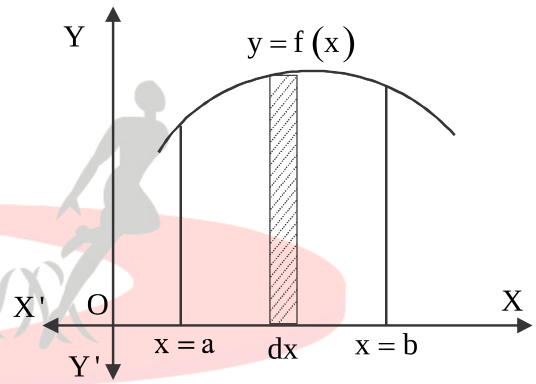
Chapter- 8

Application of Integrals

Introduction:

Integral has a large number of applications in science and engineering. Here we will study a specific application of integral to find the area under simple curves area between lines and area of circles, parabolas, and ellipse. In geometry, we have learned to formulate to calculate areas of various geometrical figures such as triangles, rectangles, circles, etc. But these formulae are inadequate to find the area enclosed by many curves. The concept of the definite integral can be conveniently used to find the area enclosed by curves.



The area under Simple curves:-

Area of the region bounded by x-axis:-

Let $f(x)$be a continuous function defined on$[a, b]$.

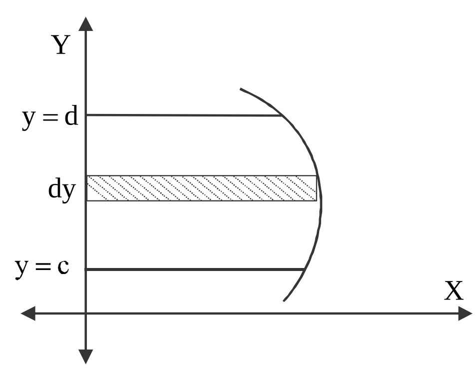
Then the area bounded by the curve $y = f(x)$,

the x-axis and the ordinates $x = a$ and $x = b$ are given by.

Area = $\int_a^b y dx = \int_a^b f(x) dx$

Example:- Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis

Area of Region bounded by Y-axis



The area bounded by the curve $x = f(y)$,

the y-axis and the lines $y = c$ and $y = d$ are given by.

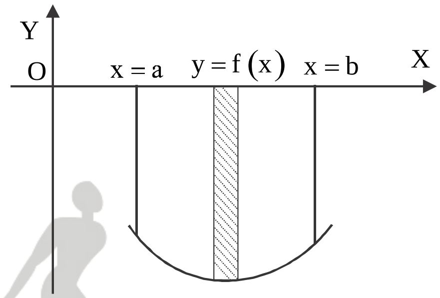
Area = $\int_c^d dA = \int_c^d x dy = \int_c^d f(y) dy$

Note:- Area of any bounded region cannot be negative

Example:- Find the area of the region by the curve $x = y^2$ and the lines $y = 1, y = 4$ and the y-axis.

Area of the region when the curve is below the x-axis:-

If the curve $y = f(x)$ lies below the x-axis then the area bounded by the curve $y = f(x)$, x-axis, and the lines $x = a$ and $x = b$ come out to be negative. But the only numerical value of the area is taken into consideration. Thus if the area is negative then we take its absolute value.



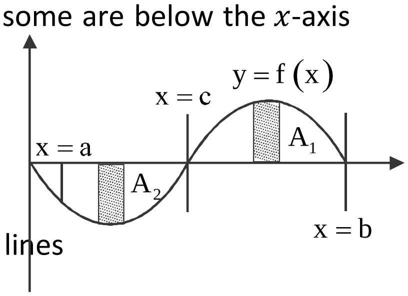
i.e $\left|\int_a^b f(x)\,dx\right|$

Area = $\left|\int_a^b y dx\right| = \left|\int_a^b f(x) dx\right|$

Example:- Find the area of the region bounded by the line $2x - y = 2$ and lines $x = \frac{1}{4}$ and $x = \frac{3}{4}$ and x-axis.

Area of the region when the curve is above and below the x –axis:-

Generally, some portions of the curve may be above the x-axis and some are below the x-axis which is shown in the figure given below.



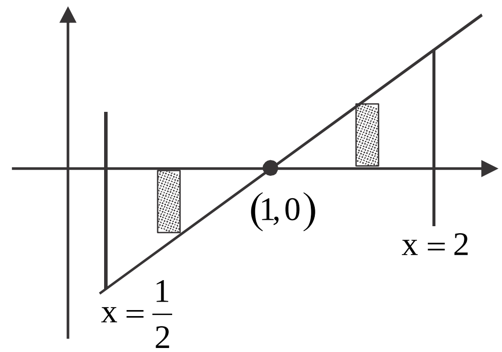
Here, $A_1 > 0$ and $A_2 < 0$.

Therefore the area A bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ is given by

$$A = A_1 + |A_2| = \int_c^b f(x)dx + \left|\int_a^c f(x)dx\right|$$

Example:- The area of the region bounded by the line $3x - y = 3$ and the lines $x = \frac{1}{2}$ and $x = 2$

Solution:-



Area = $\left|\int_{\frac{1}{2}}^{1} y\,dx\right| + \int_{1}^{2} y\,dx$

$$= \left|\int_{\frac{1}{2}}^{1} (3x-3)dx\right| + \int_{1}^{2} (3x-3)dx$$

$= \frac{15}{8} sq$ units.

Questions:-

(1) Find the area of the region bounded by the parabola $y^2 = 4ax$, its axis, and two ordinates $x = 4$ and $x = 9$ in the first quadrant.

(2) Using integration find the area of the region bounded by the line $2x + y = 8$, the y-axis, and the lines $y = 2$ and $y = 4$

(3) Find the area of the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in the fourth quadrant.

(4) Find the area bounded by the line $y = x$, the x-axis, and the lines $x = -1$ and $x = 2$.

Area of Symmetric Region:-

Sometimes the bounded region for which we have to calculate area is symmetrical about x-axis or y-axis or both x-axis and y-axis or origin. While determining the area of the symmetrical region, we have to check the symmetry of a curve is as given below.

(a) If the equation of the curve contains only

(i) Even powers of x, then it is symmetrical about the y-axis

(ii) Even powers of y, then it is symmetrical about the x-axis

(iii) The equation of the curve remains unchanged when x and y are replaced by $-x$ and $-y$ respectively then the curve is symmetrical in opposite quadrants

Area of the region bounded by a curve and a line:-

To find the area of the region bounded by a line and a circle, a line, and a parabola a line and an ellipse we use the following steps.

Step -I:- We draw the rough sketch of given curves and identify the region for which we have to find the area.

Step – II:- Find the point of intersection of the line and curve.

Step – III:- (i) If the region is symmetrical then draw a vertical (or horizontal) strip and take suitable limits.

(ii) If the region is not symmetrical then draw a vertical strip (or horizontal strip) in the required area and take suitable limits.

(iii) To find the number of the vertical (horizontal) strip we generally draw perpendicular lines from the intersection points of the line and curve to the x −axis or y-axis.

Step – IV:- If these perpendicular lines divide the region into two (or more) parts then take two (or more) vertical (or horizontal) strips and find the area by using suitable formula.

For two vertical strips.

Area = $\int_{x=a}^{c} y_1 dx + \int_{x=c}^{b} y_2 dx$, where y_1 & y_2 represents the height of vertical stripes.

For two horizontal strips

Area = $\int_{y=a}^{c} x_1 dy + \int_{y=c}^{b} x_2 dy$, where x_1 and x_2 represent the lengths of the horizontal strips.

Examples:-

01. Sketch the region $\{(x,0): y = \sqrt{4-x^2}\}$ and x-axis. Find the area of the region using integration

02. Using integration find the area endorsed by the circles $x^2 + y^2 = a^2$

03. Find the area of the circle $x^2 + y^2 - 2x = 0$

04. Find the area of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

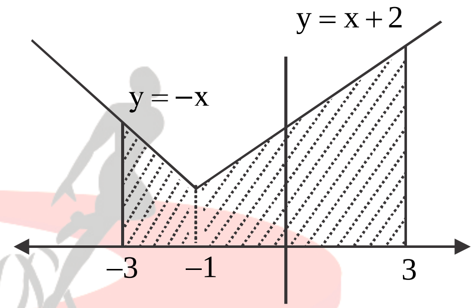
05. Find the area bounded by $y=4x^2, x=0, y=1$, and $y=4$.

06. Find the area of the parabola $y^2=4ax$ bounded by its latus rectum.

07. Find the area of the portion of the parabola $y^2=4x$ bounded by the double ordinate through $(3,0)$.

08. Draw a rough sketch of the given curve $y=1+|x+1|, x=-3, x=3$ and $y=0$ and find the area of the region bounded by them using integration.

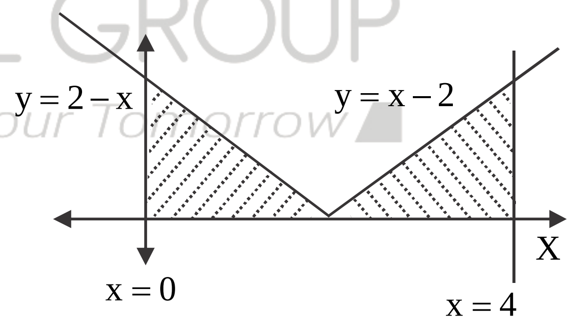
Solution:-



$$y=1+|x+1|=\begin{cases}2+x, & x\geq -1\\ -x, & x<-1\end{cases}$$

Required area = $\int_{-3}^{-1}-x\,dx+\int_{-1}^{3}(x+2)dx=16$ sq.units

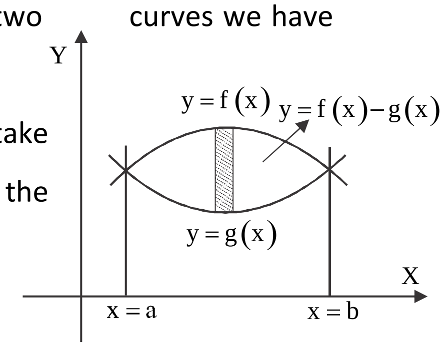
09. Draw a rough sketch of the curve $y=|x-2|$. Find the area under the curve and the line $x=0$ and $x=4$



Area = $\int_0^2(2-x)dx+\int_2^4(x-2)dx=4\ sq.units$

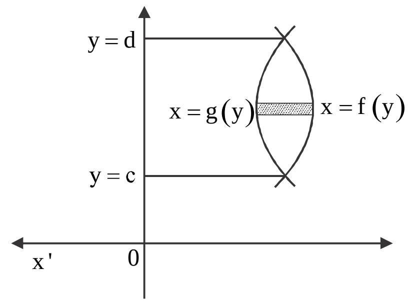
The area between two curves:-

Suppose two curves are given by $y=f(x)$ and $y=g(x)$ and their points of intersection are given by $x=a$ and $x=b$, then for finding the area between these two curves we have two conditions.



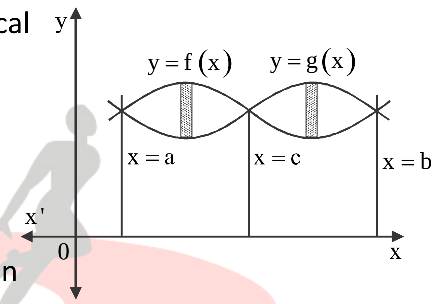
(a) When $\boldsymbol{f(x)\geq g(x)}$ in $[\boldsymbol{a},\boldsymbol{b}]$. Suppose $f(x)\geq g(x)$ in $[a,b]$ then take a vertical strip whose width is dx and length is $f(x)-g(x)$. Then the area $\quad A=\int_a^b[f(x)-g(x)]\,dx$

Note:- suppose two curves $x = f(y)$ and $x = g(y)$ are such that $g(y) \geq f(y)$.



Area = $\int_c^d (f(y) - g(y))dy$

(b) When $f(x) \geq g(x)$ in $[a, c]$ and $0 \leq f(x) \leq g(x)$ in $[c, b]$ where a < c < b. Suppose $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$ then we take two vertical strips, one strip in $[a, c]$ and the second strip in $[c, b]$.



Area = $\int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$

Working Rule:-

Suppose two curves $y = f(x)$ and $y = g(x)$ are given to us, then for finding the area of the region between two curves we use the following steps

(a) Firstly, draw the rough sketches of both given curves and identify the region bounded by them

(b) Find the point of intersection of both curves

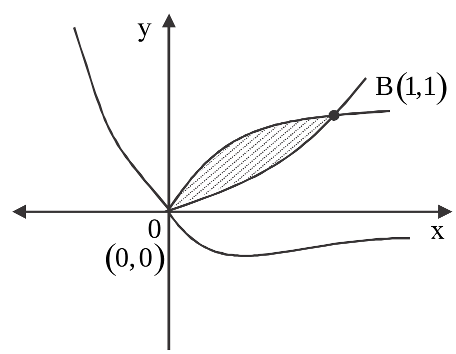
(c) Take the limit for the bounded region and draw the strips (one or two)

(d) Now find the area by using a suitable formula.

Examples:-

(a) Find the area of the region bounded by two parabolas $y = x^2$ and $y^2 = x$

Solution:-



We have two parabolas $y = x^2$ and $y^2 = x$

Point of intersection $y = x^2, y^2 = x$

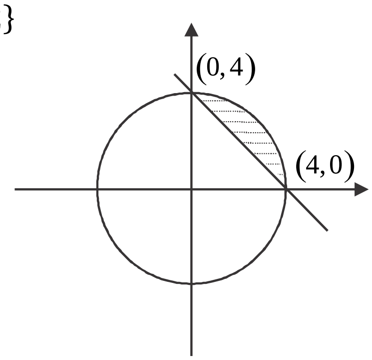
$$\therefore y^2 = x^4$$

$\Rightarrow x = x^4 \Rightarrow x^4 - x = 0$

$\Rightarrow x = 0, x = 1 \therefore y = 0, y = 1$

Required area = $\int_0^1 (\sqrt{x} - x^2)\, dx = \frac{1}{3}$ sq units

(b) Find the area of the region $\{(x,y): x^2+y^2 \le 4, x+y \ge 2\}$



Solution:-

Curves $x^2+y^2=4, x+y=2$

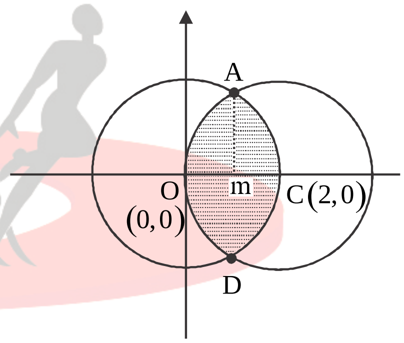
Area = $\int_0^4 \left(\sqrt{4x^2-(2-x)}\right) dx$

$= \pi - 2$ sq units

(c) Find the area of the region enclosed between the two circles $x^2+y^2=4$ and $(x-2)^2+y^2=4$

Solution:-

We have two circles $x^2+y^2=4$ and $(x-2)^2+y^2=4$



Point of intersection $x=1, y=\pm\sqrt{3}, (1,\sqrt{3}), (1,-\sqrt{3})$

Required area = area of region OACDO

= 2 × area of region OACO

= 2[Area of region OAMO + Area of region AMCA]

$= 2\left(\int_0^1 \sqrt{4-(x-2)^2}\right) dx + \int_1^2 \sqrt{4-x^2} dx$

$= \frac{8\pi}{3} - 2\sqrt{3}$ sq units

(d) Using integration finds the area of the region bounded by a parabola $y^2=4x$ and circle $4x^2+4y^2=9$.

OR

Find the area of the region $\{(x,y): y^2 \le 4x, 4x^2+4y^2 \le 9\}$ using integration

(e) Using integration find the area of the region bounded by the lines $2x+y=4, 3x-2y=6$ and $x-3y+5=0$