^0 with each other. Work done by the road $W = Fd\cos\theta = 200\times10\times\cos\pi = -2000\,J$. (Negative since cycle will stop in accordance with WE theorem)

(b) From Newton's 3rd law, an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. The road undergoes no displacement. Thus work done by cycle on the road is zero.

QUESTION:-

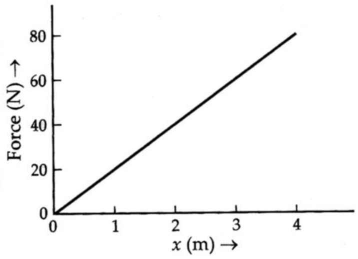
A force $\vec{F} = \hat{i} + 5\hat{j} + 7\hat{k}$ acts on a particle and displaces it through $\vec{S} = 6\hat{i} + 9\hat{k}$. Find the work done if the force is in Newton and displacement in meter.

$W = \vec{F}.\vec{S}$

$= (\hat{i} + 5\hat{j} + 7\hat{k}).(6\hat{i} + 0\hat{j} + 9\hat{k}) = 6 + 63 = 69\text{ joule}$

QUESTION:-

Find work done in moving the object from x = 2m to x = 3m from the following graph.



Work done = Area under F~x graph between x = 2 and x = 3 m

$= = \frac{1}{2}(40 + 60)\times(3 - 2) = 50\,J$

QUESTION:-

A particle moves along X-axis from x = 0 to x = 3 under the influence of a force given $F = 7 - 2x + 3x^2$. Find the work done in the process.

Solution:- $W = \int_0^5 F\,dx = \int_0^5 (7 - 2x + 3x^2).abc$

$= \int_0^5 7dx - \int_0^5 2x\,dx + \int_0^5 3x^2\,dx = [7x - x^2 + x^3]_0^5$

$$=7(5-0)-[24-0]+(125-0)=35-25+125=135\,J$$

Numerical:-

1. A force $F=(5+0.5x)$ acts on a particle in X-direction. Where F is in newton and x in meter. Find the work done by this force during displacement from x = 0 to x = 2m.
2. What is the work done by a person in carrying a suitable weighting 10 kg f on his head when he travels a distance of 5m in the (i) vertical direction (ii) horizontal direction? (Take g = 9.8 m/s^2)
3. A man weighing 50 kg of supports a body of 25 kgf. On his head what is the work done when he moves a distance of 20 m up an incline of 1 on 10? (g = 9.8 m/s^2)
4. A man moves on a straight horizontal road with a block of mass 2kg in his hand. If he covers a distance of 40 m with an acceleration of 0.5 m/s^2, find the work done by the man on the block during the motion.
5. A cluster of clouds at a height of 1000m above the earth burst and enough rain fell to cover an area of $10^6 m^2$ with a depth of 2cm. how much work would have been done in raising water to a height of clouds?

Kinetic Energy :

The energy possessed by a body by virtue of its motion is called kinetic energy.

Example (1) A fast-moving stone can break a windowpane.

(2) The kinetic energy of air is used to run windmills.

(3) Sailing ships employ the kinetic energy of wind.

(4) A hammer drives a nail into the wood.

Expression For Kinetic Energy :

Consider a body of mass m initially at rest. A force $\vec{F}$ applied to the body produces displacement $\overrightarrow{ds}$ in the same direction. Now small work done.

$$dw = \vec{F}.\overrightarrow{ds} = F\,ds$$

$$F = ma = m\frac{dv}{dt}$$

$$\Rightarrow dw = m\frac{dv}{dt}ds$$

$$= m\left(\frac{ds}{dt}\right)dv \qquad = mv\,dv$$

$$\left(\therefore v = \frac{ds}{dt}\right)$$

Total work done to increase its velocity from 0 to v is :

$$w = \int dw = \int_0^v mvdv = m\int_0^v vdv = m\left[\frac{v^2}{2}\right]_0^v = \frac{1}{2}mv^2$$

This work done appears as kinetic energy (k) of the body.

$$K = \frac{1}{2}mv^2$$

$\Rightarrow$ Kinetic energy is a **scalar** quantity

$\Rightarrow$ Same unit as work.

The relation between K.E. and linear momentum.

$$p = mv, k = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

And $p = \sqrt{2mk}$

Work-Energy Theorem For Constant Force:-

Statement – 1 ;

Work done by the net force acting on a body is equal to the change produced in the kinetic energy of the body.

Proof: Let a force F acting on a body produces acceleration a. Let the velocity of the body changes from u to v after a distance s.

$$v^2 - u^2 = 2as$$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas \text{ (By multiplying } \frac{1}{2}m)$$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = fs = w \qquad (\because f = ma)$$

$$\Rightarrow k_f - k_i = w$$

$\Rightarrow$ Change in K:E: of the body = Work done on the body by the Net force.

Work-Energy Theorem For Variable Force:-

Let a variable force $\vec{F}$ acts on a body of mass m which produces displacement $\overrightarrow{ds}$ in its direction.

$$\Rightarrow dw = \vec{F}.\overrightarrow{ds} = F\, ds \cos 0^0 = Fds$$

From Newtons 2nd law of motion,

$$F = ma = m\frac{dv}{dt}$$

$$\Rightarrow dw = m\frac{dv}{dt}.ds = m\left(\frac{ds}{dt}\right)dv = mvdv \qquad (\because v = ds(dt))$$

If $w =$ Total Work done on the body,

$$w = \int_0^w dw = \int_u^v mvdv = m\int_u^v vdu = m\left[\frac{v^2}{v}\right]_u^v$$

$$\Rightarrow w = \frac{1}{2}mv^2 = \frac{1}{2}mu^2$$

$$\Rightarrow w = k_f - k_i = \text{change in K.E. of body}$$

This is work-energy theorem for variable force.

Example

A block of mass m = 1kg moving on a horizontal surface with speed $v_i = 2m/s$ enters a rough patch ranging from x = 0.10m to x = 2.01 m. The retarding force F_S on the block in this range is inversely proportional to x over this range.

$F_S = -k/x$ from $0.1 < x < 2.01m$

= 0 for x < 0.1 m and x > 2 .01 m

Where k = 0.6 J. What is the final kinetic energy and speed of v_f the block as it crosses this patch)?

Solution: $K_f - K_i = \int_{X_i}^{X_f} F\,dx$

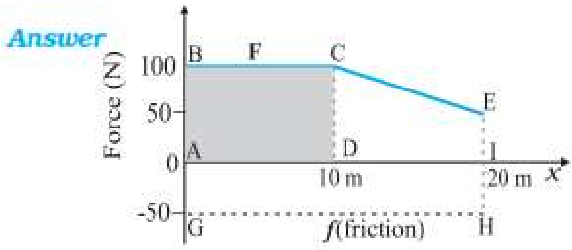
$$\Rightarrow K_f = K_i = \int_{X_i}^{X_f} F\,dx = \frac{1}{2}mV_i^2 + \int_{0.1}^{2.01} -\frac{K}{X}dx \qquad \Rightarrow K_f = \frac{1}{2}mV_i^2 = K\int_{0.1}^{2.01}\frac{dx}{x} = \frac{1}{2}m\,V_i^2 - K[\ln x]_{0.1}^{2.01}$$

$$\Rightarrow K_f = \frac{1}{2}mV_i^2 - K\,\ln\left[\frac{2.01}{0.1}\right] = 2 - 0.5\ln(20.1) \qquad \Rightarrow K_f = 2 - 1.5 = 0.5J$$

$$\Rightarrow \frac{1}{2}mV_f^2 = 0.5 \Rightarrow V_f = \sqrt{\frac{2K_f}{m}} = 1m/s.$$

Example – 2

A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100N. Over a distance of 10m. Then she gets tired and her applied force reduces linearly with distance to 50N. The total distance moved by trunk is 20m. Plot the force applied by the women and the frictional force. Which is 50N versus displacement? Find the work done by the two forces over 20m.



Answer: Work done by the women is $W_f \rightarrow$ Area of the rectangle ABCD + Area of trapezium CEID

$$\Rightarrow W_f = (100\times10) + \frac{1}{2}(100+50)\times10$$

$$= 1000 + 750$$

$$= 1750J$$

Work done by the frictional force is

$W_f \rightarrow$ Area of the rectangle AGHI. $\Rightarrow W_f = (-50)\times20 = -1000J$.

The area on the negative side of the force axis has a negative sign.

Example:

If the linear momentum of a body increases by 20% what will be the % increase in kinetic energy of the body.

Solution : $K = P^2/2m$

Final momentum $P + \left(\frac{20}{100}\right)P = 6P/5$

$$K_f = \frac{(6P/5)^2}{2m} = \frac{36}{25}\left(\frac{p^2}{2m}\right) = \frac{36}{25}k$$

Increase in kinetic energy.

% Increase in Kinetic Energy.

$$=\left(\frac{K^1-K}{K}\right)\times 100=\frac{\frac{11}{25}K}{K}\times 100=44\%$$

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Question for Practice:

1. In a ballistics demonstration, a police officer fires a bullet of mass 50.0 g with a speed 200 m/s On wood of thickness 2.00cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet?
2. A shot travelling at the rate of 100 m/s in just able to pierce a plank 4cm thick. What velocity is required to just pierce a plank 9cm thick?
3. Two identical 5kg blocks are moving with the same speed 2m/s towards each other along with the frictionless horizontal surface. The two blocks collide, stick together, and come to rest. Consider the two blocks as a system. Find work done by (i) External Force (ii) Internal Force.
4. In linear momentum of a body increases by 20% what will be the % increase in the kinetic energy of the body?
5. Two bodies of masses 1g and 16g are moving with equal kinetic energies. Find the ratio of the magnitudes of their linear moment.

Potential Energy:-

The energy stored in a body or a system by virtue of its position or by its configuration is called potential energy.

Example:-

(a) P.E of water stored in dams is used to run turbines.

(b) A body on the roof of a building has some potential energy.

(c) A stretched bow possesses potential energy.

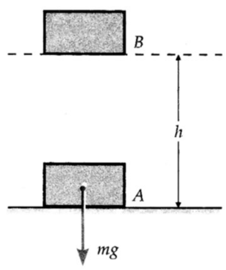
Types of Potential Energy:-

(a) Gravitational potential energy:- The Potential energy associated with the state of separation of two bodies that attract one another by gravitational force.

(b) Elastic Potential energy:- The Potential energy associated with the state of compression and extension on an elastic object. (Ex- spring)

(c) Electrostatic Potential Energy:- The energy due to the interaction between two electric charges is Electrostatic Potential Energy.

Derivation of Gravitational potential Energy:-



Let m = mass of the body at height h from the surface of the earth,

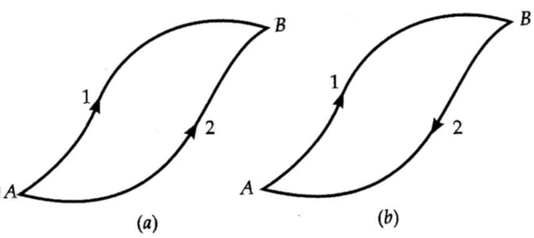
If $h << R$

$\Rightarrow$ Force needed to lift the body F= mg

Work was done on the body in raising through height h, w = F.h = mgh

Work done against gravity is stored as gravitational potential energy (U) of the body

$\Rightarrow \boldsymbol{U = mgh}$



Conservative Force:-

If the work done by the force in moving from one point to another is independent of the path followed by the particle and depends only on endpoints, then it is conservative.

$$W_{AB}(\text{in path }1) = W_{AB}(\text{in path }2)$$

$$\Rightarrow W_{AB}(\text{in path }1) = W_{BA}(\text{in path }2) \qquad \Rightarrow W_{AB}(\text{in path }1) + W_{BA}(\text{in path }2) = 0$$

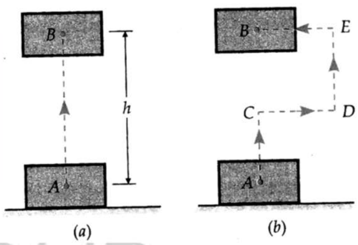
$$W_{\text{closed path}} = 0$$

i.e work done by the force in moving around a closed path is zero in cased of a conservative force.

Gravitational Force is a Conservative Force:-

If a body is taken vertically through height h from A to B

$\Rightarrow W = mg.AB = mgh$ ……………………. (1)



Let the body taken from A to B along the path ACDEB. Total work done is

$$W = W_{AC} + W_{CD} + W_{DE} + W_{EB}$$

$$= mg \times AC + 0 + mg \times DE + 0$$

$$= mg(AC + DE) = mg \times h = mgh$$

Work done against gravity is independent of the path and depends on the initial and final position of the body. So Gravitational force is a conservative force.

The relation between Potential Energy and Conservative force:-

When conservative force F(x) acts on a particle, the change in potential energy ΔU is negative of work done by the conservative force.

But $W = \int_{x_i}^{x_f} F(x)\,dx \quad \Rightarrow \Delta U = -\int_{x_i}^{x_f} F(x)\,dx$

Differentiating the above equation,

$$\frac{dU(x)}{dx} = -F(x)$$

$$\Rightarrow F(x) = -\frac{dU(x)}{dx}$$

The negative gradient of potential energy (U) is a conservative force F(x)

$$-\frac{d}{dh}U(h) = -\frac{d}{dh}(mgh) = -mg = F$$

The negative gradient of potential energy is equal to the gravitational force

Gravitational force is conservative in nature.

Example:-

Find the velocity of the bob of a simple pendulum in its means position if it can rise to a vertical height of 10 cm. (g = 9.8 m/s^2)

Solution:-

By constant of energy, K.E of the bob at mean = P.E of the bob at the highest position

$$\Rightarrow \frac{1}{2}mv^2 = mgh$$

$$\Rightarrow v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 0.10} = 1.4 m/s$$

Example:-

A vehicle of mass 15 quintal climbs up a hill 200m high. It then moves on a level road with a speed of 30 m/s. Find the Potential energy gained by it and its total mechanical energy while running on top of the hill.

Solution:-

$m = 1500kg, g = 9.8m/s^2, h = 200m$

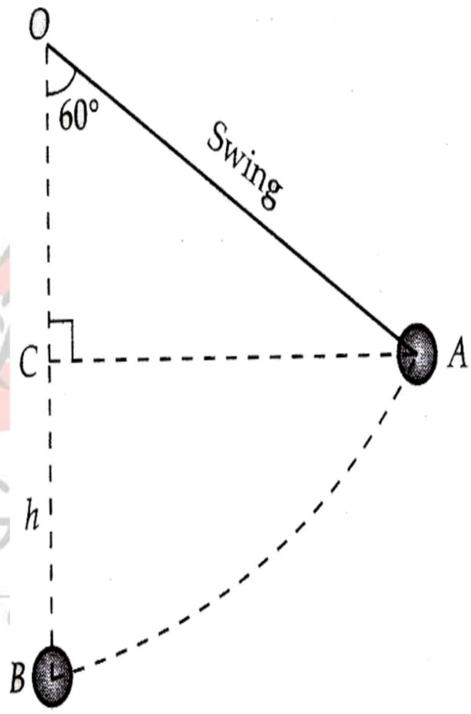
$U = mgh = 1500 \times 9.8 \times 200 = 2.94 \times 10^6 J$

Now $K = \frac{1}{2}mv^2 = \frac{1}{2} \times 1500 \times (30)^2 = 0.675 \times 10^6 J$

$E = K + U = (0.675 + 2.94) \times 10^6 = 3.615 \times 16^6 J$

Example:-

A girl of mass 40 kg sits in as wing formed by a rope of 6m length. A person pulls the swing to a side so that the rope makes an angle of 60^0 with the vertical. What is the gain in the potential energy of girl?



Solution:-

M = 40 kg, OA = OB = 6m

In OCAC, $\cos 60^0$ = OC/OA

$\Rightarrow OC = OA\cos 60^0 = 6 \times \frac{1}{2} = 3m$

h = CB = OB – OC = 6 – 3 = 3m

P.E gained by girl = mgh = $40 \times 9.8 \times 3 = 1176J$

Numerical:-

01. A ball is thrown vertically up with a velocity of 20m/s. At what height will its K.E be half its original value?

02. Find the work done in lifting a 300 N weight to a height of 10 m with an acceleration 0.5 m/s^2. (Take g = 10 m/s^2)

03. A bullet of mass 10 g travels horizontally with a speed of 100 m/s and is absorbed by a wooden block of mass 500 g suspended by a string. Find the vertical height through which the block rises. $\left(g = 10 m/s^2\right)$

04. A ball falls under gravity from a height of 10m with an initial downward velocity u. it collides with the ground and loses 50% of its energy in the collision and then rises back to the same height. Find the initial velocity u.

05. The potential energy of a 2kg particle free to move along the X-axis is given by $U(x) = \left(\frac{x}{b}\right)^4 - 5\left(\frac{x}{b}\right)^2$ J. Where b = 1m. Plot this potential, identifying the extremism points. Find the region where the particle is found to maximum speed given that the total mechanical energy is (i) 36 J.

Power:-

The rate of doing work is called power. It is denoted by P.

$Mathematically,\ P = \frac{w}{t}$ (where w= work done and t= time taken)

Power is a scalar quantity

Dimension:- $P = \frac{W}{t} = \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$

Units:-

Watt (w):- The power of an agent is one watt if it does work at the rate of 1 joule per second.

$$1\,\text{Watt} = \frac{1\,\text{Joule}}{1\,\text{second}}$$

The bigger unit is $(KW\,\&\,HP)$

$1kw = 10^3\,w$

$1\,hp = 746\,w$

$1\,MW = 10^6\,w$

Instantaneous Power:-

The power of an agent may not be constant during a time interval. The limiting value of average power as the time interval approaches zero is called instantaneous power.

$$P = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

Again $dw = \vec{F}.\vec{ds}$

$$\Rightarrow P = \frac{dw}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$$

$$\Rightarrow P = \vec{F}.\vec{V} \qquad (As\, \vec{V} = \frac{ds}{dt})$$

Power is equal to the dot product of the **force** and **velocity vector** at that instant.

Example:-

A car of mass 2000 kg is lifted a distance of 30 m by a crane in 1 min. A 2nd crane does the same job in 2 min. Do the cranes consume the same or different amounts of fuel? What is the power supplied by each crane? (Neglect power dissipation against friction)

Solution:- M = 2000 kg, S = 30 m, t_1 = 60 sec, t_2 = 120 sec

Work is done by each crane

$$W = FS = mgs = 2000 \times 9.8 \times 30 = 5.88 \times 10^5 J$$

Now$P_1 = \frac{w}{t_1} = \frac{5.88 \times 10^5}{60} = 9800w$ and

$$P_2 = W / t_2 = 5.88 \times 10^5 / 120 = 4900w$$

Example:-

An electric motor is used to lift an elevator and its load (total mass = 1500 kg) to a height of 20 m. The time taken for the job is 20 s. What is the work done? What is the rate at which work is done? If the efficiency of the motor is 75% at which rate is the energy supplied to the motor?

Solution:-

M = 1500kg, h = 20m

η= 75% , t = 20s

W = mgh = 1500 x 9.8 x 20 = $2.94 \times 10^5 J$

$$P = \frac{W}{t} = \frac{2.94\times10^5}{20} = 1.47\times10^4 \text{ w}$$

$$\eta = \frac{P_{out}}{P_{int}} \Rightarrow \frac{75}{100} = \frac{1.47\times10^4}{P_{int}}$$

$$\Rightarrow P_{int} = \frac{1.47\times10^4\times100}{75} = 1.96\times10^4 \text{ w}$$

Example:-

Find the horsepower of a man who can chew ice at the rate of 30 g per minute?

Given 1 hp = 746w and J = 4.2 J/cal

Solution:-

m = 30, L = 80 cal/g, Heat to melt ice it = ML = 30 x 80 cal

Work done W = J mL = 4.2 x 30 x 80 J

$$\Rightarrow P = \frac{w}{t} = \frac{240\times4.2}{60} = 168\text{W} = \frac{168}{746} = 0.225\, hp$$

Example:-

A car of mass 1000 kg accelerates uniformly from rest to a velocity of 54 km/h in 5 seconds. Find (i) its acceleration (ii) it's gain in K.E (iii) Average power of the engine during this period by neglecting friction.

Solution:- M = 1000 kg, u = 0, v = 54 km/h = 15 m/s , t = 5 sec

(a) $a = \frac{v-u}{t} = \frac{15-0}{5} = 3 m/s^2$

(b) Gain in K.E = $\frac{1}{2}m\left(v^2-u^2\right) = \frac{1}{2}\times1000\left(15^2-0^2\right) = 1.125\times10^5 J$

(c) $P = w/t = \frac{1.125\times10^5}{5} = 22500w$

Example:-

A body is moving unidirectionally under the influence of a source of constant power. Its displacement in time t is proportional to ___?

Solution:- $W = p\times t = \frac{1}{2}mv^2$

$\Rightarrow v^2 = \frac{2pt}{m}$ $\qquad \Rightarrow v = \sqrt{\frac{2pt}{m}}$

$\Rightarrow \frac{ds}{t} = \sqrt{\frac{2pt}{m}}$

On integration, $S = \sqrt{\frac{2p}{m}} \times \frac{2}{3} t^{3/2}$

$\Rightarrow S \propto t^{3/2}$

Example:-

Power applied to a particle varies with time as $P = \left(3t^2 - 2t + 1\right)$ a watt. where t is in second. Find the change in its kinetic energy between t = 2s and t = 4 sec.

Solution:- $\Delta K = w = \int P\,dt = \int_2^4 \left(3t^2 - 2t + 1\right)dt$

$= \left[t^3 - t^2 + t\right]_2^4 = (64-8) - (16-4) + (4-2)$

$= 56 - 12 + 2$

$= 46\,J$

Example:-

A man weighing 60 kg climbs up a staircase carrying a load of 20 kg on his head. The staircase has 20 steps each of height 0.2m. If he takes 10s to climb, find his power?

Solution:- M = 60+20=80 kg, h = 20 x 0.2 = 4m

g = 9.8 m/s^2, t = 10s

$$P = \frac{w}{t} = \frac{mgh}{t} = \frac{80 \times 9.8 \times 4}{10} = \frac{3136}{10} = 313.6w$$

Problems for Practice:-

01. A well 20 m deep and 3m in diameter contains water to a depth of 14m. how long will 5 hp engine take to empty it?

02. A pump on the ground floor of a building can pump up water to fill a tank of volume 30 m^3 in 15 min. If the tank is 40 m above the ground and the efficiency of the pump is 30%, how much electric power is consumed by the pump?

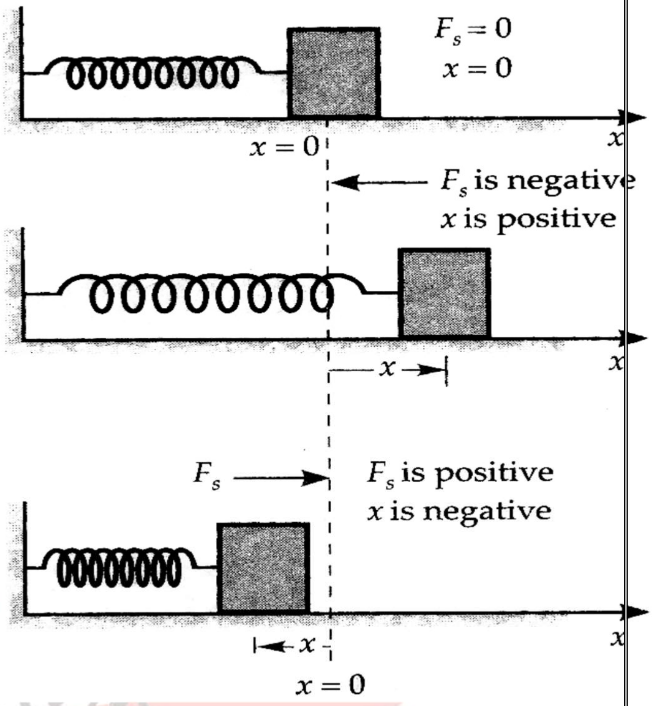
03. A particle of mass 0.5 kg travels in a straight line with velocity $V = ax^{3/2}$, where $a = 5m^{-1/2}.s^{-1}$. What is work done by the net force during its displacement from x = 0 to x = 2m.

04. A particle of mass m is moving in a horizontal circle of radius r, under a centripetal force equal to –k/r^2 where K is constant. What is the total energy of the particle?

05. A body of mass M is moved along a straight line by a machine delivering a constant power P. Find the expression for the distance moved by the body in terms of M, P, and t?

Potential Energy of spring and Conservative force:-

Consider an elastic spring of small mass with one end attached to a support and another end to a body of mass m which slides on the surface.



x = displacement of the block from the equilibrium position.

F_S = Spring force acting on spring towards the equilibrium position.

From Hooke's law, $F_S \propto x$ or $F_S = -Kx$

[K is proportionality constant called spring constant. The unit of K is N/m. If K is large then spring is stiff]

Work done by spring force for small extension dx is $dw_s = F_s dx = -Kxdx$

If the block is moved from initial displacement x_i to final displacement x_f, the work done by the spring force is.

$$W_s = \int dW_s = -\int_{x_i}^{x_f} Kxdx = -K\left[\frac{x^2}{2}\right]_{x_i}^{x_f}$$

$$\Rightarrow W_s = \frac{1}{2}Kx_i^2 - \frac{1}{2}Kx_f^2$$

Work is done by an external force (F) to pull the block outward,

$W = \frac{1}{2}Kx_f^2 - \frac{1}{2}Kx_i^2$ (since applied force is equal and opposite to F_s)

This work done is equal to the increase in P.E of spring

$\Rightarrow \Delta u = \frac{1}{2}Kx_f^2 - \frac{1}{2}Kx_i^2$. If we take $u(x) = 0$ in the equilibrium position,

$$\Rightarrow U(x) - 0 = \frac{1}{2}Kx^2$$

$$\Rightarrow U(x) = \frac{1}{2}Kx^2$$

Conclusion:-

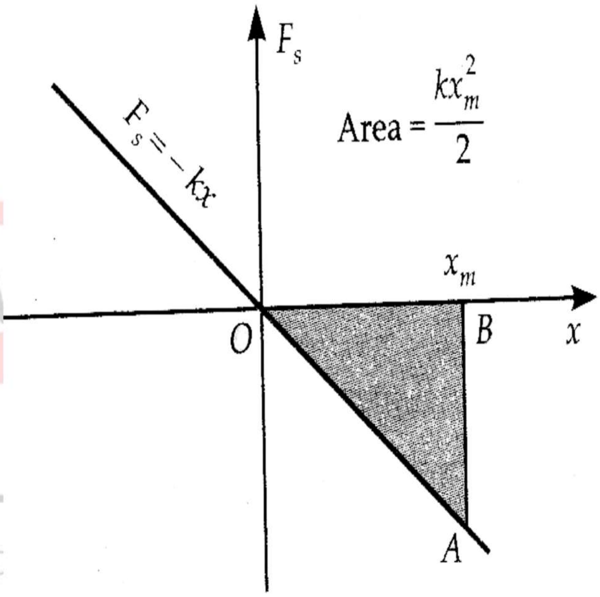
(a) Spring force is position-dependent.

(b) Work done by spring force depends on the initial and final position.

(c) Work done by spring force in a cyclic process is zero.

Thus spring force is **conservative** in nature.

The graph between spring force (F_s) and displacement (x):-



The area of shaded triangle OAB represents the work done by the spring force.

$$W_s = \text{Area of OAB}$$

$$= \frac{1}{2} \times AB \times OB \qquad = \frac{1}{2} \times F_x \times X_m$$

$$= \frac{1}{2}(-Kx_m) \times X_m \qquad (\because F_s = -Kx_m)$$

$W_s = -\frac{1}{2}KX_m^2$.(Due to the opposite signs of F_x and x, W_s is negative).

Conservation of Energy in Elastic Spring:-

When a spring is stretched total energy is Potential Energy. When it is released, all P.E is converted into Kinetic energy until it stops at position x = $-x_m$, where all K.E is converted into P.E, i.e the spring oscillates about mean position and total mechanical energy remains constant.

At extreme:- Here $x = \pm x_m \,\&\, V = 0$

$\Rightarrow K = 0 \,\&\, U = \frac{1}{2}KX_m^2 (Maximum)$

At intermediate position:-

If $-x_m < x < x_m$ $\Rightarrow$ Total energy is partly kinetic and partly potential.

Total energy = $E_k + E_p$

$$\Rightarrow \frac{1}{2}KX_m^2 = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2$$

$$\Rightarrow K = \frac{1}{2}mv^2 = \frac{1}{2}K\left(x_m^2 - x^2\right)$$

$$\Rightarrow V = \sqrt{\frac{K}{m}\left(x_m^2 - x^2\right)}$$

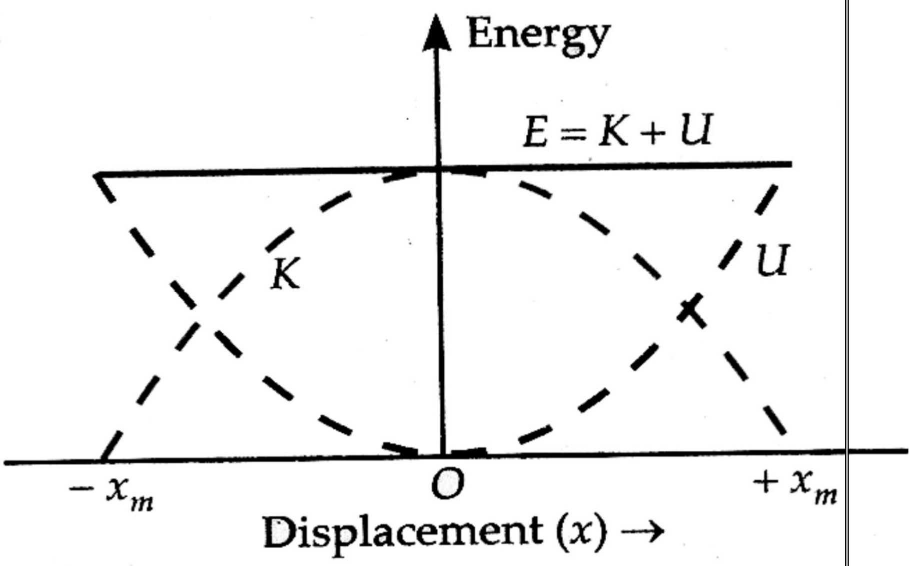
At equilibrium, $x = 0$

$$\Rightarrow U = \frac{1}{2}K(0)^2 = 0$$

$$K = \frac{1}{2}mv_m^2 = \frac{1}{2}Kx_m^2$$

$$\Rightarrow V_m = \sqrt{\frac{K}{m}}x_m$$

V_m = The maximum speed of spring.



The graph between K.E, P.E, and total energy with displacement x is shown below.

Example:-

Two springs have force constants K_1 and K_2 ($K_1 > K_2$). On which spring is more work done, if

(i) they stretched by the same force and

(ii) they are stitched by the same amount?

Solution:- $F = K_1 x_1 = K_2 x_2$

$$\Rightarrow \frac{W_1}{W_2} = \frac{\frac{1}{2}K_1 x_1^2}{\frac{1}{2}K_2 x_2^2} = \frac{F.x_1}{F.x_2} = \frac{x_1}{x_2} = \frac{K_2}{K_1}$$

As $K_1 > K_2 = W_2 > W_1$

Let tow springs are stitched by same distance r, then $\frac{W_1}{W_2} = \frac{\frac{1}{2}K_1 x^2}{\frac{1}{2}K_2 x^2} = \frac{K_1}{K_2}$

As $K_1 > K_2 \Rightarrow W_1 > W_2$

Example:-

A spring gun has a spring constant of 80 N/cm. The spring is compressed 12 cm by a ball of mass 15g. How much is the potential energy of the spring? If the trigger is pulled, what will be the velocity of the ball be?

Solution:-

Here K = 80 N/cm = 80 x 100 N/m

$x = 12\text{cm} = 0.12\text{m}, m = 15\text{g} = 15 \times 10^{-3}\text{kg}$

$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 80 \times 100(0.12)^2 = 57.6\,J$

K.E of ball = P.E of spring

$$\Rightarrow \frac{1}{2}mv^2 = U = 57.6 \text{ joule}$$

$$\Rightarrow V^2 = \frac{2\times 57.6}{15\times 10^{-3}} = 7680$$

$$V = 87.6\, m/s$$

Example:-

To simulate car accidents, Auto manufacturers study the collisions of moving cars with mounted springs of different spring constants. Consider a typical simulation with a car of mass 1000 kg moving with a speed 18.0 km/h on a smooth road and colliding with a horizontally mounted spring of spring constant $6.25\times 10^3\,\text{Nm}^{-1}$. What is the maximum compression of the spring?

Solution:-

At maximum compression, the kinetic energy of the car is converted entirely into the potential energy of the spring. The kinetic energy of the moving car is

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}\times 10^3 \times 5\times 5$$

$$K = 1.25\times 10^4\,\text{J}$$

Where we have converted 18kmh^{-1} to $5\,\text{ms}^{-1}$. At maximum compression x_m, the potential energy of the spring is equal to the kinetic energy K of the moving car from the principle of conservation of mechanical energy.

$$V = \frac{1}{2}kx_m^2$$

$$= 1.25\times 10^4\,\text{J}$$

We obtain $x_m = 2.00\,m$

Example:-

Consider Example – 5, taking the coefficient of friction, μ to be 0.5, and calculate the maximum compression of the spring. **(For solution, NCERT – 6.9 page no. 125)**

Numericals for Practice:-

01. A solid mass 2kg moving with a velocity of 10 m/s strikes an ideal weightless spring and produces a compression of 25 cm in it. Find the force constant of the spring.

02. A block of mass 2kg initially at rest is dropped from a height of 1m into a vertical spring having a force constant 490 N/m. Find the maximum distance through which the spring will be compressed.

03. A block of mass 2kg is dropped from a height of 40 cm on a spring whose force constant is 1960 N/m. What will be the maximum distance x through which the spring is compressed?

Conservation of Mechanical Energy:-

Statement:- *If only the conservative forces are doing work on a body, then its total mechanical energy (K.E + P.E) remains constant.*

Proof:- From the work-energy theorem, the charge in K.E is $\Delta K = F(x)\Delta x$.

For conservative force, the charge in P.E is

$\Delta U =$ Negative of work done = $-F(x)\Delta x$

$\Rightarrow \Delta K = -\Delta U$

$\Rightarrow \Delta K + \Delta U = 0$

$\Rightarrow \Delta(K + U) = 0$

$\Rightarrow K + U = \text{Constant}$

The sum of the total mechanical energy of the system remains constant under the conservative force.

Conservation of mechanical energy in the free failing body:-

Let a body of mass m lying at position A at a height h above the ground.

At point A:-

$E_k = 0$

P.E of body $E_P = mgh$

Total mechanical energy $E_A = E_K + E_P = 0 + mgh = mgh$

At point B:-

If a body falls through height x and reaches point B,

$V^2 - u^2 = 2gx \ (\therefore u = 0, h = x)$

$$\Rightarrow V^2 = 2gx \Rightarrow E_k = \frac{1}{2}mv^2 = \frac{1}{2}m \times 2gx = mgx$$

Similarly, potential energy at B Is

$$E_P = mg(h-x)$$

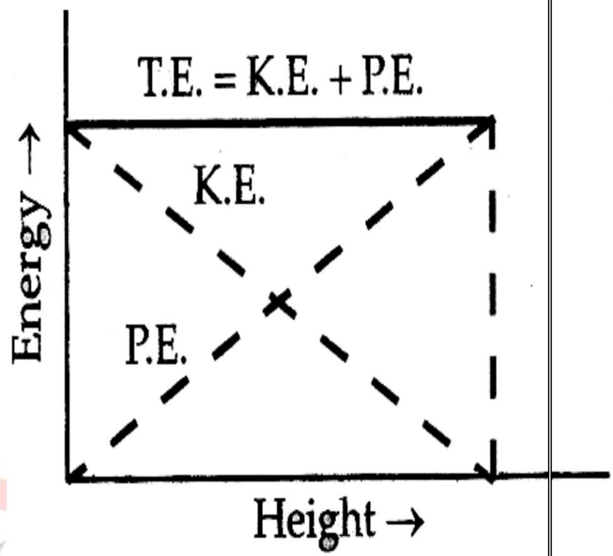
$$E_B = E_k + E_P = mgx + mg(h-x) = mgh$$

<u>At point C</u>:- Let V = velocity of the body at point C

$\Rightarrow V^2 = 2gh$ $\quad\quad \Rightarrow E_K = \frac{1}{2}mv^2 = mgh$

$E_P = 0$ $\quad\quad \Rightarrow E_C = E_K + E_P = mgh$

When the body falls, its P.E decreases, and K.E increases and total mechanical energy remains constant at all points. i.e total mechanical energy is conserved during the free fall of a body. The variation of K.E and P.E and the total energy with height are shown below.



<u>Properties of conservative Force</u>:-

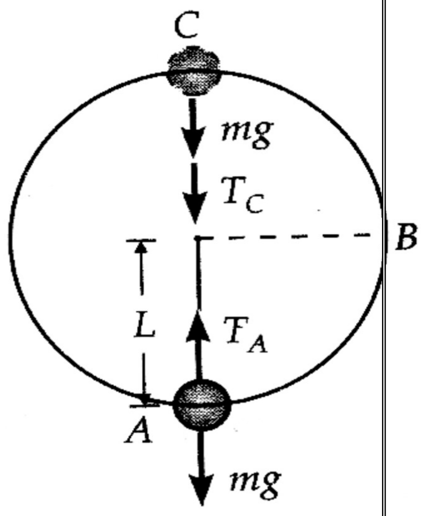
(a) A force F is conservative if it is defined from the scalar potential energy function U(x) by the relation $F(x) = -\frac{dU(x)}{dx}$

(b) The work done by a conservative force on a body is path independent and depends only on endpoints. $W = \int_{x_i}^{x_f} F(x)dx = K_f - K_i = U_i - U_f$

(c) The work done by the conservative force is zero if the body is moving around any closed path returns to its initial position $W_{closed\,path} = \oint F(x)dx = 0$

(d) If only the conservative forces are acting on the body, then its mechanical energy is conserved.

Example:-



A bob of mass m is suspended by a light string of length L. It is imparted a horizontal velocity v_0 at the lowest point A such that it completes a semi-circular trajectory in the vertical plane with the string becoming slack only on reaching the topmost point C. This is shown in figure obtain an expression for (i) v_0 (ii) the speeds at points B and C. This is shown in the figure. obtain an expression for (i) v_0 (ii) the speeds at points B and C (iii) the ratio of the kinetic energies (K_B/K_C) at B and C. Comment on the nature of the trajectory of the bob after it reaches the point C.

Solution:- There are two external forces on the bob gravity and tension T in the string. The total mechanical energy E of the system is conserved. We take the potential energy of the system is zero at the lowest point A. Thus at A

$E = \frac{1}{2}mv_0^2$ .... (1)

$T_A - mg = \frac{mv_0^2}{L}$ (Newton's Second Law)

Where T_A is the tension in the string at A. At the highest point C. The string slackens, as the tension in the string (T_A) becomes zero. Thus at C

$E = \frac{1}{2}mv_c^2 + 2mgL$.......................... (2)

$mg = \frac{mv_c^2}{L}$ (Newton's Second Law) .... (3)

Where v_c is the speed at C. From equation (2) and (3)

$E = \frac{5}{2}mgL$

Equating this to the energy at A

$$\frac{5}{2}mgL = \frac{m}{2}v_0^2$$

Or $v_0 = \sqrt{5gL}$

it is clear from equation (3),

$$v_c = \sqrt{gL}$$

At B the energy is $E = \frac{1}{2}mv_B^2 + mgL$

Equating this to the energy at A and employing the result from (i). namely $v_0^2 = 5gL$

$$\frac{1}{2}mv_B^2 + mgL = \frac{1}{2}mv_0^2$$

$$= \frac{5}{2}mgL$$

$$\therefore v_B = \sqrt{3gL}$$

(iii) The ratio of the kinetic energies at B and C is

$$\frac{K_B}{K_C} = \frac{\frac{1}{2}mv_B^2}{\frac{1}{2}mv_c^2} = \frac{3}{1}$$

If the string is cut at point C it will execute a projectile motion otherwise the bob will continue its circular path and complete revolution.

Example:-

A person trying to lose weight lifts a 10 kg mass, one thousand times, to a height of 0.5 m each time. Assume that the potential energy lost each time she lowers the mass is dissipated. (a) How much work does she do against the gravitational force? (b) Fat supplies 3.8×10^7 J of energy per kilogram which is converted to mechanical energy with a 20% efficiency rate. How much fat will the dieter use up?

Solution:- $m = 10kg, h = 0.5m, n = 1000, g = 9.8m$

Work was done against the gravitational force $W = n \times mgh = 1000 \times 10 \times 9.8 \times 0.5$

$\Rightarrow W = 49000\,J$

Mechanical energy supplied by 1kg of fat

$= 20\%$ of $3.8 \times 10^7\,J \quad = \frac{20}{100} \times 3.8 \times 10^7 = 76 \times 10^5\,J$

$\Rightarrow$ Fat consumed for $76 \times 10^5\,J$ of energy = 1 kg

$\Rightarrow$ Fat consumed for $49,000J$ of energy $= \frac{49000}{76 \times 10^5} = 6.45 \times 10^{-3}\,kg$

Example:-

A bolt of mass 0.3 kg falls from the ceiling of an elevator moving down with a uniform speed of $7ms^{-2}$. It hits the floor of the elevator (length of the elevator = 3m) and does not rebound. What is the heat produced by the impact? Would your answer be different if the elevator were stationary?

Solution:-

As the elevator is moving down with a uniform speed (a = 0), g remains the same for m = 0.3 kg, h=3m, g = 9.8 m/s^2

P.E lost by the bolt = $mgh = 0.3 \times 9.8 \times 3 = 8.82\,J$

As the bolt does not rebound, the energy is converted into heat.

$\Rightarrow$ The heat produced = 8.82 J

Even if the elevator were stationary, the same amount of heat would have produced

Numerical:-

01. A stone of mass 0.4 kg is thrown vertically up with a speed of 9,8 ms^{-1}. Find the potential and kinetic energies after half second

02. A ball is thrown vertically up with a velocity of $20ms^{-1}$. At what height, will its K.E be half its original value?

03. 230 joules were spent in lifting a 10 kg weight to a height of 2m. Calculate the acceleration with which it was raised. Take g = $10ms^{-2}$

04. Calculate the work done in lifting a 300 N weight to a height of 10 m with an acceleration $0.5ms^{-2}$. Take $g = 10ms^{-2}$

05. A bullet of mass 10 g travels horizontally with a speed of $100ms^{-1}$ and is absorbed by a wooden block of mass 990 g suspended by a string. Find the vertical height through which the block rises. Take $g = 10\,ms^{-2}$

06. A simple pendulum of length 1m has a wooden bob of mass 1kg. It is struck by a bullet of mass 10^{-2} kg moving with a speed of $2\times10^{2}\,ms^{-1}$. The bullet gets embedded into the bob. Obtain the height to which the bob rise before swinging back. Take $g = 10\,ms^{-2}$

Non-conservative Force and Motion in a circle---

If the amount of work done in moving a body against a force from one point to another depends on the path along which the body moves, then that force is called Non-conservative Force.

Examples- Force of Friction and Viscosity

Property--

(1) Work done by this force for a particle moving in a closed path is not Zero.

(2) Work done by this force on any object is path-dependent.

(3) This force is not a negative gradient of Potential Energy.

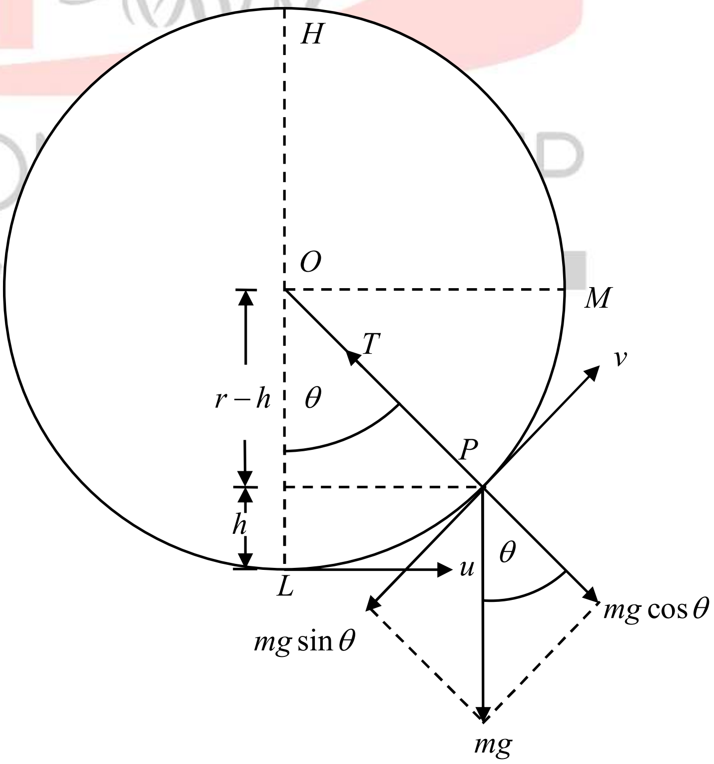
MOTION IN A VERTICAL CIRCLE—

Consider a body of mass m tied to one end of string rotating in a vertical circle of radius r,

Let u= velocity at the lowest point.

V= Velocity at highest Point,

h= Vertical distance from L to N.



Now from conservation of energy

$(K.E+P.E)_{at\ L} = (K.E+P.E)_{at\ P}$

$\Rightarrow \frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mgh$

$\Rightarrow u^2 = v^2 + 2gh$

$\Rightarrow v^2 = u^2 - 2gh$

$\Rightarrow v = \sqrt{u^2 - 2gh}$-----------------(1)

TENSION (T)-

The net centripetal force acting on the object in the string is

T-mg cosθ = $\frac{mv^2}{r}$

$\Rightarrow$ T= mg cosθ + $\frac{mv^2}{r}$ -----------(2)

In OPN right angle triangle

Cos θ =$\frac{ON}{OP}$ = $\frac{r-h}{r}$ ===============(3)

Using eq (1) and (3) equation (2) becomes.

T= $mg\left(\frac{r-h}{r}\right) + m/r(u^2 - 2gh)$

$\Rightarrow$ T= m/r(u^2 +gr -3gh) -----------(4)

This is tension along the string at any point of the circle,

<u>At Lowest Point</u> L , h=0

$\Rightarrow$ T_L = m/r (u^2 + gr) ---------------------(5)

<u>At highest Point</u> H, h= 2r

$\Rightarrow$ T_H = m/r(u^2 – 5gr) -------------(6)

$\Rightarrow$ T_L- T_H = m/r(u^2+ gr-(u^2 -5gr) = 6 mg

<u>Minimum Velocity of Projection at Lowest Point for looping the Loop</u>-

The body will cross highest Point H without any slackening of string if T_H is Positive, I,e

$T_H \geq 0 \Rightarrow \frac{m}{r}(u^2 - 5gr) \geq 0$

$\Rightarrow u^2 \geq 5gr$. Or $u \geq \sqrt{5gr}$

$\sqrt{5gr}$ is the minimum velocity which the body must possess at the bottom of the circle to looping the loop.

<u>Minimum Velocity at top of Loop-</u>

If v= Velocity of the body at highest Point H,

In case of no slackening of string,

$V^2 = u^2 - 2g.2r$

$\Rightarrow V^2 = 5gr - 4\,gr$ ($u = \sqrt{5gr}$)

$\Rightarrow V^2 = gr$

$\Rightarrow$ **V =** $\sqrt{gr}$

This is the minimum velocity at the highest point of the loop.

EXAMPLE-1

In a Circus, the diameter of the globe of death is 20m, From what minimum height must a cyclist start to go round the globe successfully?

Solution—

When the cyclist rolls down the incline;

Loss in Potential energy = Gain in Kinetic Energy.

$\Rightarrow mgh = ½\ mv^2$

$\Rightarrow V = \sqrt{2gh}$

The minimum velocity at the lowest point should be $\sqrt{5gr}$.

$\Rightarrow \sqrt{2gh} = \sqrt{5gr}$

$\Rightarrow 2gh = 5gr$

$\Rightarrow h = 5r/2 = 25m.$

EXAMPLE-2

A Body weighing 0.4 Kg is whirled in a vertical circle making 2 revolutions per second. If the radius of the circle is 1.2 m Find the tension in the string when the body is (1) at the bottom of the circle(2) at the top of the circle.

SOLUTION : m=0.4 kg,r = 1.2 m , v = 2 rps, ω = 2πv = 4π rad/second.

(1) <u>When the body is at the bottom</u> of the circle,

Let T_1 = Tension in the string

$\Rightarrow T_1 - mg = mr\omega^2$

$\Rightarrow T_1 = m(r\omega^2 + g) = 0.4(1.2 \times 16\pi^2 + 9.8)$

$\Rightarrow T_1 = 0.4(1.2 \times 16 \times 9.87 + 9.8) = 79.32$ N

(2) <u>When body at the tip of the Circle</u>—

Let T_2 = Tension in the string.

$T_2 + mg = mr\omega^2$

$\Rightarrow T_2 = m(r\omega^2 - g) = 0.4(1.2 \times 16\pi^2 - 9.8)$

$= 0.4(189.5 - 9.8)$

$\Rightarrow T_2 = 71.88$ N.

EXAMPLE =3

A weightless thread can bear tension up to 3.7 Kgwt. A stone of mass 500 g is tied to it and revolves in a circular path of radius 4 m in a vertical plane .if g= 10 m/s^2 , then what will be the maximum angular velocity of the stone?

Solution- T_{max} =37 N, m=0.5 kg, r =4m.

$\Rightarrow T_{max} = \frac{mv^2}{r} + mg$

$\Rightarrow \frac{mv^2}{r} = T_{max} - mg$

$= 37 - 0.5 \times 10$

$= 32.$

$\Rightarrow V^2 = \frac{32r}{m} = \frac{32 \times 4}{0.5} = 256$

$\Rightarrow V = 16$ m/s

$\Rightarrow \omega = \frac{V}{r} = 4$ rad/second.

PROBLEMS FOR PRACTICE

(1) An aeroplane flying in the skydives with a speed of 360 km/h in a vertical circle of radius 200 m, the weight of the pilot sitting in it is 75 Kg. Calculate the force with which the Pilot presses his seat when the aeroplane is (1) at the highest position (2) at the lowest position of the circle, Take g = $10 m/s^2$.

(2) A body weighing 0.5 kg tied to a string is projected with a velocity of 10 m/s. The body starts whirling in a vertical circle. If the radius of the circle is 0.8 m find the tension in the string when the body is (1)at the top of the circle and (2) at the bottom of the circle.

(3) A child revolves a stone of mass 0.5 kg tied to the end of a string of length 40 cm in a vertical circle. The speed of the stone at the lowest point of the circle is 3 m/s. Calculate the tension in the string at this point.

(4) A stone is tied to a weightless string and revolved in a vertical circle of radius 5m.(i)What should be the minimum speed of the stone at the highest point of the circle so that the string does not slack? (ii) What should be the speed of the stone at the lowest point in this situation? Take g =$9.8 m/s^2$

(5) The railway bridge over a canal in the form of an arc of a circle of radius 20m.What is the minimum speed with which a car can cross the bridge without leaving contact with the ground at the highest point? Take g=$9.8 m/s^2$.

What is Collision?

A Collision is said to occur between two bodies either if they physically collide against each other or if the path of one is affected by the force exerted by others.

TYPES OF COLLISION

(1) ELASTIC COLLISION-

(a) Both kinetic energy and momentum are conserved.

(b) Total Energy is conserved.

(c) Forces in this collision are Conservative.

(d) Mechanical Energy is not conserved.

(e) Example- Collision between glass balls, Collision between subatomic Particles.

(2) INELASTIC COLLISION-

(a) Momentum is conserved.

(b) The total energy is conserved but Kinetic energy is not conserved.

(c) Forces involved in the collision are Non-conservative.

(d) Mechanical Energy is converted into heat, light, sound, etc.

(e) Example- Collision between two vehicles, collision between ball and floor.

(3) Head-on Collision-

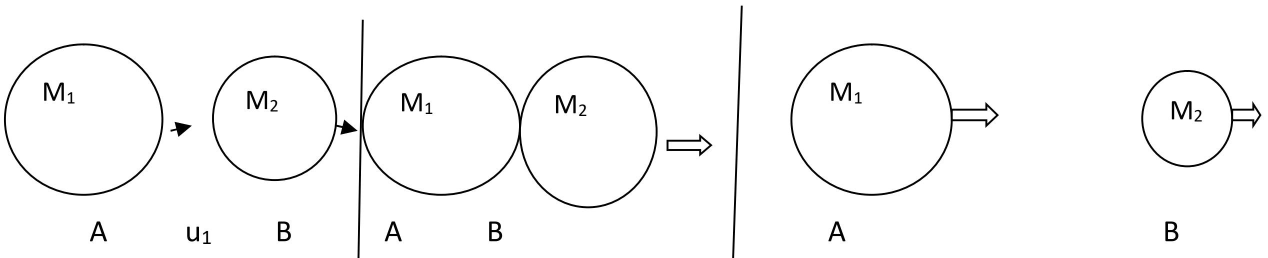
The collision where the colliding bodies move in a straight line before and after the collision is called a dimensional head-on Collision.

(4) Perfectly Inelastic Collision-

If two bodies stick together after the collision and move as a single body with a common velocity, then it is a perfectly inelastic collision.

Example-- Mud was thrown to the wall and sticking on it etc.

EXPRESSION OF VELOCITIES OF TWO BODIES IN ONE DIMENSIONAL ELASTIC COLLISION-



(Before Collision) (During collision) (After Collision)

Consider two bodies A and B of masses m_1 and m_2 moving along a straight line with velocities u_1 and u_2 respectively. Let $u_1 > u_2$.

Two bodies collide head-on and move in the same direction with velocities v_1 and v_2 respectively. As momentum is conserved,

$\Rightarrow m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$\Rightarrow m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2$

$\Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2)$ ----------------(1)

As kinetic energy is conserved,

$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

$\Rightarrow m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$

$\Rightarrow m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$

$\Rightarrow m_1(u_1+v_1)(u_1-v_1) = m_2(v_2+u_2)(v_2-u_2)$ -----------(2)

Dividing equation (2) by equation (1),

$u_1+v_1 = v_2+u_2$

$\Rightarrow u_1 - u_2 = v_2 - v_1$ --------------------(3)

Relative velocity of approach = Relative velocity of separation.

Now from equation (3) ,

$V_2 = u_1 - u_2 + v_1$

Putting v_2 in equation (1)

$m_1(u_1 - v_1) = m_2(u_1 - u_2 + v_1 - u_2) = m_2(u_1 + v_1 - 2u_2)$

$\Rightarrow m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1 - 2m_2u_2$

$\Rightarrow (m_1 + m_2)v_1 = (m_1 - m_2)u_1 + 2m_2u_2$

$v_1 = \left(\frac{m1 - m2}{m_1 + m_2}\right) u_1 + \frac{2m_2}{m_1 + m_2} u_2$ ------------(4)

Similarly by putting values of v_1 from equation (3) in equation (1)

$V_2 = \left(\frac{m1 - m2}{m_1 + m_2}\right) u_2 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1$ -----------(5)

CASE-1 (When two bodies of equal masses Collide)]

Let $m_1 = m_2 = m$ (say)

From equation (4)

$V_1 = \frac{2mu_2}{2m} = u_2$

From equation (5), $v_2 = u_1$.

When two bodies of equal masses suffer a one-dimensional elastic collision, their velocities are exchanged after the collision.

CASE-2(When a body collides against a stationary Body of Equal Mass)

Let $m_1 = m_2 = m$ and $u_2 = 0$

From equation (4) **$V_1 = 0$ and from equation (5) $V_2 = U_2$**

When two elastic bodies of equal mass collide with each other and the second body is at rest, then the first body comes to rest and the second body moves with an initial velocity of the first body.

CASE-3(**When a light body collides against a massive stationary body).**

If $m_2 >> m_1$ and $u_2 = 0$

Now **$V_1 = -U_2$ and $V_2 = 0$.**

If a light body collides with a heavy body at rest then the light body rebounds after the collision with an equal and opposite velocity while the heavy body remains at rest.

CASE -4(**When a heavy body collides against a light stationary body, $m_1 >> m_2$ and $u_2 = 0$)**

From equation 4 and 5 we get,

$V_1 = u_2$ and $v_2 = 2u_1$

When a heavy body collides light body at rest, the velocity of a heavy body is unchanged while the light body starts moving with twice the Velocity of a massive body.

EXAMPLE-(1)

In a nuclear reactor, a neutron of high speed must be slowed down so that it can have a high probability of interacting with Isotope $^{238}_{92}U$ and causing it in fission. Show that a neutron can lose most of its kinetic energy in an elastic collision with a light nuclei like deuterium or carbon which has a mass only a few times the neutron mass. The material making up the light nuclei usually heavy water (D_2O) or graphite, is called a moderator.

Solution- The initial kinetic energy of the neutron is

$K_{1i} = \frac{1}{2}m_1 v_{1i}^2$

While its final kinetic energy is

$K_{1f} = \frac{1}{2}m_1 v_{1f}^2 = \frac{1}{2}m_1\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 v_{1i}^2$

The fractional kinetic energy lost is

$F_i = \frac{k_{1f}}{k_{1i}} = \frac{m_1 - m_2}{m_1 + m_2}$

Example-2

Two ball bearings of mass m each moving in opposite directions with equal speed v collides head-on with each other. Predict the outcome of the Collision, assuming it to be perfectly elastic.

Solution-

Here $m_1=m_2=m$, $u_1 = v$, $u_2 = -v$

$v_1 = \left(\frac{m1 - m2}{m_1 + m_2}\right)u_1 + \frac{2m_2}{m_1 + m_2}u_2$

$= \left(\frac{m - m}{m + m}\right)v + \frac{2m}{m + m}(-v)$

$= 0 - v$

$= -v$

$V_2 = \left(\frac{m1 - m2}{m_1 + m_2}\right)u_2 + \left(\frac{2m_1}{m_1 + m_2}\right)u_1$

$= \left(\frac{m-m}{m+m}\right)(-v) + \left(\frac{2m}{m+m}\right)v$

= v.

Two balls bounce back with equal speeds after the collision.

Numericals for Practice—

(1) Two bodies of masses 5 kg and 3 kg moving in the same direction along the same straight line with velocities 5 m/s and 3 m/s respectively suffer a one-dimensional elastic collision. Find their velocities after the collision.

(2) A 10 kg ball and 20 kg ball approach each other with velocities 20m/s and 10 m/s respectively. What are their velocities after the collision if the collision is perfectly elastic.

(3) A railway carriage of mass 9000 kg moving with a speed of 36 km/h collides with a stationary carriage get coupled and move together. What is their common speed after collision? What type of collision is this?

(4) A vehicle of mass 30 quintals moving with a speed of 18 km/h collides with another vehicle of mass 90 quintals moving with a speed of 14.4 km/h in the opposite direction. What will be the velocity of each after the collision?

(5) A body of mass m strikes a stationary body of mass M and undergoes an elastic collision. After collision m has a speed one-third of its initial speed. What is the ratio M/m?

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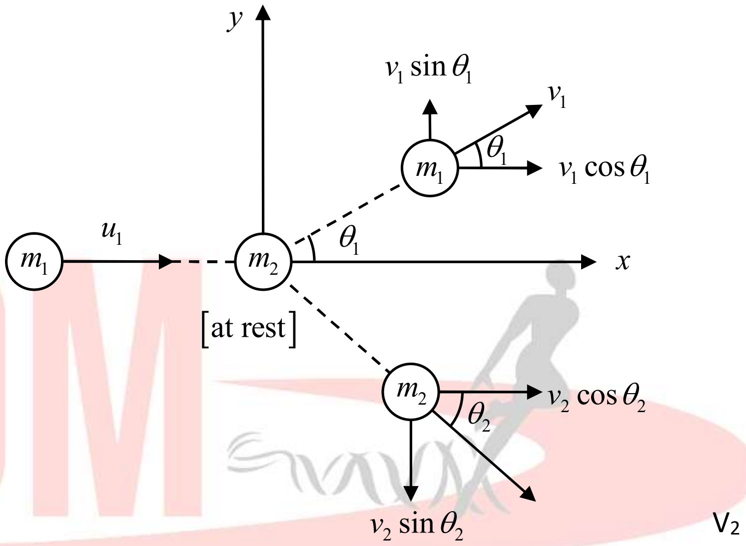
(ELASTIC COLLISION IN TWO DIMENSION)

Let a particle of mass m_1 moving along X-axis with velocity u_1 collides with another particle of mass m_2 at rest.

Let two particles move with velocities v_1 and v_2 making angle θ_1 and θ_2 with X-axis after the collision.



From the principle of conservation of momentum along X-axis,

$m_1u_1 = m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2$ -----------------(1)

In Y-axis, Similarly

$0 = m_1v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2$ ----------------(2).

As K.E is conserved,

$\frac{1}{2} m_1u_1^2 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2$ -------------(3)

Four unknown quantities v_1, v_2, θ_1, and θ_2 can not be calculated using 3 equations(1),(2),(3). By measuring one of the four unknowns experimentally, the values of the other three unknowns can be solved.

CASE- 1 (Glancing collision)

Here $\theta_1 = 0$ and $\theta_2 = 90$.

From equation (1) and (2), we get $u_1 = v_1$ and $v_2 = 0$

The incident practice does not lose any kinetic energy and is scattered undeflected.

CASE- 2 (Elastic collision for two identical particles)

Let $m_1 = m_2 = m$(say),

again $\frac{1}{2} mu^2 = \frac{1}{2} mv_1^2 + \frac{1}{2} mv_2^2$

$\Rightarrow u_1^2 = v_1^2 + v_2^2$

Again $\overrightarrow{mu_1} = \overrightarrow{mv_1} + \overrightarrow{mv_2}$

$\Rightarrow \vec{u}_1 = \vec{v_1} + \vec{v_2}$

$\Rightarrow \overrightarrow{u1}.\overrightarrow{u1} = (\overrightarrow{v1} + \overrightarrow{v2}).(\overrightarrow{v1} + \overrightarrow{v2})$

$\Rightarrow u_1^2 = v_1^2 + v_2^2 + 2\,\overrightarrow{v1}.\overrightarrow{v2}$

$\Rightarrow 2\vec{v_1}.\vec{v_2} = 0$ (since $u_1^2 = v_1^2 + v_2^2$)

The angle between v_1 and v_2 is 90° .

Coefficient of restitution

The ratio of the magnitude of the relative velocity of separation after a collision to the magnitude of the relative velocity of approach before the collision.

$e = \left|\frac{v1 - v2}{u1 - u2}\right|$

Case -1 if e=1 ⇒ perfectly elastic collision

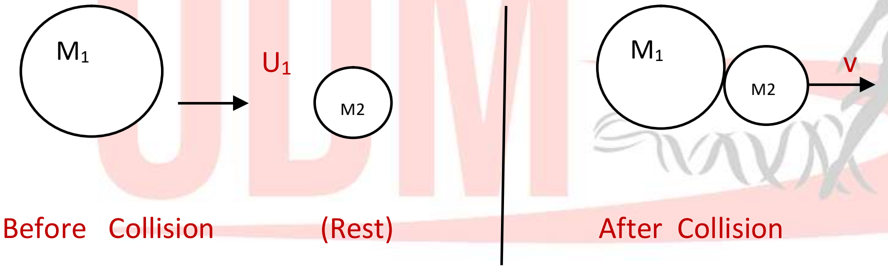
Case-2 for 0<e<1 ⇒ in elastic collision

Case-3 if e=0⇒perfectly inelastic collision

Perfectly inelastic Collision In One Dimension

When two colliding bodies stick together and move as a single body with a common velocity after the collision, the collision is perfectly inelastic.

A body of mass m_1 moves with velocity u_1 collides head-on with another body of mass m_2 at rest. Two bodies move together with a common velocity v after the collision.



As linear momentum is conserved, so $m_1u_1+m_2x0=(m_1+m_2)v$

$\Rightarrow v = (\frac{m1}{m1+m2})u_1$

The loss in kinetic energy on collision is

$\Delta k=k_i - k_f = \frac{1}{2} m_1 u_1^2 - \frac{1}{2}(m_1+m_2)v^2$

$= \frac{1}{2} m_1u_1^2 - \frac{1}{2}(m1+m2) [\frac{m1}{m1+m2} u1]^2$

$= \frac{1}{2} m_1u_1^2 - \frac{1}{2}\frac{m1^2}{m1+m2} u_1^2$

$= \frac{1}{2} m_1u_1^2 [1 - \frac{m1}{m1+m2}]$

$= \frac{1}{2} \frac{m1m2}{m1+m2} u_1^2$

KE lost in form of heat and sound

$\Rightarrow \frac{Kf}{Ki} = \frac{\frac{1}{2}(m1+m2)v^2}{\frac{1}{2}\,m_1 {u_1}^2} = \frac{m1+m2}{m1}\,\frac{v^2}{u_1^2} = \frac{m1+m2}{m1}\,.[\frac{m1}{m_1}]^2$

$\Rightarrow \frac{Kf}{Ki} = \frac{m1}{m1+m2} < 1$

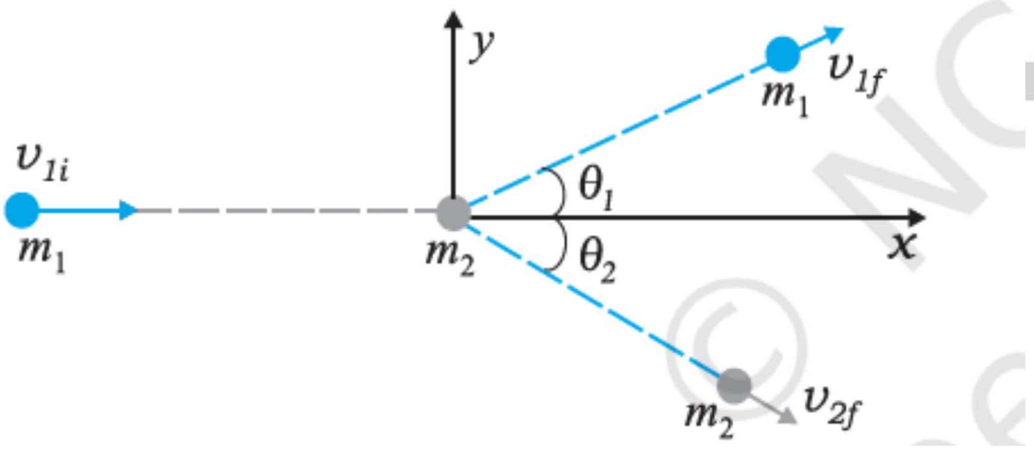
$\Rightarrow$ kinetic energy after the collision is less than the kinetic energy before collision.

Case-1 if $m_2 >> m_1 \Rightarrow \frac{Kf}{Ki} = 0 \Rightarrow K_f \approx 0$

When a light moving body collides against the heavy body at rest and sticks to it, practically all of its kinetic energy is lost.

EXAMPLE -1. (NCERT)

Consider the collision depicted in Fig to be between two billiard balls with equal masses $m_1=m_2$, The first ball is called cue and the second ball is called target. The billiard player wants to sink the target ball in a corner pocket which is angle θ_2= 37 degrees, Assume the collision to be elastic and the friction and rotational motion are not important. Obtain θ_1.



Solution-- -- From momentum conservation, since the masses are equal,

$\overrightarrow{v_{1i}} = \overrightarrow{v_{1f}} + \overrightarrow{v_{2f}}$

Or $v_{1i}^2 = (\overrightarrow{v_{1f}} + \overrightarrow{v_{2f}}).(\overrightarrow{v_{1f}} + \overrightarrow{v_{2f}})$

$= v_{1f}^2 + v_{2f}^2 + 2\,\overrightarrow{v_{1f}}.\overrightarrow{v_{2f}})$

$= v_{1f}^2 + v_{2f}^2 + 2\,v_{1f}\,v_{2f}\,\cos(\Theta_1 + 37^0)$ ------------(1)

Since the collision is elastic and $m_1 = m_2$ it follows from conservation of kinetic energy that

$v_{1i}^2 = v_{1f}^2 + v_{2f}^2$ ------------(2)

Comparing equation 1 and 2 ,we get

$\text{Cos}\,(\Theta_1 + 37^0) = 0$

Or $\Theta_1 + 37^0 = 90^0$

Or $\Theta_1 = 53^0$.

Conclusion- When two equal masses undergo glancing elastic collision with one of them at rest, after the collision, they will move at the right angle to each other.

EXAMPLE -2

What percentage of the kinetic energy of a moving particle is transferred to a stationary particle, when the moving particle strikes with a stationary particle of mass (i) 9 times in mass(ii) equal in mass (iii) 1/19 th of its mass.

Problems for practice

(1) Two particles of masses 0.5kg and 0.25 kg moving with velocities 4 m/s and 3 m/s collide head-on in a perfectly inelastic Collision. Find (1) the velocity of the composite particle after the collision and (ii) the kinetic energy lost in the collision.

(2) What percentage of the K.E of a moving particle is transferred to a stationary particle when it strikes the stationary particle of four times of its mass.

(3) What percentage of the kinetic energy of a moving particle is transferred to a stationary particle when moving particle strikes with a stationary particle of mass (i) 19 times it's a mass (ii) equal in mass (iii)1/9 th of its mass.

(4) Show that when a moving body collides with the stationary body of mass m or 1/m times its mass, then the moving body transfers $\frac{4m}{(1+m)^2}$ part of its kinetic energy to the stationary body.

(5) A ball is dropped from a height h on to a floor. If the coefficient of restitution is e, calculate the height to which the ball first rebounds.