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EXPONENTS AND POWERS

INTRODUCTION

The multiplication statement $3 \times 5 = 15$ has two parts : the two numbers that are being multiplied, and the answer. The answer (15) is called the product, and the numbers that are being multiplied (3 and 5) are called **factors.** In the expression $3 \times 3 \times 3 \times 3 \times 3$ the number 3 is used as a factor five times. We call 3 a repeated factor. To express a repeated factor, we can use an **exponent**.

An exponent is used to indicate repeated multiplication. It tells how many times the base is used as a factor.

LAWS OF EXPONENT FOR REAL NUMBERS

Positive Integral power : For any real number a and a positive integer 'n' we define a^n as

$a^n = a \times a \times a \times \ldots\ldots\ldots\ldots \times a$ (n times)

a^n is called the n^{th} power of a. The real number 'a' is called the base and 'n' is called the exponent of the n^{th} power of a. Eg. : $2^3 = 2 \times 2 \times 2 = 8$

For any non-zero real number 'a' we define $a^0 = 1$. Thus, $3^0 = 1, 5^0 = 1, \left(\frac{3}{4}\right)^0 = 1$ and so on.

Negative Integral power : For any non-zero real number 'a' and a positive integer 'n' we define $a^{-n} = \frac{1}{a^n}$

Thus we have defined a^n for all integral values of n, positive, zero or negative. a^n is called the n^{th} power of a.

Law of integral exponents :

First law : If 'a' is any real number and m, n are positive integers, then $a^m \times a^n = a^{m+n}$.

Second law : If 'a' is non-zero real number and m, n are positive integers, then $\frac{a^m}{a^n} = a^{m-n}$.

Case I : When $m > n$ $\quad a^m \div a^n = a^{m-n}$.

Case II : When $m = n$ $\quad a^m \div a^n = a^0 = 1$

Case III : When $m < n$ $\quad a^m \div a^n = a^{-(n-m)} = \frac{1}{a^{n-m}}$

Third law : If 'a' is any real number and m, n are positive integers, then $(a^m)^n = a^{mn} = (a^n)^m$

Fourth law : If a, b are real numbers and m, n are positive integers, then

(i) $(ab)^n = a^n b^n$ $\qquad$ (ii) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$

Rational exponents of a real number :

If 'a' is positive real number and 'n' is a positive integer, then the principal n^{th} root of a is the unique positive real number. For $x^n = a$, the principal n^{th} root of a positive real number a is denote by $a^{1/n}$ or $\sqrt[a]{n}$.

Rational power (Exponents) :

For any positive real number 'a' and a rational number $\frac{p}{q}$, where $q > 0$, we define $a^{p/q} = (a^p)^{1/q}$

i.e., $a^{p/q}$ is the principal q^{th} root of a^p.

Laws of rational exponents :

The following laws hold the rational exponents

(i) $a^m a^n = a^{m+n}$ **(ii)** $a^m \div a^n = a^{m-n}$ **(iii)** $(a^m)^n = a^{mn}$

(iv) $a^{-n} = \frac{1}{a^n}$ **(v)** $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$ **i.e.,** $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

(vi) $(ab)^m = a^m b^m$ **(vii)** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ **(viii)** $a^{bn} = a^{b \times b \times b ....n \text{ times}}$

Where a, b, are positive real numbers and m, n are rational numbers.

Example 1 :

Express each of the following rational numbers in exponential form : (i) $\frac{27}{64}$ (ii) $\frac{-32}{243}$

Sol. (i) We can write, $27 = 3 \times 3 \times 3 = 3^3$ and $64 = 4 \times 4 \times 4 = 4^3$ $\therefore \frac{27}{64} = \frac{3^3}{4^3} = \left(\frac{3}{4}\right)^3$

(ii) We can write, $-32 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) = (-2)^5$ and $243 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$

$\therefore \frac{-32}{243} = \frac{(-2)^5}{3^5} = \left(\frac{-2}{3}\right)^5$

Example 2 :

Simplify : $13^{\frac{1}{5}}.17^{\frac{1}{5}}$

Sol. $13^{\frac{1}{5}}.17^{\frac{1}{5}} = (13 \times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$

Example 3 :

Simplify : $\pi^{3/4}.\pi^{1/2}$

Sol. $\pi^{3/4}.\pi^{1/2} = \pi^{\frac{3}{4}+\frac{1}{2}} = \pi^{\frac{5}{4}}$

Example 4 :

Simplify : $\left(\frac{2^a}{2^b}\right)^{a+b} \left(\frac{2^b}{2^c}\right)^{b+c} \left(\frac{2^c}{2^a}\right)^{c+a}$

Sol. $\left(\frac{2^a}{2^b}\right)^{a+b} \left(\frac{2^b}{2^c}\right)^{b+c} \left(\frac{2^c}{2^a}\right)^{c+a} = 2^{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)} = 2^0 = 1$

Example 5 :

Simplify: $\left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{3}\right)^{-2} + \left(\frac{1}{4}\right)^{-2}$

Sol. We have, $\left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{3}\right)^{-2} + \left(\frac{1}{4}\right)^{-2} = \left(\frac{2}{1}\right)^{2} + \left(\frac{3}{1}\right)^{2} + \left(\frac{4}{1}\right)^{2} = (2^2 + 3^2 + 4^2) = (4 + 9 + 16) = 29$

Example 6 :

Find the reciprocal or the multiplicative inverse of the following :

(i) $\frac{-7}{8}$ (ii) $\left(\frac{3}{4}\right)^{3}$ (iii) $\left(\frac{-2}{3}\right)^{2}$ (iv) $\left(\frac{1}{3}\right)^{-3}$

Sol. (i) The reciprocal of $\frac{-7}{8}$ is $\left(\frac{-7}{8}\right)^{-1} = \frac{-8}{7}$ (ii) The reciprocal of $\left(\frac{3}{4}\right)^{3}$ is $\left(\frac{4}{3}\right)^{3} = \frac{64}{27}$

(iii) The reciprocal of $\left(\frac{-2}{3}\right)^{2}$ is $\left(\frac{-2}{3}\right)^{-2} = \left(\frac{-3}{2}\right)^{2} = \frac{9}{4}$ (iv) The reciprocal of $\left(\frac{1}{3}\right)^{-3}$ is $\left(\frac{3}{1}\right)^{3} = 3^3 = 27$

SELF CHECK-1

Q.1 Evaluate :

(i) $\left(\frac{-2}{3}\right)^{5}$ (ii) $\left(\frac{1}{6}\right)^{-2}$ (iii) $\left(\frac{2}{3}\right)^{0}$

Q.2 Evaluate : (i) 5^{-3} (ii) $\left(\frac{1}{3}\right)^{-4}$ (iii) $\left(\frac{3}{2}\right)^{-3} \times \left(\frac{3}{2}\right)^{-2}$ (iv) $\left(\frac{8}{5}\right)^{-3} \times \left(\frac{8}{5}\right)^{2}$

Q.3 Simplify : $\left(\frac{2}{3}\right)^{-3} \div \left(\frac{4}{3}\right)^{-2}$

Q.4 Evaluate : (i) $\left(\frac{1}{8}\right)^{3} \times \left(\frac{16}{15}\right)^{3}$ (ii) $(-10)^4 \times \left(\frac{2}{5}\right)^{4}$

Q.5 Express $\frac{27}{64}$ and $\frac{-8}{27}$ as the powers of rational numbers.

Q.6 Simplify : (i) $\left(\frac{-3}{5}\right)^{3} \times \left(\frac{-3}{5}\right)^{2}$ (ii) $\left(\frac{2}{3}\right)^{4} \times \frac{2}{3}$

Q.7 Evaluate : $\left(\frac{-1}{4}\right)^{-3} \times \left(\frac{-1}{4}\right)^{-2}$

Q.8 Find the reciprocal of the following : (i) 4^3 (ii) $(-2)^2$

Q.9 Express each of the following rational numbers in exponential form : (i) $\frac{-1}{343}$ (ii) $\frac{-27}{512}$

Q.10 Simplify the express with positive exponents : (i) $\left(\frac{2}{5}\right)^{-5} \div \left(\frac{2}{5}\right)^{-2}$ (ii) $\left(\frac{-7}{9}\right)^{-2} \div \left(\frac{-7}{9}\right)^{4}$ (iii) $\left[\left(\frac{-5}{3}\right)^{-2}\right]^{-3}$

USE OF EXPONENTS TO EXPRESS SMALL NUMBERS IN STANDARD FORM

Observe the following facts.

1. The distance from the Earth to the Sun is 149,600,000,000 m.

2. The speed of light is 300,000,000 m/sec.

3. The average diameter of a Red Blood Cell is 0.000007 m.

4. The distance of moon from the Earth is 384, 467, 000 m (approx).

5. The size of a plant cell is 0.00001275 m.

6. Average radius of the Sun is 695000 km.

7. The height of Mount Everest is 8848 m.

You will find some numbers are very large and some are very small numbers. To express them in standard form we may write, $149,600,000,000 \text{ m} = 1.496 \times 10^{11}$ m. ; $300,000,000$ m/sec. $= 3 \times 10^{8}$ m/sec

$0.000007 \text{ m.} = 7 \times 10^{-6}$ m ; $384, 467, 000 \text{ m} = 3.84467 \times 10^{8}$ m

$0.00001275 \text{ m} = 1.275 \times 10^{-5}$ m ; $695000 \text{ km} = 6.95 \times 10^{5}$ km

$8848 \text{ m} = 8.848 \times 10^{3}$ m

Example 7 :

Write the following numbers in standard form :

(i) 0.0000021 (ii) 0.0000000003 (iii) 200000000

Sol. (i) $0.0000021 = 2.1 \times 10^{-6}$ (ii) $0.0000000003 = 3.0 \times 10^{-10}$ (iii) $200000000 = 2.0 \times 10^{8}$

COMPARING VERY LARGE AND VERY SMALL NUMBERS

Exponents are very useful in comparing various numbers.

For example, the diameter of the Sun is 1.4×10^{9} m and the diameter of the Earth is 1.2756×10^{7} m.

Suppose we want to compare the diameter of the Earth, with the diameter of the Sun.

Diameter of the Sun = 1.4×10^{9} m

Diameter of the earth = 1.2756×10^{7} m

Therefore, $\frac{1.4 \times 10^{9}}{1.2756 \times 10^{7}} = \frac{1.4 \times 10^{9-7}}{1.2756} = \frac{1.4 \times 100}{1.2756}$ which is approximately 100

So, the diameter of the Sun is about 100 times the diameter of the earth.

Let us compare the size of a Red Blood cell which is 0.000007 m to that of a plant cell which is 0.00001275 m.

Size of Red Blood cell = 0.000007 m = 7×10^{-6} m

Size of plant cell = 0.00001275 = 1.275×10^{-5} m

Therefore, $\frac{7 \times 10^{-6}}{1.275 \times 10^{-5}} = \frac{7 \times 10^{-6+5}}{1.275} = \frac{7 \times 10^{-1}}{1.275} = \frac{0.7}{1.275} = \frac{0.7}{1.3} = \frac{1}{2}$ (approx.)

So a red blood cell is half of plant cell in size.

Example 8 :

Mass of earth is 5.97×10^{24} kg and mass of moon is 7.35×10^{22} kg. What is the total mass?

Sol. Total mass = 5.97×10^{24} kg + 7.35×10^{22} kg.

$= 5.97 \times 100 \times 10^{22} + 7.35 \times 10^{22}$

$= 597 \times 10^{22} + 7.35 \times 10^{22} = (597 + 7.35) \times 10^{22} = 604.35 \times 10^{22}$ kg.

SELF CHECK-2

Q.1 Write the number in standard form :

(i) 0.00000952 (ii) 93200

Q.2 Write the charge of electron is 0.00000000000000000016 coulomb in standard form.

Q.3 By what number should $\left(\frac{-3}{2}\right)^{-3}$ be divided so that the quotient is $\left(\frac{9}{4}\right)^{-2}$?

USEFUL TIPS

* **Important results**

(i) $a^m a^n = a^{m+n}$ **(ii)** $a^m \div a^n = a^{m-n}$ **(iii)** $(a^m)^n = a^{mn}$

(iv) $a^{-n} = \frac{1}{a^n}$ **(v)** $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$ **i.e.,** $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

(vi) $(ab)^m = a^m b^m$ **(vii)** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ **(viii)** $a^{b^n} = a^{b \times b \times b ....n \text{ times}}$

* **Unit digit in exponential expressions :**

(a) When there is 2 in unit's place of any number.

Since in 2^1 unit's place of any number.

Since in 2^1 unit digit is 2

in 2^2 unit digit is 4

in 2^3 unit digit is 8

in 2^4 unit digit is 6

after than the unit digits repeats.

(b) When there is 3 in unit's place of any number

Since in 3^1 unit digit is 3

in 3^2 unit digit is 9

in 3^3 unit digit is 7

in 3^4 unit digit is 1

after that the unit digits repeats.

(c) When there is 4 in unit's place of any number

Since in 4^1 unit digit is 4

in 4^2 unit digit is 6

after that the unit digits repeats.

(d) When there is 5 in unit's place of any number

Since in 5^1 unit digit is 5

in 5^2 unit digit is 5 & so on

(e) When there is 6 in unit's place of any number
Since in 6^1 unit digit is 6
in 6^2 unit digit is 6 & so on

(f) When there is 7 in unit's place of any number
Since in 7^1 unit digit is 7
in 7^2 unit digit is 9
in 7^3 unit digit is 3
in 7^4 unit digit is 1
after that the unit digits repeats.

(g) When there is 8 in unit's place of any number
Since in 8^1 unit digit is 3
in 8^2 unit digit is 4
in 8^3 unit digit is 4
in 8^4 unit digit is 6
after that the unit's digits repeats after a group of 4.

(h) When there is 9 in unit's place of any number
Since in 9^1 unit's digit is 9
in 9^2 unit's digit is 1
after that the unit's digits repeats after a group of 2.

(i) When there is zero in unit's place of any number
There will always be zero in unit's place.
after that the unit's digits repeats after a group of 4.

Example :

(a) In $(23)^{13}$ unit digit is 3 **(b)** In $(34)^{14}$ unit digit is 6 **(c)** In $(97)^{99}$ unit digit is 3