Chapter- 9

Differential Equations

Definition, Order, and Degree

An equation involving differentials of the variables or differential coefficients of the independent variable is called a differential equation.

Example:

i. $y^2dx + x\,dy = sinx$

ii. $\frac{dy}{dx} + y\,cosx = x$

iii. $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 - 5y = 0$

iv. $\frac{d^3y}{dx^3} = e^{\frac{dy}{dx}}$

v. $\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt{1 + y\left(\frac{dy}{dx}\right)^2}$

Order of a Differential Equation:

The order of the highest order derivative of the dependent variable w. r. t. the independent variable involved in the differential equation is called the order of the differential equation.

$e.g., \frac{dy}{dx} + y = c\,,\ \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ Involve derivatives whose highest orders are 1 and 2 respectively.

Degree of a Differential Equation:

When a differential equation is a polynomial equation in derivatives, the highest power (positive integral index) of the highest order derivative is known as the degree of the differential equation.

$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + y = c$, the highest derivative is $\frac{dy}{dx}$, its positive integral power is 2. So its degree is 2.

Example 1:

Find the order and degree of the following differential equations.

i. $\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^4 + 4\left(\frac{dy}{dx}\right)^7 + 3y = 0$

ii. $\frac{dy}{dx} - \frac{x}{\frac{dy}{dx}} + y^3 = 0.$

iii. $\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt{5y + \left(\frac{d^2y}{dx^2}\right)^6}$

iv. $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$

v. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

vi. $\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^2 = 3\log\left(\frac{d^2y}{dx^2}\right)$

Example 2:

Choose the correct answer from the given options.

1. The degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$

   a) 4 b) $\frac{3}{2}$ c) 2 d) not defined

2. The order and degree of the differential equation $y = x\frac{dy}{dx} + \frac{2}{\frac{dy}{dx}}$ are

   a) 1, 3 b) 1, 2 c) 2, 1 d) 1, 1

3. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$ is

   a) 1 b) 2 c) 3 d) not defined

4. Degree of the differential equation $\left(\frac{d^3y}{dx^2}\right)^{\frac{2}{3}} = x$ is

   a) 1 b) 2 c) 3 d) doesn't exist

5. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{4}} + x^{\frac{1}{5}} = 0$ respectively, are

   a) 2 and not defined b) 2 and 2 c) 2 and 3 d) 3 and 3

Example 3:

i. Write the sum of the order and degree of the differential equation $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0.$

ii. Write the sum of the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0.$

Example 4:

i. Write the degree of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$

ii. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0.$

iii. What is the degree of the following differential equation? $5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = logx$

General and particular solution of a differential equation

An equation containing the dependent variable and independent variable and free from derivatives, which satisfies the differential equation is called the solution or primitive of the differential equation.

For example, $y = e^x$ is a solution of the differential equation $\frac{dy}{dx} - y = 0$.

Also, $y = 3e^x$ is a solution of the differential equation $\frac{dy}{dx} - y = 0$.

There are two types of solutions to a differential equation which are general solution and particular solution.

General Solution:

The solution which contains as many arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

For example, $y = A\cos x + B\, sinx$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Particular Solution:

The solution which is obtained from the general solution of a differential equation by assigning particular values to the arbitrary constants is called a particular solution.

For example, $y = 2\cos x + 3\, sinx$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Example 1:

State whether the following statements are true or false.

i. $x + y = \tan^{-1} y$ is a solution of the differential equation $y^2 \frac{dy}{dx} + y^2 + 1 = 0$.

Answer: TRUE

ii. $y = x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$.

Answer: FALSE

Example 2:

Choose the correct answer from the given options.

1. The differential equation for $y = A\cos \alpha x + B\, sin\alpha x$, where A and B are arbitrary constants is

   a) $\frac{d^2y}{dx^2} - \alpha^2 y = 0$ b) $\frac{d^2y}{dx^2} + \alpha^2 y = 0$

   c) $\frac{d^2y}{dx^2} + \alpha y = 0$ d) $\frac{d^2y}{dx^2} - \alpha y = 0$

   Answer: (b)

2. The number of arbitrary constants in the particular solution of the differential equation of the third order is

   a) 3 b) 2 c) 1 d) 0

Answer: (a)

3. If $y = e^{-x}(A\,cosx + B\,sinx)$, then y is a solution of

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ d) $\frac{d^2y}{dx^2} + 2y = 0$

Answer: (c)

4. The number of solutions to $\frac{dy}{dx} = \frac{y+1}{x-1}$ when $y(1) = 2$ is

a) None b) one c) two d) infinite

Answer: (a)

5. $y = ae^{mx} + be^{-mx}$ satisfies which of the following differential equation?

a) $\frac{dy}{dx} + my = 0$ b) $\frac{dy}{dx} - my = 0$ c) $\frac{d^2y}{dx^2} - m^2y = 0$ d) $\frac{d^2y}{dx^2} + m^2y = 0$

Answer: (c)

Example 3:

Verify that $y = cx + 2c^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$

Answer:

Given $y = cx + 2c^2$

$\Rightarrow \frac{dy}{dx} = c + 0$

Now, $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y$

$= 2c^2 + xc - cx - 2c^2 = 0$

Hence verified

Example 4:

Show that $y = Ax + \frac{B}{x}, x \neq 0$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

Answer:

Given $y = Ax + \frac{B}{x}, x \neq 0$

$\Longrightarrow \frac{dy}{dx} = A - \frac{B}{x^2}$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{2B}{x^3}$

Now, $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$

$$= x^2\frac{2B}{x^3} + x\left(A - \frac{B}{x^2}\right) - Ax - \frac{B}{x} = \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

Hence verified.

Formation of the Differential equation whose general solution is given

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation.

If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

To formulate a differential equation from a given relation containing the independent variable(x), dependent variable(y), and some arbitrary constants, we may follow the following algorithm:

ALGORITHM

STEP I: Write the given equation involving independent variable x (say), dependent variable y (say) and arbitrary constants.

STEP-II: Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.

STEP III: Differentiate the relation in Step I n times w.r.t. x.

STEP IV: Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in Step III and an equation in Step I. The equation so obtained is the desired differential equation.

Remember:

The number of arbitrary constants present in the equation is equal to the order of the resulting differential equation.

Example -1:

Form the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

Answer:

Given $y = mx - - - - - (1)$

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = m$$

Putting the value of m in equation (1)

$$y = \frac{dy}{dx}x$$

Which is required differential Equations.

Example-2:

Form the differential equation representing the family of curves $y = c_1 \, cosx + c_2 \, sinx$, where c_1 and c_2 are arbitrary constants.

Answer:

Given $y = c_1 \, cosx + c_2 \, sinx$

Differentiating w.r.t. x we get

$$\Rightarrow \frac{dy}{dx} = -C_1 \sin x + C_2 \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -C_1 \cos x - C_2 \sin x = -y \quad from \ (1)$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Which is required differential Equations.

Example-3:

Form the differential equation representing the family of curves $x^2 - y^2 = a^2$, where a is an arbitrary constant.

Answer:

Given $x^2 - y^2 = a^2$

Differentiating w.r.t. x we get

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Which is required differential Equations.

Example-4:

Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

Answer:

Given $y = ae^{bx+5}$------------(1)

Differentiating w.r.t. x we get

$\Rightarrow \frac{dy}{dx} = y_1 = ab\ e^{bx+5} = by$ from equation (1)

$\Rightarrow \frac{y_1}{y} = b$

Again, Differentiating w.r.t. x we get

$\Rightarrow \frac{y(y_2)-(y_1)^2}{y^2} = 0 \Rightarrow y(y_2)-(y_1)^2 = 0$

Which is required differential Equations.

Example-5:

Form the differential equation representing the family of curves $y = A\cos(x+B)$, where A and B are arbitrary constants.

Answer:

Given $y = A\cos(x+B)$

Differentiating w.r.t. x we get

$\frac{dy}{dx} = -A\sin(x+B)$

Again, Differentiating w.r.t. x we get

$\frac{d^2y}{dx^2} = -A\cos(x+B) = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0$

Which is required differential Equations.

Example-6:

Form the differential equation representing the family of curves given by $y = Ae^{2x} + Be^{5x}$, where A and$B$ are arbitrary constants.

Answer:

Given $y = Ae^{2x} + Be^{5x}$-------------------------------(1)

Differentiating w.r.t . x we get

$y_1 = 2Ae^{2x} + 5Be^{5x}$ ---------------------------------(2)

Again, Differentiating w.r.t. x we get

$y_2 = 4Ae^{2x} + 25Be^{5x}$ --------------------------------(3)

Solving equations (1) and (2) we get

$A = \frac{3y-y_1}{e^{2x}}$ and $B = \frac{y_1-2y}{3e^{5x}}$

Putting the value of A and B in equation (3) we get

$$y_2 = \frac{3y - y_1}{e^{2x}} 4e^{2x} + \frac{y_1 - 2y}{3e^{5x}} 25e^{5x}$$

$$\Rightarrow y_2 = (3y - y_1)4 + \frac{(y_1 - 2y)25}{3} \Rightarrow 3y_2 = 36y - 12y_1 + 25y_1 - 50y$$

$\Rightarrow 3y_2 - 13y_1 + 14y = 0$ which is the required differential equation.

Example-7:

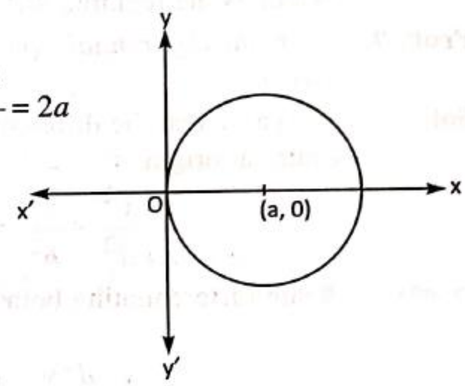
Form the differential equation of the family of circles touching the y-axis at origin and centre lies on the x-axis.

Answer:

We know that any circle which touches y-axis at the origin must have its centre on the x-axis.

Let $A(a, 0)$ be the centre of the circle.

Then the equation of the circle is



$$(x-a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 = 2ax \Rightarrow \frac{x^2+y^2}{x} = 2a$$

$$\Rightarrow \frac{\left(2x + 2y\frac{dy}{dx}\right)x - (x^2+y^2).1}{x^2} = 0$$

$$\Rightarrow 2x^2 + 2xy.\frac{dy}{dx} - x^2 - y^2 = 0$$

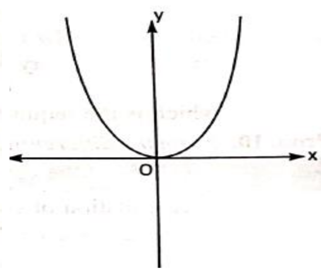
$$\Rightarrow 2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

Example-8:

Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

Answer:

We know that the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis is



$$x^2 = 4ay \Rightarrow \frac{x^2}{y} = 4a$$

differentiating both sides w.r.t. x, we get

$$\frac{2xy - x^2\frac{dy}{dx}}{y^2} = 0 \Rightarrow 2xy - x^2\frac{dy}{dx} = 0 \Rightarrow x\frac{dy}{dx} - 2y = 0$$

Example- 9:

Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

We know that the equation of the family of ellipses having foci on y-axis and centre at origin is

Answer:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

$\Rightarrow$ $$\frac{2x}{b^2}+\frac{2y}{a^2}\frac{dy}{dx}=0$$

$\Rightarrow$ $$\frac{y}{x}\frac{dy}{dx}=\frac{-a^2}{b^2}$$

Again differentiating both sides w.r.t. x, we have

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left[\frac{x\frac{dy}{dx}-y}{x^2}\right]=0$$

$\Rightarrow$ $$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{1}{x}\left(\frac{dy}{dx}\right)^2-\frac{y}{x^2}\frac{dy}{dx}=0$$

$\Rightarrow$ $$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

Which is the required differential equation.

The solution of Differential equations by the method of separation of variables.

A general differential equation of first order and first degree is of the form $f\left(x,y,\frac{dy}{dx}\right)=0$. ------ (i)

The general solution of (i) represents the equation of the family of curves in one arbitrary constant.

In this section, we shall discuss several techniques of obtaining solutions to the following type of differential equations.

Differential Equations of the type $\frac{dy}{dx}=f(x)$

We have $\frac{dy}{dx}=f(x)\Leftrightarrow dy=f(x)dx$

Integrating both sides, we have $\int dy=\int f(x)\,dx$

$\Rightarrow y=\int f(x)\,dx+C$, which is the general solution.

Example 1:

Solve the following differential equations.

i. $\frac{dy}{dx}=\frac{x}{x^2+1}$

ii. $\frac{dy}{dx}=xlogx$

iii. $x\frac{dy}{dx}+1=0; y(1)=0.$

Answer:

(i) Given $\frac{dy}{dx}=\frac{x}{x^2+1}$

$$\Rightarrow dy = \frac{x}{x^2+1}dx$$

Integrating both sides we get

$$\Rightarrow \int dy = \int \frac{x}{x^2+1}dx \text{ ----------(1)}$$

Let $x^2+1=t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

So equation (1) becomes

$$\Rightarrow y = \int \frac{dt}{2t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t + K$$

$$\Rightarrow y = \frac{1}{2}\log(x^2+1)+k$$

ii) Given $\frac{dy}{dx} = x\log x$

$$\Rightarrow \int dy = \int x\log x dx$$

$$\Rightarrow y = (\log x)\int x\,dx - \int \frac{1}{x}\frac{x^2}{2}dx$$

$$\Rightarrow y = \frac{x^2}{2}(\log x) - \frac{1}{2}\int x\,dx$$

$$\Rightarrow y = \frac{x^2}{2}(\log x) - \frac{1}{4}x^2 + k$$

iii) Given $x\frac{dy}{dx}+1=0, \quad y(1)=0$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x} \Rightarrow \int dy = -\int \frac{1}{x}dx \Rightarrow y = -\log x + k \text{ ----------(1)}$$

Given $y(1)=0$

$$\Rightarrow 0 = -\log 1 + k \Rightarrow k = 0$$

Hence equation (1) becomes $y = -\log x$

Differential Equations of the type $\frac{dy}{dx} = f(y)$

We have $\frac{dy}{dx} = f(y) \Leftrightarrow dx = f(y)dy$

Integrating both sides, we have $\int dx = \int \frac{1}{f(y)}\,dy$

$\Rightarrow x = \int \frac{1}{f(y)}\,dy + C$, which is a general solution.

Example 2:

Solve the following differential equations.

i. $\frac{dy}{dx} = sin^2 y$

ii. $\frac{dy}{dx} + 2y^2 = 0, y(1) = 1$

Answer:

(i) Given $\frac{dy}{dx} = \sin^2 y$

$$\Rightarrow \int \frac{dy}{\sin^2 y} = \int dx \Rightarrow \int \cos ec^2 y\, dy = x \Rightarrow -\cot y = x + k$$

(ii) Given $\frac{dy}{dx} + 2y^2 = 0,\ \ y(1) = 1$

$$\Rightarrow \int \frac{dy}{2y^2} = -\int dx \Rightarrow \frac{-1}{2y} = -x + k \quad \text{-------(1)}$$

Given $y(1) = 1$

$$\Rightarrow \frac{-1}{2} = -1 + k \Rightarrow k = \frac{1}{2}$$

Putting the value of $k = \frac{1}{2}$ in equation (1) we get

$$\Rightarrow \frac{-1}{2y} = -x + \frac{1}{2} \Rightarrow x - \frac{1}{2y} = \frac{1}{2}$$

Differential Equations in the variable separable form:

A differential equation is said to have separable variables if it is of the form $f(x)dx = g(y)dy$.
Such types of equations can be solved by integrating both sides.
The general solution is given by $\int f(x)dx = \int g(y)dy + C$, where C is an arbitrary constant.

Remember:

There is no need of introducing arbitrary constants of integration on both sides as they can be combined to give just one arbitrary constant.

Example 3:

Solve the following differential equations.

i. $2(y+3) - xy\frac{dy}{dx} = 0$

ii. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

iii. $e^x\sqrt{1-y^2}\, dx + \frac{y}{x}\, dy = 0$

iv. $(1+y^2)\tan^{-1}x\,dx + 2y(1+x^2)dy = 0$

Answer:

(i) Given $2(y+3) - xy\dfrac{dy}{dx} = 0$

$$\Rightarrow 2(y+3) = xy\frac{dy}{dx} \Rightarrow \int \frac{2}{x}dx = \int \frac{y}{y+3}dy$$

$$\Rightarrow \int \frac{2}{x}dx = \int \frac{y+3-3}{y+3}dy \Rightarrow 2\log x = \int 1dy - 3\int \frac{1}{y+3}dy$$

$$\Rightarrow 2\log x = y - 3\log(y+3) + k$$

(ii) The given differential equation is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\,dy = (1+y^2)\,dx$$

$$\Rightarrow \quad \frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}x + \tan^{-1}C$$

$$\Rightarrow \quad \tan^{-1}y - \tan^{-1}x = \tan^{-1}C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1}C$$

$$\Rightarrow \quad \frac{y-x}{1+xy} = C$$

$\Rightarrow \quad y - x = C\,(1+xy)$, which is the required solution.

iii) We are given that

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$$

$$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}\,dy$$

$$\Rightarrow \quad x\,e^x\,dx = -\frac{y}{\sqrt{1-y^2}}\,dy$$

$$\Rightarrow \quad \int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx = -\int \frac{y}{\sqrt{1-y^2}}\,dy$$

$$\Rightarrow \quad xe^x - \int e^x\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{ where } t = 1-y^2$$

$$\Rightarrow \quad xe^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{t} + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{1-y^2} + C \text{ is the required solution.}$$

Example 4:

Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$.

Answer:

Given that $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2 = (1+x^2) + y^2(1+x^2) = (1+x^2)(1+y^2)$

$\Rightarrow \int \frac{dy}{1+y^2} = \int (1+x^2)dx \Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + c$ -----------(1)

Putting x =0 and y = 1 in above equation we get

$\Rightarrow \frac{\pi}{4} = c$

Hence, equation (1) becomes

$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$ which is a particular solution.

Example 5:

Find the solution of the differential equation $e^x tany\,dx + (1-e^x)sec^2y\,dy = 0$.

Answer:

Given $e^x tany\,dx + (1-e^x)sec^2y\,dy = 0$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y}dy = \int \frac{-e^x}{1-e^x}dx$$

$$\Rightarrow \log(\tan y) = \log(1-e^x) + \log c$$

$$\Rightarrow \log|\tan y| = \log|c(1-e^x)|$$

Example 6:

Find the equation of a curve passing through the point $(-2, 3)$, given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Answer:

A.T.Q. $\frac{dy}{dx} = \frac{2x}{y^2} \Rightarrow \int y^2 dy = \int 2x dx$

$\Rightarrow \frac{y^3}{3} = x^2 + k$ -------------**(1)**

A.T.Q. curve passing through (-2, 3)

Therefore, $9 = 4 + k \Rightarrow k = 5$

Hence equation (1) becomes

$\Rightarrow \frac{y^3}{3} = x^2 + 5$

Example 7:

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹ 100 double itself in 10 years. (use $\log_e 2 = 0{\cdot}6931$*)*

Answer:

Let P be the principal and rate of interest = $r\%$ per year

Then $\frac{dP}{dt} = \frac{r}{100} . P \Rightarrow \frac{dP}{P} = \frac{r}{100} dt \Rightarrow \int \frac{dP}{P} = \int \frac{r}{100} dt$

$\therefore \log P = \frac{r}{100} t + \log C \Rightarrow \log P - \log C = \frac{rt}{100} \Rightarrow \log \frac{P}{C} = \frac{rt}{100}$

$\therefore \frac{P}{C} = e^{rt/100} \Rightarrow P = Ce^{rt/100}$

When $t = 0$, $P = 100$ then $100 = Ce^{\circ} \Rightarrow C = 100$

$\therefore P = 100\, e^{rt/100}$

When $t = 10$, $P = 200$

$\therefore 200 = 100\, e^{r \times 10/100} \Rightarrow 2 = e^{r/10} \Rightarrow \log 2 = \frac{r}{10}$

$\Rightarrow r = 10 \times \log 2 \Rightarrow r = 10 \times 0{\cdot}6931$

$\Rightarrow r = 6{\cdot}931\%$ per year.

Equations Reducible to Variable separable form

Let the differential equation be of the form $\frac{dy}{dx} = f(ax + by + c)$

It can be reduced to variable separable form by the substitution $z = ax + by + c$

Example 8:

Solve the following.

i. $\frac{dy}{dx} = (4x + y + 1)^2$

ii. $\frac{dy}{dx} = \cos(x + y)$

Answer:

i) We are given that

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let $4x + y + 1 = v$, Then,

$$4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Putting $4x + y + 1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 4$ in the given differential equation, we get

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \quad \frac{dv}{dx} = v^2 + 4$$

$$\Rightarrow \quad dv = (v^2 + 4)\, dx$$

$$\Rightarrow \quad \frac{1}{v^2 + 4} dv = dx$$

$$\Rightarrow \quad \int \frac{1}{v^2 + 4} dv = \int 1 \cdot dx$$

$$\Rightarrow \quad \frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + C$$

$$\Rightarrow \quad \frac{1}{2} \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = x + C,$$ which is the required solution.

(ii) We are given that

$$\frac{dy}{dx} = \cos(x + y)$$

Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\frac{dv}{dx} - 1 = \cos v$$

$\Rightarrow \quad \frac{dv}{dx} = 1 + \cos v$

$\Rightarrow \quad \frac{1}{1+\cos v} dv = dx$

$\Rightarrow \quad \frac{1}{2}\sec^2 \frac{v}{2} \, dv = dx.$

$\Rightarrow \quad \int \frac{1}{2}\sec^2 \frac{v}{2} dv = \int 1 \cdot dx$

$\Rightarrow \quad \tan \frac{v}{2} = x + C$

$\Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + C$, which is the required solution.

Homogeneous Differential equations of the first order and first degree

Definition: Homogeneous Function

A function $f(x,y)$ is said to be a homogeneous function of degree $n$if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$.
Example: Test the homogeneity of the function $f(x,y) = 3x^2 - 2y^2 + 7xy$

$$f(\lambda x, \lambda y) = 3(\lambda x)^2 - 2(\lambda y)^2 + 7(\lambda x)(\lambda y)$$
$$= \lambda^2(3x^2 - 2y^2 + 7xy) = \lambda^2 f(x,y)$$

Remember:

- If the sum of powers of x and y in each term is the same, then the function is homogeneous.
- If $f(x,y)$ is a homogeneous function of degree n, then we write

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$
$$= y^n \Psi\left(\frac{x}{y}\right)$$

Definition:

A differential equation of the form $f(x,y)dx + g(x,y)dy = 0 \ldots\ldots(i)$
Is said to be homogeneous if $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree.

Equation(i) can be written as $\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)} = F(x,y) \ldots\ldots(ii)$
Equation (ii) is a homogeneous differential equation if $F(x,y)$ is a homogeneous function of degree 0.
To solve equation (ii), put $y = vx$, where v is a function of x.
$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
Then equation (ii) becomes, $v + x\frac{dv}{dx} = F(x,vx) = h(v)$
Separating the variables, $\frac{dv}{h(v)-v} = \frac{dx}{x}$

On integrating, we get $\int \frac{dv}{h(v)-v} = \int \frac{dx}{x} + C$, where C is an arbitrary constant of integration. After integration, v is replaced by $\frac{y}{x}$ to get the complete solution.

Remember:

If a homogeneous differential equation is of the form $\frac{dx}{dy} = G(x,y) = y^n \Psi\left(\frac{x}{y}\right)$, then we have to substitute $\frac{x}{y} = v$ or $x = vy$, where v is a function of y.

Example 1:

Show that the differential equation $(x-y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Answer:

The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \qquad \ldots (1)$$

Let $$F(x, y) = \frac{x+2y}{x-y}$$

Now $$F(\lambda x, \lambda y) = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^0 \cdot f(x, y)$$

Therefore, $F(x, y)$ is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad \ldots (2)$$

R.H.S. of differential equation (2) is of the form $g\left(\frac{y}{x}\right)$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or $$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or $$x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

or $$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int \frac{v-1}{v^2+v+1}dv = -\int \frac{dx}{x}$$

or $$\frac{1}{2}\int \frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\int \frac{2v+1}{v^2+v+1}dv - \frac{3}{2}\int \frac{1}{v^2+v+1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right| - \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right|-\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=-\log|x|+C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log\left|(y^2+xy+x^2)\right|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log\left|(x^2+xy+y^2)\right|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 2:

Show that the differential equation $(x^2-2y^2)dx+2xydy=0$ is homogeneous and solve it.

Answer:

We have,

$$(x^2-2y^2)\,dx+2xy\,dy=0 \Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{2xy}$$

This is a homogeneous differential equation.

Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v^2 - 1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{2v}$$

$$\Rightarrow \quad 2v\,dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \int 2v\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad v^2 = -\log|x| + C$$

$$\Rightarrow \quad y^2 = -x^2\log|x| + Cx^2$$

It is given that $y(1) = 1$ i.e. when $x = 1$, $y = 1$. Putting $x = 1, y = 1$ in (i), we get:

$$1 = 0 + C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get: $y^2 = -x^2\log|x| + x^2$ as the required solution.

Example 3:

Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it.

Answer:

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \quad \ldots (1)$$

It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

Here $$F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda[y\cos\left(\frac{y}{x}\right) + x]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0[F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \ldots (2)$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \ldots (3)$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or $$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or $$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or $$\cos v\, dv = \frac{dx}{x}$$

Therefore $$\int \cos v\, dv = \int \frac{1}{x}\, dx$$

or $$\sin v = \log|x| + \log|C|$$

or $$\sin v = \log|Cx|$$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log|Cx|$$

which is the general solution of the differential equation (1).

Homogeneous Differential equations of the first order and first degree

Example 1:

Show that the differential equation $2ye^{\frac{x}{y}}dy + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$, when $y = 1$.

Answer:

The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}} \qquad \text{... (1)}$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0[F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \qquad \dots (2)$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^v}$$

or $$2e^v\,dv = \frac{-dy}{y}$$

or $$\int 2e^v \cdot dv = -\int \frac{dy}{y}$$

or $$2\,e^v = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2e^{\frac{x}{y}} + \log|y| = C \quad \dots (3)$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2e^0 + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 2:

Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ is given by $x^2 - y^2 = cx$.

Answer:

We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \quad \dots (1)$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int\frac{2v}{v^2-1}dv = -\int\frac{1}{x}dx$$

or $$\log|v^2-1| = -\log|x| + \log|C_1|$$

or $$\log|(v^2-1)(x)| = \log|C_1|$$

or $$(v^2-1)\,x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm C_1$$

or $$(y^2-x^2) = \pm C_1\,x \text{ or } x^2-y^2 = Cx$$

Example 3:

Solve:

$$y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}dx - x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}dy = 0$$

Answer:

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}} \qquad \text{...(i)}$$

It can be checked that RHS does not change when x is replaced by λx and y by λy. So, the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx\{x\cos v + vx\sin v\}}{x\{vx\sin v - x\cos v\}} = \frac{v\{\cos v + v\sin v\}}{\{v\sin v - \cos v\}}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\frac{dx}{x} \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad -\int \frac{\cos v - v\sin v}{v\cos v}\,dv = 2\int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad -\log|v\cos v| = 2\log|x| + \log C$$

$$\Rightarrow \quad \log\frac{1}{|v\cos v|} = \log|x^2| + \log C$$

$$\Rightarrow \quad \left|\frac{1}{v\cos v}\right| = |C|\,x^2$$

$$\Rightarrow \quad \left|\frac{x}{y}\sec\left(\frac{y}{x}\right)\right| = |C|\,x^2$$

$$\Rightarrow \quad |xy\cos(yx)| = \frac{1}{|C|}$$

$$\Rightarrow \quad |xy\cos(y/x)| = k, \text{ where } k = 1/|C|$$

Hence, $\left|xy\cos\left(\frac{y}{x}\right)\right| = k,\ x \neq 0,\ k > 0$ gives the required solution.

Example 4:

Solve the differential equation $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0.$

Answer:

We have,

$$(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1+e^{x/y}}{e^{x/y}\left(1-\frac{x}{y}\right)} \Rightarrow \frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$$

Putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$, we get

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{-e^v + v e^v - v - v e^v}{1+e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v}$$

$$\Rightarrow \quad \frac{1+e^v}{v+e^v}\,dv = -\frac{1}{y}\,dy$$

$$\Rightarrow \quad \int \frac{1+e^v}{v+e^v}\,dv = -\int \frac{1}{y}\,dy \qquad \text{[On integrating]}$$

$$\Rightarrow \quad \log(v+e^v) = -\log y + \log C$$

$$\Rightarrow \quad v+e^v = \frac{C}{y} \Rightarrow \frac{x}{y} + e^{x/y} = \frac{C}{y} \Rightarrow x + y\,e^{x/y} = C$$

Example 5:

Solve each of the following initial value problems:

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0,\ y(e) = e$$

Answer:

We have,

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v - v^2}{2}$$

$$\Rightarrow 2x\frac{dv}{dx} = -v^2$$

$$\Rightarrow -\frac{2}{v^2}\,dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{-2}{v^2}\,dv = \int \frac{1}{x}\,dx \qquad \text{[On integrating]}$$

$$\Rightarrow \frac{2}{v} = \log|x| + C$$

$$\Rightarrow \frac{2x}{y} = \log|x| + C \qquad \text{...(i)}$$

It is given that $y(e) = e$ i.e. $y = e$ when $x = e$.

Putting $x = e$ and $y = e$ in (i), we get

$$2 = 1 + C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$\frac{2x}{y} = \log|x| + 1 \Rightarrow y = \frac{2x}{1 + \log|x|}$$

Hence, $y = \dfrac{2x}{1 + \log|x|}$ gives the required solution.

linear differential equations and solution procedure

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \dots\dots (1)$$

whereP, Q are functions of x or constants.

Equation (1) is also known as Leibnitz's linear equation.

To solve it, we multiply both sides $e^{\int P\,dx}$, by getting

$\frac{dy}{dx} e^{\int P\,dx} + y\left(e^{\int P\,dx}\right) P = Q\, e^{\int P\,dx}$

$$\frac{d}{dx}\left(y e^{\int P\,dx}\right) = Q\, e^{\int P\,dx}$$

Integrating both sides, we have

$y e^{\int P\,dx} = \int Q e^{\int P\,dx}\, dx + C$, which is the required solution.

Remember:

- The factor $e^{\int P\,dx}$ on multiplying by which the L.H.S. of (1) becomes the differential coefficient of a single function is called the integrating factor.
- Also, a linear differential equation is of the form $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only or constants. The integrating factor in this case is $e^{\int P\,dy}$.

Steps involved to solve first order linear differential equation:

Step-1 Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where$P$, Q are functions of x or constants.

Step-2 Find the integrating factor $(I.F.) = e^{\int P\,dx}$

Step-3 Write the solution of the given differential equation as $y(I.F.) = \int Q(I.F.)\, dx + C$.

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + P_1x = Q_1$, where P_1, Q_1 are functions of y only or constants. Then $I.F. = e^{\int P_1 dy}$ and the solution of the differential equation is given by $x(I.F.) = \int Q_1(I.F.)dy + C$.

Example1:

Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

Answer:

We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This is a linear differential equation with $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$$

Multiplying both sides of (i) by I.F. $= e^{2\sqrt{x}}$, we get

$$\frac{dy}{dx}e^{2\sqrt{x}} + \frac{y\,e^{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Integrating both sides with respect to x, we get

$$ye^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\,dx + C$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = 2\sqrt{x} + C$$

$\Rightarrow \quad y = (2\sqrt{x} + C)e^{-2\sqrt{x}}$, which gives the required solution.

Example 2:

Find the general solution of the differential equation $\frac{dy}{dx} - y = cosx$.

Answer:

Given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -1 \text{ and } Q = \cos x$$

Therefore $\quad I.F = e^{\int -1\,dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or
$$\frac{d}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\,dx + C \qquad \text{... (1)}$$

Let
$$I = \int e^{-x}\cos x\,dx$$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int(-\sin x)\,(-e^{-x})\,dx$$

$$= -\cos x\,e^{-x} - \int \sin x\,e^{-x}\,dx$$

$$= -\cos x\,e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\,(-e^{-x})\,dx\right]$$

$$= -\cos x\,e^{-x} + \sin x\,e^{-x} - \int \cos x\,e^{-x}\,dx$$

or
$$I = -e^{-x}\cos x + \sin x\,e^{-x} - I$$

or
$$2I = (\sin x - \cos x)\,e^{-x}$$

or
$$I = \frac{(\sin x - \cos x)e^{-x}}{2}$$

Substituting the value of I in equation (1), we get

$$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + C$$

or
$$y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{x}$$

which is the general solution of the given differential equation.

Example 3:

Find the general solution of the differential equation $y\,dx - (x + 2y^2)dy = 0$.

Answer:

The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$, where $P_1 = -\frac{1}{y}$ and

$Q_1 = 2y$. Therefore $I.F = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log(y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right) dy + C$$

or $$\frac{x}{y} = \int (2dy) + C$$

or $$\frac{x}{y} = 2y + C$$

or $$x = 2y^2 + Cy$$

which is a general solution of the given differential equation.

Example 4:

Solve the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)\, dx$.

Answer:

The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \quad \dots (1)$$

Now (1) is a linear differential equation of the form $\frac{dx}{dy} + P_1\, x = Q_1$,

where, $P_1 = \frac{1}{1+y^2}$ and $Q_1 = \frac{\tan^{-1} y}{1+y^2}$.

Therefore, $I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus, the solution of the given differential equation is

$$x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2} \right) e^{\tan^{-1} y}\, dy + C \qquad \text{... (2)}$$

Let $$I = \int \left(\frac{\tan^{-1} y}{1+y^2} \right) e^{\tan^{-1} y}\, dy$$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2} \right) dy = dt$, we get

$$I = \int t\, e^t dt = t\, e^t - \int 1 \,.\, e^t\, dt = t\, e^t - e^t = e^t\,(t-1)$$

or $$I = e^{\tan^{-1} y}\,(\tan^{-1} y - 1)$$

Substituting the value of I in equation (2), we get

$$x \,.\, e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

or $$x = (\tan^{-1} y - 1) + C\, e^{-\tan^{-1} y}$$

which is the general solution of the given differential equation.

Example 5:

Find the particular solution of the differential equation $\frac{dy}{dx} + y\, cotx = 2x + x^2\, cotx\,, x \neq 0$ given that $y = 0$ when $x = \frac{\pi}{2}$.

Answer:

The given equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$,

where $P = \cot x$ and $Q = 2x + x^2 \cot x$. Therefore

$$I.F = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x\, dx + C$$

or $$y \sin x = \int 2x \sin x\, dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = \sin x \left(\frac{2x^2}{2} \right) - \int \cos x \left(\frac{2x^2}{2} \right) dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x\, dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = x^2 \sin x + C \qquad \text{... (1)}$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 6:

Find the equation of a curve passing through the point $(0, 1)$, if the slope of the tangent to the curve at any point (x, y) is equal to the sum of x coordinate and the product of the x coordinate and y coordinate of that point.

Answer:

We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \quad \dots (1)$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

Therefore, $$\text{I.F} = e^{\int -x\,dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x)\left(e^{\frac{-x^2}{2}}\right)dx + C \qquad \ldots (2)$$

Let $$I = \int (x)\, e^{\frac{-x^2}{2}}\, dx$$

Let $\frac{-x^2}{2} = t$, then $-x\,dx = dt$ or $x\,dx = -\,dt$.

Therefore, $I = -\int e^t\,dt = -e^t = -e^{\frac{-x^2}{2}}$

Substituting the value of I in equation (2), we get

$$y\,e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + C$$

or $$y = -1 + C\,e^{\frac{x^2}{2}} \qquad \ldots (3)$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -1 + C \,.\, e^0 \quad \text{or} \quad C = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\,e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

Related problems on linear differential equation

1. Find the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$

   Answer:

   O=2, Degree not defined.

2. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is

   a) 3 b) 2 c) 1 d) not defined

   Answer: (d)

3. Solve the differential equation $y\,e^{\frac{x}{y}}\,dx = \left(x\,e^{\frac{x}{y}} + y^2\right)dy \;(y \neq 0).$

   Answer:

The given equation is $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy$.

$\therefore \quad \dfrac{dx}{dy} = \dfrac{xe^{x/y} + y^2}{ye^{x/y}}$, is a homogeneous differential equation. ...(i)

put $x = vy$ so that $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

$$\therefore \quad v + y\frac{dv}{dy} = \frac{vye^v + y^2}{ye^v} \Rightarrow v + y\frac{dv}{dy} = \frac{y(ve^v + y)}{ye^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{ve^v + y}{e^v} - v \Rightarrow y\frac{dv}{dy} = \frac{ve^v + y - ve^v}{e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{y}{e^v} \Rightarrow e^v\,dv = dy$$

$$\Rightarrow \quad \int e^v\,dv = \int dy \Rightarrow e^v = y + C \Rightarrow e^{x/y} = y + C$$

4. Find the general solution of $x\log x\,\frac{dy}{dx} + y = \frac{2}{x}\,logx$.

Answer:

Given equation $x\log x\dfrac{dy}{dx} + y = 2\log x$ can be expressed as $\dfrac{dy}{dx} + \dfrac{1}{x\log x}\cdot y = \dfrac{2}{x}$

Which is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$ where $P = \dfrac{1}{x\log x}$, $Q = \dfrac{2}{x}$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{1/x}{\log x}dx} = e^{\log\log x} = \log x$$

Hence, the solution of differential equation is

$$y(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + C$$

$$\Rightarrow \quad y\log x = \int \frac{2}{x}\log x dx + C \Rightarrow y\log x = 2\int \frac{1}{x}.\log x dx + C$$

$$= 2\frac{(\log x)^2}{2} + C \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1\right]$$

$$\Rightarrow \quad y\log x = (\log x)^2 + C.$$

5. Find the particular solution of $\frac{dy}{dx} - 3y\,cotx = sin2x$; $y = 2$ when $x = \frac{\pi}{2}$.

Answer:

The given equation is $\frac{dy}{dx} - 3y \cot x = \sin 2x$ which is linear differential equation of the type

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -3 \cot x \text{ and } Q = \sin 2x$$

$\therefore$ Integrating factor (I.F.) $= e^{\int -3 \cot x dx} = e^{-3 \log \sin x} = e^{\log \operatorname{cosec}^3 x} = \operatorname{cosec}^3 x$

Thus, the required solution of the given differential equation is

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

$$y.\operatorname{cosec}^3 x = \int \sin 2x . \operatorname{cosec}^3 x dx + C \Rightarrow y.\operatorname{cosec}^3 x = \int 2 \sin x \cos x . \frac{1}{\sin^3 x} . dx + C$$

$$\Rightarrow \quad y.\operatorname{cosec}^3 x = 2\int \operatorname{cosec} x \cot x dx + C \Rightarrow y \operatorname{cosec}^3 x = -2 \operatorname{cosec} x + C$$

$$\Rightarrow \quad y = -2 \sin^2 x + C \sin^3 x, \text{ Putting } x = \frac{\pi}{2} \text{ and } y = 2$$

$$\therefore \quad 2 = -2 \sin^2 \frac{\pi}{2} + C \sin^3 \frac{\pi}{2}$$

$$\Rightarrow \quad 2 = -2 + C \quad \Rightarrow \quad C = 4$$

$$\therefore \quad y = -2 \sin^2 x + 4 \sin^3 x.$$

6. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Answer:

Let $\frac{dy}{dx}$ be the slope of tangent at point (x, y).

Then, $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

Which is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where, $P = -1$ and $Q = x$

$\therefore$ Integrating factor (I.F.) $= e^{\int -1 dx} = e^{-x}$

Thus, required solution of the given differential equation is

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

$$y.e^{-x} = \int xe^{-x} dx + C \Rightarrow x\left(\frac{e^{-x}}{-1}\right) - \int 1.\frac{e^{-x}}{-1} dx = -xe^{-x} + \int e^{-x} dx$$

$$\Rightarrow \quad ye^{-x} = -xe^{-x} - e^{-x} + C \Rightarrow y = -x - 1 + Ce^{x}$$

putting $x = 0$ and $y = 0$

$$\therefore \quad 0 = 0 - 1 + Ce^0 \Rightarrow C = 1 \Rightarrow y = -x - 1 + e^x$$

$$\Rightarrow \quad x + y + 1 = e^x.$$

7. Find a particular solution of the differential equation $(x-y)(dx+dy)=dx-dy$ given that $y=-1$, when $x=0$.

Answer:

The given equation is $(x-y)(dx+dy)=dx-dy$

$\therefore \quad (x-y-1)\,dx+(x-y+1)\,dy=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{(x-y-1)}{(x-y+1)}$

$\Rightarrow \quad \frac{dy}{dx}=\frac{-(x-y-1)}{(x-y+1)}$

Put $x-y=t \quad \Rightarrow \quad 1-\frac{dy}{dx}=\frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=1-\frac{dt}{dx}$

$\therefore \quad 1-\frac{dt}{dx}=-\frac{(t-1)}{(t+1)} \quad \Rightarrow \quad \frac{dt}{dx}=1+\frac{t-1}{t+1} \quad \Rightarrow \quad \frac{dt}{dx}=\frac{t+1+t-1}{t+1}$

$\Rightarrow \quad \frac{dt}{dx}=\frac{2t}{t+1} \quad \Rightarrow \quad \frac{t+1}{2t}dt=dx \quad \Rightarrow \quad \int\frac{t+1}{2t}dt=\int dx$

$\Rightarrow \quad \frac{1}{2}\int\left(1+\frac{1}{t}\right)dt=\int dx$

$\therefore \quad \frac{1}{2}t+\frac{1}{2}\log|t|=x+C \Rightarrow \quad \frac{1}{2}[(x-y)+\log|x-y|]=x+C$

Putting $x=0$ and $y=-1$

$\therefore \quad \frac{1}{2}[(0+1)+\log|0+1|]=0+C \quad \Rightarrow \quad C=\frac{1}{2}$

$\therefore \quad \frac{1}{2}[(x-y)+\log|x-y|]=x+\frac{1}{2} \quad \Rightarrow \quad x-y+\log|x-y|=2x+1$

$\Rightarrow \quad \log|x-y|=2x+1-x+y \quad \Rightarrow \quad \log|x-y|=x+y+1.$

8. The population of a village increases continuously at a rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?

Answer:

Let y be the population of village at any time t. It is given that,

rate of increase of population $\propto$ number of inhabitants

$\therefore \quad \frac{dy}{dt}\propto y \quad \Rightarrow \quad \frac{dy}{dt}=Ky$ where K is constant of proportionality.

$\therefore \quad \frac{dy}{y}=Kdt \quad \Rightarrow \quad \int\frac{dy}{y}=K\int dt \Rightarrow \quad \log y=Kt+C$

In 1999, $t = 0$ and $y = 20{,}000$

$\therefore \quad \log 20{,}000 = K \times 0 + C \quad \Rightarrow \quad C = \log 20{,}000$

$\therefore \quad \log y = Kt + \log 20{,}000 \quad \Rightarrow \quad \log y - \log 20{,}000 = Kt$

$$\Rightarrow \quad K = \frac{\log y / 20{,}000}{t}$$

In 2004, $t = 5$ and $y = 25{,}000$

$$K = \frac{\log 25{,}000/20{,}000}{5} = \frac{1}{5}\log\frac{5}{4}$$

$$\therefore \quad (i) \Rightarrow \log\frac{y}{20{,}000} = \left(\frac{1}{5}\log\frac{5}{4}\right)t$$

In 2009, $t = 10$

$$\therefore \quad \log\frac{y}{20{,}000} = \left(\frac{1}{5}\log\frac{5}{4}\right)\times 10 \quad \Rightarrow \quad \log\frac{y}{20{,}000} = 2\log\frac{5}{4}$$

$$\log\frac{y}{20{,}000} = \log\left(\frac{5}{4}\right)^2 \quad \Rightarrow \quad \frac{y}{20{,}000} = \frac{25}{16} \quad \Rightarrow \quad y = \frac{25}{16}\times 20{,}000 = 31{,}250$$

Thus population of village will be 2009 is 31,250.

7 Solve the differential equation $(x+y)\frac{dy}{dx} = 1$

Answer:

The given equation is $(x+y)\frac{dy}{dx} = 1$.

$$\Rightarrow \quad \frac{dx}{dy} = x + y \quad \Rightarrow \quad \frac{dx}{dy} - x = y$$

Integrating factor (I.F.) $= e^{\int -1 dy} = e^{-y}$

Thus, required solution of the given differential equation is

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

$$x.e^{-y} = \int y.e^{-y}\,dy + C$$

$$\Rightarrow \quad xe^{-y} = y\left(\frac{e^{-y}}{-1}\right) - \int 1.\left(\frac{e^{-y}}{-1}\right)dy + C$$

which is a linear differential equation of the type $\frac{dx}{dy} + Px = Q$, where $P = -1$ and $Q = y$

$\Rightarrow \quad xe^{-y} = -ye^{-y} + \int e^{-y}\, dy + C$

$\Rightarrow \quad xe^{-y} = -ye^{-y} - e^{-y} + C$

$\Rightarrow \quad x = -y - 1 + Ce^{y} \Rightarrow x + y + 1 = Ce^{y}.$