Chapter-4

# Determinants

## STUDY NOTE

It is the value (or number) associated with a square matrix. Only square Matrices have their determinants. The matrices which are not square do not have their determinants.

If $A = \left[a_{ij}\right]$ is a square matrix of order 'n', then the determinant of A is denoted by det(A) or |A| and

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$$

**Determinant of a square matrix of order 1: -**

If $A = \left[a_{11}\right]$ is a square matrix of order 1, then $|A| = |a_{11}| = a_{11}$

**Example:-** If $A = [7]$ and $B = [-4]$ find $|A|$ and $|B|$

Here, $|A| = 171 = 7$, $|B| = |-4| = -4$

**Determinant of a square matrix of order – 2: -**

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then the determinant of a is defined as $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

**Example:-** Evaluate (i) $\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix}$ (ii) $\begin{vmatrix} -a & b \\ -b & -a \end{vmatrix}$

**Solution:-** (i) $\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix} = 2(-5) - (-3)(4) = -10 + 12 = 12$

(ii) $\begin{vmatrix} -a & b \\ -b & -a \end{vmatrix} = a^2 + b^2$

**Determinant of a square matrix of order – 3: -**

The determinants of a square matrix of order -3 is the sum of the product of each element $a_{ij}$ in the ith row (or jth column) with $(-1)^{i+j}$ times the determinant of the matrix obtained by leaving the ith row and jth column.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

(i) Using elements of the first row

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 6 & -2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 5 \\ -2 & 3 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 \\ 6 & -2 \end{vmatrix}$$

$$= 2(0+10) - 3(12-30) - 1(-8-0)$$

$$= 20 + 54 + 8 = 82$$

(ii) Using elements of the second column

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 6 & -2 & 3 \end{vmatrix} = -3\begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix}$$

$$= -3(12-30) + 0 + 2(10+4)$$

$$= 54 + 28 = 82$$

**Example:-1** Evaluate $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

**Solution:-** $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

**Example:-2** Solve for $x : \begin{vmatrix} 2x & 5 \\ 1 & 3 \end{vmatrix} = 7$

**Solution:-** $\begin{vmatrix} 2x & 5 \\ 1 & 3 \end{vmatrix} = 7$

$\Rightarrow 6x - 5 = 7$

$\Rightarrow 6x = 12$

$\Rightarrow x = 2$

**Example – 3** If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, find x

**Solution:-** $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$

$\Rightarrow 2x^2 - 40 = 18 - 40$

$\Rightarrow 2x^2 = 18$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

**Example – 4** If $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$, then show that $|3A|=9|A|$

**Solution:-** Now $3A=3\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}6 & 9\\ 3 & 12\end{bmatrix}$

$\Rightarrow |3a|=\begin{vmatrix}6 & 9\\ 3 & 12\end{vmatrix}=6\times12-9\times3=72-27=45$ ........................ (1)

And $|A|=\begin{vmatrix}2 & 3\\ 1 & 4\end{vmatrix}=2\times4-1\times3=8-3=5$

$\Rightarrow 9|A|=9\times5=45$ .............................. (2)

From equations (1) and (2) we get $|3A|=9|A|$

**Example:-** If $A=\begin{bmatrix}2 & -3 & 1\\ -1 & 2 & 3\\ 3 & 1 & -4\end{bmatrix}$, prove that $|2A|=8|A|$

**Solution:-** $2A=2\begin{bmatrix}2 & -3 & 1\\ -1 & 2 & 3\\ 3 & 1 & -4\end{bmatrix}=\begin{bmatrix}4 & -6 & 2\\ -2 & 4 & 6\\ 6 & 2 & -8\end{bmatrix}$

$\Rightarrow |2A|=\begin{vmatrix}4 & -6 & 2\\ -2 & 4 & 6\\ 6 & 2 & -8\end{vmatrix}$ Expanding by $R_1$, we get

$=4\begin{vmatrix}4 & 6\\ 2 & -8\end{vmatrix}-(-6)\begin{vmatrix}-2 & 6\\ 6 & -8\end{vmatrix}+2\begin{vmatrix}-2 & 4\\ 6 & 2\end{vmatrix}$

$=4(-32-12)+6(16-36)+2(-4-24)$

$=-176-120-56=-352$ .................................. (1)

And $|A|=\begin{vmatrix}2 & -3 & 1\\ -1 & 2 & 3\\ 3 & 1 & -4\end{vmatrix}$ Expanding by $R_1$ we get.

$=2\begin{vmatrix}2 & -3\\ 1 & -4\end{vmatrix}+\begin{vmatrix}-1 & 3\\ 3 & -4\end{vmatrix}+1\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix}$

$=2(-8-3)+3(4-9)+1(-1-6)$

$=-22-15-7=-44$

$\Rightarrow 8|A|=8(-44)=-352$ .......................... (2)

From (1) and (2) we get $|2A|=8|A|$

**Properties of Determinants:-**

**Property – 1**

The value of the determinant remains unchanged its rows and columns are interchanged.

**Example:-** Verify property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & 7 \end{vmatrix}$

**Solution:-** Expanding the determinant along with the first row we have

$$\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & 7 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= 2(0-20) + 3(-42-4) + 5(30-0)$$

$$= -40 - 138 + 150 = -28$$

By interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix}$$ Expanding by the first column

$$= 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(0-20) + 3(-42-4) + 5(30-0)$$

$$= -40 - 138 + 150 = -28$$

Clearly $\Delta = \Delta_1$

Hence, property 1 proved

**Property – 2**

If any two rows (or columns) of a determinant are interchanged, then the sign of determinants changes.

**Example:-** Verify property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

**Solution:-** $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = -28$ (As shown in the previous example)

Interchanging rows $R_2$ and $R_3$ e.g $R_2 \leftrightarrow R_3$ we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

Expanding the determinant $\Delta_1$ along the first row, we have.

$$\Delta_1 = 2\begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}$$

$$= 2(20-0) + 3(4+42) + 5(0-30)$$

$$= 40 + 138 - 150 = 28$$

Hence property 2 is verified.

**Property – 3**

If any two rows (or columns) of a determinant are identical (all corresponding elements are the same), then the value of the determinant is zero.

**Proof:-** If we interchange identical rows (or columns) of a determinant $\Delta$, then $\Delta$ does not change. However, by property – 2, it follows that $\Delta$ has changed its sign.

$$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} \qquad R_1 \leftrightarrow R_3$$

$$\therefore\ \Delta = -\Delta$$

$$\Rightarrow 2\Delta = 0$$

$$\Rightarrow \Delta = 0$$

**Example:-** Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

**Solution:-** $\Delta = 3(6-6) - 2(6-9) + 3(4-6)$

$$= 0 - 2(-3) + 3(-2)$$

$$= 6 - 6 = 0$$

Here $R_1$ and $R_3$ are identical

**Property – 4**

If each element of a row (or a column) of a determinant is multiplied by a constant K, then its value gets multiplied by K.

**Verification:-** Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding by $R_1$

$\Delta = a_1(b_2c_3 - b_3c_1) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ........................................ (1)

Now $\Delta_1$ be the determinant obtained by multiplying the elements of the first row by K.

$$\Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding by the first row

$$\Rightarrow \Delta_1 = K_1a_1(b_2c_3 - b_3c_2) - Kb_1(a_2c_3 - a_3c_2) + Kc_1(a_2b_3 - a_3b_2)$$

$$\Rightarrow \Delta_1 = K[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$\Rightarrow \Delta_1 = K\Delta$ From equation (1)

Hence, $\begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

**Remember:-**

By this property, we can take any common factor from any one row or any one column of a given determinant

If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero.

**Example:-** $\Delta = \begin{vmatrix} b_1 & b_2 & b_3 \\ Kb_1 & Kb_2 & Kb_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$ ($\because$ $R_1$ and $R_2$ are proportional)

**Example:-** Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

**Solution:-** $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

$$\begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

**Property – 5**

If some or all elements of a row or column of a determinant are expressed as the sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

**Example:- (1)**

(i) $\begin{vmatrix} a_1+a_2 & b_1 & c_1 \\ a_3+a_4 & b_2 & c_2 \\ a_5+a_6 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_2 & c_2 \\ a_5 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_1 & c_1 \\ a_4 & b_2 & c_2 \\ a_6 & b_3 & c_3 \end{vmatrix}$

(ii) $\begin{vmatrix} a_1+k_1 & a_2+k_2 & a_3+k_3 \\ b_1 & b_2 & c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(iii) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+\lambda_1 & b_2 & b_3+\lambda_2 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda_1 & 0 & \lambda_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$

**Example – 2** Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

**Solution:-** $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

$= 0 + 0 = 0$ (by property 5 and property 3 and 4)

**Example:-** Using properties of determinants without expanding prove that $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

**Solution:-** $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$

$= 0 + 0 = 0$ RHS

**Property – 6**

If to each element of any row or column of a determinant, the equimultiples of corresponding elements of another row (or column) are added, then the value of the determinant remains the same, i.e the value of the determinant remains the same if we apply the operation

$R_i \to R_i + KR_j$ or $C_i \to C_i + KC_j$

**Verification:-** Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix}$ Now $\Delta_1$ can be written as

$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} \quad = \Delta + K\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$

$\Rightarrow \Delta_1 = \Delta + K.0$

$\Rightarrow \Delta_1 = \Delta$

**Note:-**

(a) We can add or subtract any two rows or any two columns of a determinant then the value of the determinant remains unchanged.

(b) We also can add the entire row or all columns at a time of a determinant remain unchanged.

**Example:-** $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+a_2 & a_2 & a_3 \\ b_1+b_2 & b_2 & b_3 \\ c_1+c_2 & c_2 & c_3 \end{vmatrix}$ $\qquad C_1 \to C_1 + C_2$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1-c_1 & b_2-c_2 & b_3-c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ $\qquad R_2 \to R_2 - R_3$

$= \begin{vmatrix} a_1+a_2+a_3 & a_2 & a_3 \\ b_1+b_2+b_3 & b_2 & b_3 \\ c_1+c_2+c_3 & c_2 & c_3 \end{vmatrix}$ $\qquad C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1+b_1+c_1 & a_1+b_1+c_1 & a_1+b_1+c_1 \end{vmatrix}$ $\qquad R_3 \to R_1 + R_2 + R_3$

**Example – 1** Without expanding, prove that $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

**Solution:-** $= \begin{vmatrix} x+y+z & x+y+z & z+x+y \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $\qquad R_1 \to R_1 + R_2$

Taking $x+y+z$ common from $R_1$

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ ($\because$ $R_1$ and $R_3$ are identical)

$= (x+y+z) \times 0 = 0$

**Problem – 1,** Using properties of the determinant show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

**Solution:-** LHS $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ $\qquad R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ Taking common $(a - b), (b - c)$ from $R_1$ and $R_2$ respect)

$$=\left[(a-b)(b-c)\right]\begin{vmatrix}0 & 1 & a+b\\0 & 1 & b+c\\1 & c & c^2\end{vmatrix}$$

$$=(a-b)(b-c)\left(0-0+1(b+c)-1(a+b)\right)$$ By expanding by $C_1$

$$=(a-b)(b-c)(b+c-a-b)$$

$$=(a-b)(b-c)(c-a)$$ RHS

**Problem – 2**

Using properties of determinants, show that $\begin{vmatrix}x & x^2 & yz\\y & y^2 & zx\\z & z^2 & xy\end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$

**Solution:-**

LHS $\begin{vmatrix}x & x^2 & yz\\y & y^2 & zx\\z & z^2 & xy\end{vmatrix}$ $R_1\to R_1-R_2,\ R_2\to R_2-R_3$

$$=\begin{vmatrix}x-y & x^2-y^2 & yz-zx\\y-z & y^2-z^2 & zx-xy\\z & z^2 & xy\end{vmatrix}$$ Taking $(x-y)(y-z)$ common from $R_1$ and $R_2$ respectively

$$=(x-y)(y-z)\begin{vmatrix}1 & x+y & -z\\1 & y+z & -x\\z & z^2 & xy\end{vmatrix}$$ $R_2\to R_2-R_1$

$$=(x-y)(y-z)\begin{vmatrix}1 & x+y & -z\\0 & z-x & z-x\\z & z^2 & xy\end{vmatrix}$$ Taking $(z-x)$ common from $R_2$

$$=(x-y)(y-z)(z-x)\begin{vmatrix}1 & x+y & -z\\0 & 1 & 1\\z & z^2 & xy\end{vmatrix}$$ Expanding by $C_1$

$$=(x-y)(y-z)(z-x)\left\{1(xy-z^2)-0+z(x+y+z)\right\}$$

$$=(x-y)(y-z)(z-x)\left\{xy-z^2+zx+zy+z^2\right\}$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)=\text{RHS}$$

**Problem – 3** Using properties of Determinants show that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

**Solution:-** LHS $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ $\qquad R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ Taking common $(5x+4)$ from $R_1$

$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ $\qquad C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$= (5x+4)\begin{vmatrix} 0 & 0 & 1 \\ x-4 & 4-x & 2x \\ 0 & x-4 & x+4 \end{vmatrix}$ Taking $(4-x)$ common from $C_2$

$= (5x+4)(4-x)\begin{vmatrix} 0 & 0 & 1 \\ x-4 & 1 & 2x \\ 0 & -1 & x+4 \end{vmatrix}$ Expanding by $R_1$ we get

$= (5x+4)(4-x)\{0-0+(-1)\}\{(x-4)-0\times 1\}$

$= (5x+4)(4-x)(4-x)$

$= (5x+4)(4-x)^2 = RHS$

**Problem – 4** Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

**Solution:-** LHS $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ $\qquad R_1 \to R_1 - R_2 - R_3$

$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ Expanding along $R_1$ we obtain

$= \Delta = 0 - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= 2c\left(ab + b^2 - bc\right) - 2b\left(bc - c^2 - ac\right)$

$= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc$

$= 4abc = RHS$

**Problem – 5** If x, y, z are different and $\Delta=\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$, then show that $1+xyz=0$

**Solution:-** $\Delta=\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}=0$ (By property – 5)

$\Rightarrow (-1)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}=0$

For 1st determinant using $(c_2 \leftrightarrow c_2 \text{ and then } c_1 \leftrightarrow c_2)$

For the 2nd determinant taking x, y, and z common from $R_1$, $R_2$, and $R_3$ respectively

$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)=0$ $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}=0$

Taking out a common factor $(y-x),(z-x)$ common from $R_2$ and $R_3$ respectively.

$\Rightarrow (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}=0$

$\Rightarrow (1+xyz)(y-x)(z-x)(z-y)=0$, expanding by $C_1$

Since, $\Delta=0$ and x, y, z are all different i.e $x-y\neq 0, y-z\neq 0, z-x=0$

$\therefore \Delta=(1+xyz)=0$ (Proved)

**Problem – 6** Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc+bc+ca+ab$

**Solution:-** LHS = $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking a, b, and c common from $R_1$, $R_2$, and $R_3$ respectively

$$=abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ Applying $R_1 \to R_1 + R_2 + R_3$

$$=abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from

$$=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix}$$ Expanding by $R_1$ we get

$$=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc+bc+ca+ab=RHS$$

**Problem No. – 7:-**

Prove by using properties of determinants

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}=\left(1+a^2+b^2\right)^3$$

**Solution:-** From LHS, taking b and a common from $C_1$ and $C_2$ respectively we ger.

$$=b.a\begin{vmatrix} \frac{1+a^2-b^2}{b} & 2b & -2b \\ 2a & \frac{1-a^2+b^2}{a} & 2a \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix}$$ Now multiplying $b$ and $a$ in $R_1$ and $R_2$ respectively we get

$$=\begin{vmatrix} 1+a^2-b^2 & 2b^2 & -2b^2 \\ 2a^2 & 1-a^2+b^2 & 2a^2 \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix}$$ Applying $C_1 \to C_1 + C_2, C_2 \to C_2 + C_3$

$$=\begin{vmatrix}1+a^2+b^2 & 0 & -2b^2\\ 1+a^2+b^2 & 1+a^2+b^2 & 2a^2\\ 0 & -(1+a^2+b^2) & 1-a^2-b^2\end{vmatrix}$$ Taking common $(1+a^2+b^2)$ from both $C_1$ and $C_2$ we get

$$=(1+a^2+b^2)^2\begin{vmatrix}1 & 0 & -2b^2\\ 1 & 1 & 2a^2\\ 0 & -1 & 1-a^2-b^2\end{vmatrix}$$ Now expanding by $C_1$ we get

$$=(1+a^2+b^2)^2\left\{1\left\{(1-a^2-b^2)+2a^2\right\}-1(0-2b^2)+0\right\}$$

$$=(1+a^2+b^2)(1+a^2+b^2)=(1+a^2+b^2+c^2)^3$$

**Problem No. – 8**

Prove by using properties of determinant that $\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$

**Solution:-** From LHS taking a, b, and c common from $R_1$, $R_2$, and $R_3$ respectively.

$$=abc\begin{vmatrix}\frac{a^2+1}{a} & b & c\\ a & \frac{b^2+1}{b} & c\\ a & b & \frac{c^2+1}{c}\end{vmatrix}$$ Taking a, b, and c multiplying in $C_1$, $C_2$, and $C_3$ respectively

$$=\begin{vmatrix}a^2+1 & b^2 & c^2\\ a^2 & b^2+1 & c^2\\ a^2 & b^2 & c^2+1\end{vmatrix}$$ $C_1\to C_1+C_2+C_3$

$$=\begin{vmatrix}1+a^2+b^2+c^2 & b^2 & c^2\\ 1+a^2+b^2+c^2 & b^2+1 & c^2\\ 1+a^2+b^2+c^2 & b^2 & c^2+1\end{vmatrix}$$ Taking $(1+a^2+b^2+c^2)$ common from $C_1$

$$=(1+a^2+b^2+c^2)\begin{vmatrix}1 & b^2 & c^2\\ 1 & b^2+1 & c^2\\ 1 & b^2 & c^2+1\end{vmatrix}$$ $R_1\to R_1-R_2,\ R_2\to R_2-R_3$

$$=(1+a^2+b^2+c^2)\begin{vmatrix}0 & -1 & 0\\ 0 & 1 & -1\\ 1 & b^2 & c^2+1\end{vmatrix}=(1+a^2+b^2+c^2)\times 1=1+a^2+b^2+c^2$$ (By expanding by $R_1$)

**Problem – 9:-** If a, b, and c are positive and unequal, show that the value of the determinant $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is –ve

**Solution:-** Applying $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$ Taking $(a+b+c)$ common from $C_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$ $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$ Expanding by $C_1$ we get

$$= (a+b+c)\left[(c-b)(b-c) - (a-c)(a-b)\right]$$

$$= (a+b+c)\left(-a^2 - b^2 - c^2 + ab + bc + ca\right)$$

$$= -\frac{1}{2}(a+b+c)\left(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca\right)$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

Which is –ve $a+b+c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

**Problem – 10:-** If a, b, and c are in AP find value of $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+b & 7y+9 & 10y+c \end{vmatrix}$

**Solution:-** Give a, b, c are in AP so, $2b = a + c$ ..................................... (1)

From LHS Applying $R_1 \to R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 6y+10-6y-10 & 12y+16-12y-16 & 18y+a+c-18y-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$ (From equation 1)

= 0

**Problem -11:-** Show that $\Delta=\begin{vmatrix}(y+z)^2 & xy & zx\\ xy & (x+z)^2 & yz\\ xz & yz & (x+y)^2\end{vmatrix}=2xyz(x+y+z)^3$

**Solution:-** From LHS $R_1\to xR_1, R_2\to yR_2, R_3\to zR_3$ and dividing by $xyz$ we get

$$=\frac{1}{xyz}\begin{vmatrix}x(y+z)^2 & x^2y & x^2z\\ xy^2 & y(x+z)^2 & y^2z\\ xz^2 & yz^2 & z(x+y)^2\end{vmatrix}$$

Taking common x, y, and z from $C_1$, $C_2$ and $C_3$ respectively, we get

$$=\frac{xyz}{xyz}\begin{vmatrix}(y+z)^2 & x^2 & x^2\\ y^2 & (x+z)^2 & y^2\\ z^2 & z^2 & (x+y)^2\end{vmatrix} \qquad C_2\to C_2-C_1,\ C_3\to C_3-C_1$$

$$=\begin{vmatrix}(y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2\\ y^2 & (x+z)^2-y^2 & 0\\ z^2 & 0 & (x+y)^2-z^2\end{vmatrix} \qquad \text{Taking common factors } (x+y+z) \text{ from } C_2 \text{ and } C_3$$

$$=(x+y+z)^2\begin{vmatrix}(y+z)^2 & x-(y+z) & x-(y+z)\\ y^2 & (x+z)-y & 0\\ z^2 & 0 & (x+y)-z\end{vmatrix} \qquad \text{Applying } R_1-(R_2+R_3)$$

$$=(x+y+z)^2\begin{vmatrix}2yz & -2z & -2y\\ y^2 & x-y+z & 0\\ z^2 & 0 & x+y-z\end{vmatrix} \qquad \text{Applying } C_2\to\left(C_2+\frac{1}{y}C_1\right) \text{ and } C_3\to\left(C_2+\frac{1}{z}C_1\right)$$

$$=(x+y+z)^2\begin{vmatrix}2yz & 0 & 0\\ y^2 & x+y & \frac{y^2}{z}\\ z^2 & \frac{z^2}{y} & x+y\end{vmatrix} \qquad \text{Expanding by } R_1\text{, we get}$$

$$=(x+y+z)^2\,2yz\big((x+z)(x+y)-yz\big)=(x+y+z)^2\,2yz\left(x^2+xy+xz\right)$$

$$=(x+y+z)^3\,2xyz$$

**Aria of a Triangle:-** In co-ordinate Geometry we have studied that area of a $\Delta$ whose vertices are $(x_1,y_1),(x_2,y_2)$ and $(x_3,y_3)$ is given by the formula. $\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$. Now we can express in the form of a determinant as $=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$

**Note:-**

(a) Area is always a positive quantity

(b) Area of the triangle whose vertices are three collinear points is zero.

**Example – 1**

Find the area of the triangle formed by the points $(3,8),(-4,2)\text{and}(5,1)$.

**Solution:-**

The area of the triangle formed by

$$\Delta=\frac{1}{2}\begin{vmatrix}3&8&1\\-4&2&1\\5&1&1\end{vmatrix}=\frac{1}{2}\left[3(2-1)-8(-4-5)+1(-4-10)\right]$$

$$=\frac{1}{2}[3+72-14]=\frac{61}{2}\text{sq.units}$$

**Example – 2**

Show that the points $A(a,b+c),B(b,c+a)\text{and}C(c,a+b)$ are collinear.

**Solution:-**

To prove three points collinear, we have to prove the area formed by them is zero.

$$=\frac{1}{2}\begin{vmatrix}a&b+c&1\\b&c+a&1\\c&a+b&1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}a-b&b-a&0\\b-c&c-b&0\\c&a+b&1\end{vmatrix}\qquad(\because R_1\to R_1-R_2,R_2\to R_2-R_3)$$

Expanding by $C_3$

$$=\frac{1}{2}\{0-0+1((a-b)(c-b)-(b-c)(b-a))\}$$

$$=\frac{1}{2}\{0-0+ac-ab-bc+b^2-b^2+ab+bc-ca\}$$

$$=\frac{1}{2}\times0=0$$ Hence, collinear.

**Example – 3**

Find the value of K if the area of a triangle is 4 sq. Units and verities are (K, 0), (4, 0), (0, 2).

**Solution:-** ATQ $\frac{1}{2}\begin{vmatrix}K&0&1\\4&0&1\\0&2&1\end{vmatrix}=4$

$\Rightarrow-0+0-2(K-4)=8$ (By expanding $C_2$)

$\Rightarrow-2K+8=8$

$\Rightarrow K=0$

**Example – 4**

Find the equation of the line joining (1, 2) and (3, 6) using determinant

**Solution:-** Let P(x,y) be and point on the line given A(1, 2), B (3, 6).

As A, P, and B points are collinear

$$\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1(y-6) - 2(x-3) + 1(6x-3y) = 0$$

$$\Rightarrow y - 6 - 2x + 6 + 6x - 3y = 0$$

$$\Rightarrow 4x - 2y = 0$$

$$\Rightarrow 2x - y = 0$$

Which is equal to line

**Minors and co-factors:-**

Definition of minor:- Minor of an element $a_{ij}$ of a determinant is the determinant obtained by deleting its ith row and jth column in which element $a_{ij}$ lies.

Minor of the element $a_{ij}$ is denoted by $M_{ij}$.

**Example:-** Find minor of $a_{12}$, $a_{23}$, $a_{22}$, $a_{31}$ of $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

**Solution:-**

Minor of $a_{12} = M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6$

Minor of $a_{23} = M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$

Minor of $a_{22} = M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 21 = -12$

Minor of $a_{31} = M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$

**Definition of Co-factor:-**

Co-factor of an element $a_{ij}$, denoted by $A_{ij}$ or $C_{ij}$ is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where $M_{ij}$ is minor of $a_{ij}$

**Example:-** Find the minor and cofactors of elements of $\Delta = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$

**Solution:-**

Minor of $a_{11} = M_{11} = 4$

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} M_{11} = 1 \times 4 = 4$

Minor of $a_{12} = M_{12} = 3$

Cofactor of $a_{12} = C_{12} = (-1)^{1+2} M_{12} = (-1).3 = -3$

Minor of $a_{21} = M_{21} = -2$

Cofactor of $a_{21} = A_{21} = (-1)^{2+1} M_{21} = (-1).(-2) = 2$

Minor of $a_{22} = M_{22} = 1$

Cofactor of $a_{22} = A_{22} = (-1)^{2+2} M_{22} = 1 \times 1 = 1$

**Note:-**

If elements of a row (or column) are multiplied with cofactors of any other, row (column) then their sum is zero.

**Example:-** $\Delta = a_{11}A_{21} + a_{12}A_{22} + a_{13}.A_{23}$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$ (Because $R_1$ and $R_2$ are identical)

**Note:-** If elements of a row (or column) are multiplied with cofactors of the same row (or column) then their sum is $|A|$.

**Example:-**

Find minor and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$.

**Answer:-**

$$M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = 20,\ A_{11} = (-1)^{1+1} = -20$$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46,\ A_{12} = (-1)^{1+2}(-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30,\ A_{13} = (-1)^{1+3}.30 = 30$$

$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4,\ A_{21} = (-1)^{2+1}(-4) = -4$

$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19,\ A_{22} = (-1)^{2+2}(-19) = -19$

$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13,\ A_{23} = (-1)^{2+3}(13) = -13$

$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12,\ A_{31} = (-1)^{3+1}(-12) = -12$

$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22,\ A_{32} = (-1)^{3+2}(-22) = 22$

$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18,\ A_{33} = (-1)^{3+3}(18) = 18$

So, $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2(-12) + (-3)22 + 5(18) = -24 - 66 + 90 = 0$

**Homework:-** Exercise – 4.4 NCERT Book

**Adjoint and Inverse of a Matrix: -**

Adjoint of a matrix:- The adjoint of a square matrix $A = \left[a_{ij}\right]_{n\times n}$ is defined as the transpose of the matrix $\left[A_{ij}\right]_{n\times n}$, where $A_{ij}$ is the cofactor of the element $a_{ij}$. Adjoint of a matrix A is denoted by $\text{adj}(A)$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

**Example – 1** Find adj A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

**Answer:-** We have $A_{11} = (-1)^{1+1} 4 = 4,\ A_{12} = (-1)^{1+2} 1 = -1$

$A_{21} = (-1)^{2+1}.3 = -3 \quad A_{22} = (-1)^{2+2} 2 = 2$

Hence $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

$\Rightarrow \text{adj}(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

**Note:-** For a square matrix of order 2 given by $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The $adj\ (A)$ can also be obtained by interchanging $a_{11}$ and $a_{22}$ and be changing the signs of $a_{12}$ and $a_{21}$ That is adjoint of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

**Example:-** Find the adjoint of the matrix. $A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

**Solution:-**

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}=1(3-0)=3$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}=-1(2+10)=-12$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}=1(0+6)=6$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}=(-1)(-1-0)=1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}=1(1+4)=5$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}=(-1)(0-2)=2$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix}=1(-5-6)=-11$$

$$A_{32}=(-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}=(-1)(5-4)=-1$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}=1(3+2)=5$$

$$\text{So, } adj(A)=\begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}=\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

**Theorem – 1**

If A be any given square matrix of order n, then

$A(adj(A)) = adj(A).\ A = |A|.I$

Where $I$ is the identity matrix of order 'n'.

**Proof:-**

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow adj(A)\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{33} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Now, $A.adj(A)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

$=|A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad = |A| = I$

Similarly, we can prove $adj(A).A=|A|.I$

Hence, $A.adj(A)=|A|.J=adj(A).A$

**Note:-** A square matrix A is said to be a singular matrix if |A| = 0. If $|A|\neq 0$ then A is said to be a non-singular matrix.

**Example:-** If $A=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ Here $|A|=4-4=0$

So, A is a singular matrix

If $A=\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

Here, $|A|=12-10=2\neq 0$ so, A is a non-singular matrix

**Theorem:-** If A and B are non-singular matrices of the same order then AB and BA are also non-singular matrices of the same order.

**Example:-** Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}, B=\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}_{2\times 2}$

Here $|A|\neq 0$ and $|B|\neq 0$

$AB=\begin{bmatrix} 1\times 2+2\times 3 & 1\times 4+2\times 1 \\ 3\times 2+4\times 3 & 3\times 4+4\times 3 \end{bmatrix}=\begin{bmatrix} 8 & 6 \\ 18 & 16 \end{bmatrix}$

Here $|AB|\neq 0$ i.e. $8\times 16-18\times 6=128-108=20\neq 0$

$BA=\begin{bmatrix} 2\times 1+3\times 46 & 2\times 2+4\times 4 \\ 3\times 1+1\times 3 & 3\times 2+1\times 4 \end{bmatrix}=\begin{bmatrix} 14 & 20 \\ 6 & 10 \end{bmatrix}$

Here $|BA| \neq 0$ i.e $14 \times 10 - 20 \times 6$

$= 140 - 120 = 20 \neq 0$

Hence, not singular

**Theorem:-**

The determinant of the product of matrices is equal to the product of their respective determinants i.e. $|AB| = |A|.|B|$ .where A and B are square matrices of the same order.

**Example:-** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$|A| = 2 - 6 = -4$ and $|B| = 4 - 3 = 1$

So, RHS = $|A||B| = -4 \times 1 = -4$

Now, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1\times2+2\times3 & 1\times1+2\times2 \\ 3\times2+2\times3 & 3\times1+2\times2 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ 12 & 7 \end{bmatrix}$

$\Rightarrow |AB| = 56 - 60 = -4 = LHS$

So, $|AB| = |A|\,|B|$

**Remember:-** We know that $(adjA).A = |A|.I$

$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$

Taking determinants on both sides

$|adj(A)| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$

$\Rightarrow |adj(A)| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\Rightarrow |adj(A)||A| = |A|^3 \times 1$

$\Rightarrow |adj(A)| = |A|^2$

In general, if A is a non-singular matrix of order 'n' then $|adj(A)| = |A|^{n-1}$

**Theorem – 4** A square matrix A is invertible if and only if A is a non-singular matrix

**Proof:-** Let A be an invertible matrix of order n and $I$ be the identity matrix of order n.

Hence, there exists a square matrix B of order n such that $AB = BA = I$

Now, $AB = I \Rightarrow |AB| = |I|$

$\Rightarrow |A|.|B| = 1$ $\quad (\therefore |AB| = |A|.|B| \text{ and } |I| = 1)$

Here, $|A| \neq 0$, hence A is non-singular

Conversely, let A be non-singular

Then $|A| \neq 0$

Now $A(adjA) = adj(A).A = |A|\,I$

$$\Rightarrow A\left(\frac{1}{|A|}adjA\right) = \left(\frac{1}{|A|}adj(A)\right)A = I$$

Or $AB = BA = I$, Where $B = \frac{1}{|A|}adj(A)$

This A is invertible and $A^{-1} = \frac{1}{|A|}adjA$

**Example:-** If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A \text{ adj } A = |A|\,I$. Also, find $A^{-1}$

**Answer:-** We have $|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1 \neq 0$

Cofactors

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 1(16-9) = 7$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1(4-3) = -1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1(3-4) = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -1(12-9) = -3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1(4-3) = 1$$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 1(9-12) = -3$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -1(3-3) = 0$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1(4-3) = 1$

Here $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$

$\Rightarrow \text{adj}(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A.(\text{adj}(A)) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$\Rightarrow A.\text{adj}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.I$

So, $A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

**Question – 1** If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then verify that $(AB)^{-1} = B^{-1}A^{-1}$

**Solution:-** $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$\Rightarrow |AB| = 14 - 25 = -11 \neq 0$

So, $(AB)^{-1}$ exists

$\therefore (AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Now, $|A| = -8 - 3 = -11 \neq 0$

$|B| = 3 - 2 = 1 \neq 0$

So, $A^{-1}, B^{-1}$ both exist and are given by

$A^{-1} = \frac{-1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

Now, $B^{-1}.A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$= -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

$\therefore (AB)^{-1} = B^{-1}.A^{-1}$

**Question -2**

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = 0$, where I is $2\times2$ the identity matrix and O is $2\times2$ a null matrix. Using this equation, find $A^{-1}$.

**Solution:-** We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$\therefore A^2 - 4A + I = 0 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now, $A^2 - 4A + I = 0$

$\Rightarrow A.A - 4A = -I$

Multiplying $A^{-1}$ both sides (post multiplication)

$\Rightarrow A.A(A^{-1}) - 4A.A^{-1} = -IA^{-1}$

$\Rightarrow AI - 4I = -(A^{-1})$

$\Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

**Question – 3** For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$

**Solution:-** $A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now $A^2 + aA + bI = 0$

Post multiplying $A^{-1}$ both sides

$\Rightarrow (A.A)A^{-1} + a.A.A^{-1} + bI.A^{-1} = O.A^{-1}$

$\Rightarrow A.\left(A.A^{-1}\right)+aI+bA^{-1}=0$

$\Rightarrow AI+aI+bA^{-1}=0$

$\Rightarrow A+aI=-bA^{-1}$

$\Rightarrow A^{-1}=-\frac{1}{b}(A+aI)$ ............................................. (1)

Now $A^{-1}=\frac{1}{|A|}adj(A)=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$

We have from equation (1)

$\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left[\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}+\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right]$

$\Rightarrow\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}$

$\Rightarrow\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}-\frac{(3+a)}{b} & \frac{2}{b}\\ \frac{1}{b} & \frac{1+a}{b}\end{bmatrix}$

By equality of matrix

$-\frac{1}{b}=-1\Rightarrow b=1$ and $\frac{-3-a}{b}=1\Rightarrow -3-a=1$

$\Rightarrow a=-4$

**Question – 4** For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3-6A^2+5A+11I=0$ hence, find $A^{-1}$.

**Solution:-** Given $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

So, $A^2=A.A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$

Now, $A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -2 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Now, $A^3 - 6A^2 + 5A + 11\,I = 0$

$\Rightarrow A.A.A - 6A.A + 5A + 11\,I = 0$

Post multiplying $A^{-1}$ both sides

$\Rightarrow A.A.A.A^{-1} - 6\,A.A.A^{-1} + 5A.A^{-1} + 11\,I.A^{-1} = 0$

$\Rightarrow A.A.I - 6A.I + 5I + 11A^{-1} = 0$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(-A^2 + 6A - 5\,I\right)$$

$$= \frac{-1}{11}\left(A^2 - 6A + 5\,I\right)$$

$$= -\frac{1}{11}\left[\begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix} - 6\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{-1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}$$

**Question – 5**

Let A be a non-singular square matrix of order $3 \times 3$. Then $|adjA|$ is equal to

(a) $|A|$ (b) $|A|^2$ (c) $|A|^3$ (d) $3|A|$

**Solution:-** Here n = 3

We have $|adj(A)| = |A|^{3-1}$

$\Rightarrow |adj(A)| = |A|^2$

**Question – 6**

If A is an invertible matrix of order 2 then $\det(A^{-1})$ is equal to

(a) det (A) (b) $\frac{1}{\det(A)}$ (c) 1 (d) 0

**Solution:-** As A is an invertible matrix

$\therefore$ $A^{-1}$ exists and $A^{-1} = \frac{1}{|A|}adj(A)$

As matrix A is of order 2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc$ and $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|}adj(A) = \begin{bmatrix} \frac{d}{|A|} & -\frac{b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2}(ad - bc)$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2} \times |A| = \frac{1}{|A|}$$

So, $\det\left(A^{-1}\right)=\dfrac{1}{\det(A)}$

**Theorem:-** If A and B are non-singular square matrices of the same order, then

$\text{adj}(AB)=\text{adj}(B).\text{adj}(A)$

**Proof:-**

Since A and B are non-singular square matrices of the same order. Hence, AB exists

$\therefore |AB|=|A|.|B|\neq 0 \qquad (\therefore |A|\neq 0, |B|\neq 0)$

We know that $(AB)(\text{adj}(AB))=|AB|.In$ ...................................... (1)

Also $(AB)(\text{adj}B.\text{adj}A)=A(B.\text{adj}B).\text{adj}A$

$=a.(|B|.In)\text{adj}A$

$=|B|(A \text{ adj } A)$

$=|B|(|A|In)$

$=|AB|\, I_n$

$\Rightarrow (AB)(\text{adj}B.\text{adj}A)=|AB|I_n$ ............................. (2)

From equations (1) and (2) we get

$(AB)(\text{adj}AB)=(AB).(\text{adj}B.\text{adj}A)$

Pre multiplying $(AB)^{-1}$ both sides

$\Rightarrow (AB)^{-1}((AB)(\text{adj}AB))=(AB)^{-1}((AB)(\text{adj}B.\text{adj}A))$

$\Rightarrow I\text{adj}(AB)=I(\text{adj}B.\text{adj}A)$

$\Rightarrow \text{adj}(AB)=\text{adj}B.\text{adj}A$

**Theorem:-** If A is an invertible square matrix then $\text{adj } A^T = (\text{adj } A)^T$

**Proof:-** Since A is an invertible matrix $|A| \neq 0 \Rightarrow |A^T| \neq 0$

$\Rightarrow A^T$ is invertible $\quad (\because |A^T| = |A|)$

We have

$A.\,\text{adj } A = |A|.\, I_n$

$\Rightarrow [A.\text{adj}(A)]^T = [|A|\, I_n]^T$

$\Rightarrow (\text{adj } A)^T.A^T = |A|.I_n$ ............................. (1)

Again $(\text{adj } A^T).A^T = |A|.I_n$

$\Rightarrow (\text{adj } A^T).A^T = |A|\, I_n$ ............................. (2)

From equations (1) and (2)

$(\text{adj}A^T).(A^T) = (\text{adj } A)^T (A^T)$

$\Rightarrow \text{adj } A^T = (\text{adj } A)^T$ (By canceling $A^T$ both sides)

**Theorem: -** Prove that adjoint of a symmetric matrix is also a symmetric matrix.

**Proof:-** Let A be a symmetric matrix $\Rightarrow A = A^T$

We know $(\text{adj } A)^T = (\text{adj } A^T)$

$\Rightarrow (\text{adj } A)^T = (\text{adj } A)$

$\therefore$ adj A is a symmetric matrix

**Theorem:-** If A is a non-singular square matrix, then $\text{adj}(\text{adj } A) = |A|^{n-2} A$

**Proof:-**

We have $B(\text{adj } B) = |B|.I_n$

Replacing B by adj (A)

We have $(\text{adj } A)[\text{adj}(\text{adj}(A))] = |\text{adj}A|.I_n$

$\Rightarrow (\text{adj}A)[\text{adj}(\text{adj}A)] = |A|^{n-1}.I_n$

Pre-multiplying A on both the side

$\Rightarrow A.(\text{adj}A)[\text{adj}(\text{adj } A)] = A.|A|^{n-1} I_n$

$\Rightarrow (A.\text{adj}A)(\text{adjadj}A) = |A|^{n-1} A\, I_n$

$\Rightarrow |A| I_n (\text{adj adj} A) = |A|^{n-1}.A$

$\Rightarrow |A| (I_n (\text{adj adj} A)) = |A|^{n-1}.A$

$\Rightarrow |A| (\text{adj adj } A) = |A|^{n-1}.A$

$\Rightarrow \text{adj adj } A = |A|^{n-2}.A$

**Theorem:-** If A is a non-singular matrix of order n, then $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

**Proof:-** We know that $\text{adj}(\text{adj } A) = |A|^{n-2}$

$\Rightarrow |\text{adj}(\text{adj } A)| = \left| |A|^{n-2} A \right|$

$\Rightarrow |\text{adj}(\text{adj } A)| = |A|^{n(n-2)} |A|$

$\Rightarrow |\text{adj } (\text{adj } A)| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$

**Problem:-**

01. If a is an invertible matrix of order $3 \times 3$ such that $|A| = 2$, then find $\text{adj}(\text{adj } A)$.

    **Answer:-** $\text{adj}(\text{adj } A) = |A|^{3-2} A = |A|.A = 2A$

02. If A is a square matrix of order 3 such that $|A| = 2$, then write the value of $|\text{adjadj} A|$

    **Answer:-** $|\text{adjadj} A| = |A|^{(n-1)^2} = 2^{(3-1)^2} = 2^4 = 16$

03. If $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, then find, $|\text{adj adj } A|$

    **Answer:-** Try it. (1)

**Question – 1** If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \text{and} B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ then verify that $(AB)^{-1} = B^{-1}A^{-1}$

**Solution:-** $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$\Rightarrow |AB| = 14 - 25 = -11 \neq 0$

So, $(AB)^{-1}$ exists

$\therefore (AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Now, $|A| = -8 - 3 = -11 \neq 0$

$|B| = 3-2 = 1 \neq 0$

So, $A^{-1}, B^{-1}$ both exist and are given by $A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

Now, $B^{-1}.A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$= -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

$\therefore (AB)^{-1} = B^{-1}.A^{-1}$

**Question – 2**

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4 + I = 0$, where I is $2\times 2$ the identity matrix and O is $2\times 2$ the null matrix. Using this equation, find $A^{-1}$.

**Solution:-** We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$\therefore\ A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now, $A^2 - 4A + I + 0$

$\Rightarrow A.A - 4A = -1$

Multiplying $A^{-1}$ both sides (post multiplication)

$\Rightarrow A.A(A)^{-1} - 4A.A^{-1} = -IA^{-1}$

$\Rightarrow AI - 4I = -(A^{-1})$

$\Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

**Question – 3** For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$

**Solution:-** $A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now $A^2 + aA + bI = 0$

Post multiplying $A^{-1}$ both sides

$\Rightarrow (A.A)A^{-1} + a.A.A^{-1} + bI.A^{-1} = 0.A^{-1}$

$\Rightarrow A.(A.A^{-1}) + aI + bA^{-1} = 0$

$\Rightarrow AI + aI + bA^{-1} = 0$

$\Rightarrow A + aI = -bA^{-1}$

$\Rightarrow A^{-1} = \frac{-1}{b}(A + aI)$ .................................... (1)

Now $A^{-1} = \frac{1}{|A|}adj(A) = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

We have from (1)

$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b}\left[\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right]$

$\Rightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b}\begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-(3+a)}{b} & \frac{2}{b} \\ \frac{1}{b} & \frac{1+a}{b} \end{bmatrix}$

By equality of matrix $-\frac{1}{b} = -1 \Rightarrow b = 1$ and $\frac{-3-a}{b} = 1 \Rightarrow -3 - a = 1 \Rightarrow a = -4$

**Question No. – 4** For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I = 0$ find $A^{-1}$.

**Solution:-** Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

So, $A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

Now, $A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24+42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$\Rightarrow A^3=\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$\therefore A^3-6A+5A+11I$$

$$=\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}-6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}+5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}+11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -169 \\ 32 & -13 & 58 \end{bmatrix}-\begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}+\begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix}+\begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$=\begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}-\begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0 \qquad \text{Now, } A^3-6A^2+5A+11\,I=0$$

$$\Rightarrow A.A.A.A^{-1}-6A.A.A^{-1}+5A.A^{-1}+11I.A^{-1}=0$$

$$\Rightarrow A.A.I-6A.I+5I+11A^{-1}=0$$

$$\Rightarrow 11A^{-1}=-A^2+6A-5I$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left(-A^2+6A-5I\right)$$

$$=\frac{-1}{11}\left(A^2-6A+5I\right)$$

$$=\frac{-1}{11}\left[\begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix}\right]-6\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}+5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1}=\frac{-1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}=\frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}$$

**Question No. – 05**

Let A be a non-singular square matrix of order $3\times3$. Then $|\text{adj}A|$ is equal to

(a) $|A|$ (b) $|A|^2$ (c) $|A|^3$ (d) $3|A|$

**Solution:-** We have $|\text{adj}(A)|=|A|^{3-1}$

$\Rightarrow |\text{adj}(A)|=|A|^2$

**Question No. – 06**

If A is an invertible matrix of order 2 then $\det(A^{-1})$ is equal to

(a) det (A) (b) $\dfrac{1}{\det(A)}$ (c) 1 (d) 0

**Solution:-** As a is an invertible matrix

$\therefore\ A^{-1}$ exists and $A^{-1}=\dfrac{1}{|A|}\text{adj}(A)$

As matrix a is of order 2

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A|=ad-bc$ and adj $A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So, $A^{-1}=\dfrac{1}{|A|}\text{adj}(A)=\begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$

$\Rightarrow |A^{-1}|=\dfrac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}=\dfrac{1}{|A|^2}(ad-bc)$

$\Rightarrow |A^{-1}|=\dfrac{1}{|A|^2}\times|A|=\dfrac{1}{|A|}$

So, $\det(A^{-1})=\dfrac{1}{\det(A)}$

**Theorem:-**

If A and B are non-singular square matrices of the same order, then $\text{adj}(AB)=\text{adj}(B).\text{adj}(A)$

**Proof:-** Since A and B are non-singular square matrices of the same order.

Hence, AB exists

$\therefore |AB|=|A|.|B|\neq 0 \qquad (\because |A|\neq 0, |B|\neq 0)$

We know that $(AB).(adj(AB)) = |AB|.I_n$ ........................................ (1)

Also $(AB)(adjB.adjA) = A(B.adjB).adjA$

$= A.(|B|.I_n)adjA$

$= |B|(A\ adjA)$

$= |B|(|A|I_n)$

$= |A||B|\ I_n$

$= |AB|\ I_n$

$\Rightarrow (AB)(adjB.\ adjA) = |AB|I_n$ ...................................... (2)

From equations (1) and (2) we get

$(AB)(adjAB) = (AB).(adjB.adjA)$

Pre multiplying (AB) on both sides

$\Rightarrow (AB)^{-1}((AB)(adj\ AB)) = (AB)^{-1}((AB)(adjB.adjA))$

$\Rightarrow Iadj(AB) = I(adjB.adjA)$

$\Rightarrow adj(AB) = adjB.adjA$

**Theorem:-** If A is an invertible square matrix then $adj\ A^T = (adj\ A)^T$

**Proof:-** Since A is an invertible matrix

$\therefore |A| \neq 0 \Rightarrow |A^T| = 0$

$\Rightarrow A^T$ is invertible $\quad (\because |A^T| = |A|)$

We have $A.adjA = |A|.\ I_n$

$\Rightarrow [A.adj(A)]^T = [|A|\ I_n]^T$

$\Rightarrow (adjA)^T.A^T = |A|.\ I_n$ ...................................... (1)

Again $(adj.A^T).A^T = |A^T|.\ I_n$

$\Rightarrow (adj.A^T).A^T = |A|.I_n$ ................................. (2)

From equations (1) and (2)

$(adj\ A^T).(A^T) = (adjA)^T(A)^T$

$\Rightarrow adj\ A^T = (adj\ A)^T$ By canceling $A^T$ both sides

**Theorem: -** Prove that adjoint of a symmetric matrix is also a symmetric matrix.

**Proof:-** Let a be a symmetric matrix $\Rightarrow A = A^T$

We know $(adjA)^T = (adj\ A^T)$

$\Rightarrow (adj\ A)^T = (adj\ A)$

$\therefore$ adj(A) is a symmetric matrix.

**Theorem:-** If A is a non-singular square matrix, then $adj(adj\ A) = |A|^{n-2} A$

**Proof:-** We have $B(adj\ B) = |B|\ I_n$ (for n order b matrix)

Replacing b by adj(A)

We have $(adj\ A)\left[adj(adj(A))\right] = |adj\ A|.I_n$

$\Rightarrow (adjA)\left[adj(adjA)\right] = |A|^{n-1} I_n$

Pre-multiplying A on both the side

$\Rightarrow A.(adja)\left[adj(adjA)\right] = A.|A|^{n-1}.I_n$

$\Rightarrow (A.adjA)(adjadjA) = |A|^{n-1} AI_n$

$\Rightarrow |A|\ I_n (adjadjA) = |A|^{n-1}.A$

$\Rightarrow |A|(I_n (adj\ adj\ A)) = |A|^{n-1} A$

$\Rightarrow |A|(adj\ adjA) = |A|^{n-1}.A$

$\Rightarrow adj\ adj = |A|^{n-1}.A$

**Theorem:-**

If A is a non-singular matrix of order n, then $|ad(adjA)| = |A|^{(n-1)^2}$

**Proof:-**

We know that $adj(adj\ A) = |A|^{n-2} A$

$\Rightarrow |adj(adjA)| = \left||A|^{n-2} A\right|$

$\Rightarrow |adj(adj\ A)| = |A|^{n(n-2)} |A|$

$\Rightarrow |adj(adjA)| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$

**Problems:-**

01. If A is an invertible matrix of order $3\times3$ such that $|A|=2$, then find $adj(adjA)$

Answer:- $adj(adj\ A)=|A|^{3-2}A=|A|.A=2A$

02. If a is a square matrix of order 3 such that $|A|=2$, then write the value of $|adjadjA|$

Answer:- $|adjadjA|=|A|^{(n-1)^2}=2^{(3-1)^2}=2^4=16$

03. If $A=\begin{bmatrix}3 & 0 & -1\\ 2 & 3 & 0\\ 0 & 4 & 1\end{bmatrix}$, then find, $|adj\ adj\ A|$

**Applications of determinants and Matrices:-**

We shall discuss the application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

**Consistent System:-**

A system of equations is said to be consistent if its solution (one or more) exists.

**Inconsistent system:-**

A system of equations is said to be inconsistent if its solution does not exist.

**Remark:-**

In this chapter, we restrict ourselves to the system of linear equations having unique solutions only i.e only one solution.

**Solution of a system of linear equations using the inverse of the matrix:-**

Consider the system of equations

$a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Let $A=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_3\\ a_3 & b_3 & c_3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} and B=\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}$

Then the system of equations can be written as $AX=B$

i.e. $\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}$

**Case – 1** If A is a non-singular matrix, then the inverse exists. Now

$AX = B$

$\Rightarrow A^{-1}(AX) = A^{-1}B$ (By pre-multiplying by $A^{-1}$)

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow X = A^{-1}B$

This method of solving a system of equations is known as the Matrix method.

**Case – 2** If A is a singular matrix, then $|A|=0$ in this case, we calculate $(adjA).B$

If $(adjA)B \neq 0$ (0 is a zero matrix) then the solution does not exist and the system of equations is called inconsistent.

If $(adjA)B = 0$, then the system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

**Example – 1** Solve the system of equations $2x+5y=1$, $3x+2y=7$

**Solution:-** Given $2x+5y=1$ $3x+2y=7$

Let $A=\begin{bmatrix}2 & 5\\ 3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}, B=\begin{bmatrix}1\\ 7\end{bmatrix}$

So, $AX = B$

$\Rightarrow A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$ .............................................. (1)

Here $|A| = 4 - 15 = -11 \neq 0$

So $A^{-1}$ exists

So has a unique solution

Now $adj(A) = \begin{bmatrix}2 & -5\\ -3 & 2\end{bmatrix}$

So, $A^{-1} = \frac{1}{(-11)}\begin{bmatrix}2 & -5\\ -3 & 2\end{bmatrix}$ $\left(\because A^{-1} = \frac{adj(A)}{|A|}\right)$

From equation (1) we get

$X = A^{-1}B$

$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix} = \frac{-1}{11}\begin{bmatrix}2 & -5\\ -3 & 2\end{bmatrix}\begin{bmatrix}1\\ 7\end{bmatrix} = \frac{-1}{11}\begin{bmatrix}-33\\ 11\end{bmatrix} = \begin{bmatrix}3\\ -1\end{bmatrix}$

Hence, $x = 3, y = -1$

**Example:- 2** Solve the system of equations $2x+3y=5$ $\quad 2x+3y=1$

**Solution:-** Here $|A|=\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}=6-6=0$

So, $A^{-1}$ can not be found out $\quad \therefore$ the system of the equation has no solution.

**Example:- 3**

Solve the following system of equations by matrix method $3x-2y+3z=8$, $2x+y-z=1$, $4x-3y+2z=4$

**Solution:-** Here $A=\begin{bmatrix}3 & -2 & 3\\ 4 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$

Here $|A|=3(2-3)+2(4+4)+3(-6-4)=-17\neq 0$

So, $A^{-1}$ can be found out

Co-factors:-

$A_{11}=-1$ $\quad A_{12}=-8$ $\quad A_{13}=-10$

$A_{21}=-5$ $\quad A_{22}=-6$ $\quad A_{23}=1$

$A_{31}=-1$ $\quad A_{32}=9$ $\quad A_{33}=7$

$$adj(A)=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

So, $adj(A)=\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$

$\therefore A^{-1}=\dfrac{adj(A)}{|A|}=\dfrac{1}{-17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$ $\qquad$ We have $AX=B$

$\Rightarrow A^{-1}.AX=A^{-1}B$

$\Rightarrow IX=A^{-1}B$

$\Rightarrow X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{-1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

$\Rightarrow \dfrac{-1}{17}\begin{bmatrix}-17\\ -34\\ -51\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ $\qquad$ So, By equality of matrix $\qquad x=1, y=2$ and $z=3$

**Home Work:-** Exercise – 4.6 (NCERT Book) Question number 01 to 14

**Question No. – 01**

The sum of three numbers is 6 if we multiply the third number by 3 and add the second number to it, we get 11. By adding the first and third numbers, we get double the second number, represent it algebraically and find the numbers using the matrix method.

**Solution:-** Let the first, 2nd, and 3rd numbers are $x, y$, and $z$ respectively.

ATQ $\quad x+y+0 \quad\quad y+3z=11 \quad\quad x+z=2y$

$\Rightarrow x-2y+z=0$

Let $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ -1 & -2 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}6\\ 11\\ 0\end{bmatrix}$

We have $AX=B$ ............................ (1)

Now, $|A|=1(1+6)-(0-3)+(0-1)=9\neq 0$

Now to find out adj A

Co-factors

$A_{11}1(1+6)=7 \quad\quad A_{12}=-(0-3)=3 \quad\quad A_{13}=-1$

$A_{21}=-3 \quad\quad A_{22}=0 \quad\quad A_{23}=3$

$A_{31}=2 \quad\quad A_{32}=-3 \quad\quad A_{33}=1$

So, $\text{adj } A=\begin{bmatrix}7 & 3 & -1\\ -3 & 0 & 3\\ 2 & -3 & 1\end{bmatrix}^T=\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}$

From equation (1) $\quad A^{-1}AX=A^{-1}B$

$\Rightarrow IX=A^{-1}B$

$\Rightarrow X=A^{-1}B$

So, $\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}6\\ 11\\ 0\end{bmatrix}$

$=\frac{1}{9}\begin{bmatrix}42-33+0\\ 18+0+0\\ -6+33+0\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9\\ 18\\ 27\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ So, x = 1, y = 2 and z = 3

**Problem – 2** If $A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$ then find $A^{-1}$. Using $A^{-1}$ solve the system of equations

$2x-3y+5z=11;\ 3x+2y-4z=-5; x+y-2z=-3$

**Solution:-** $|A|=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$

$=2(-4+4)+3(-6+4)+5(3-2)=0-6+5=-1\neq 0$

Therefore, A is a non-singular matrix so $A^{-1}$ exists

$C_{11}=0; C_{12}=2; C_{13}=1; C_{21}=-9; C_{23}=-5$ $\quad C_{31}=2; C_{32}=23; C_{33}=13$

$$\text{adj } A=\begin{bmatrix}0 & 2 & 1\\ -1 & -9 & -5\\ 2 & 23 & 13\end{bmatrix}^{T}=\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{-1}\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}$$

The given system can be expressed as $AX=B$, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

Now, $AX=B\Rightarrow X=A^{-1}B$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0-5+6\\ -22-45+68\\ -11-25+39\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

On equating, we get x = 1, y = 2 and z = 3

**Problem – 3**

The cost of 4 kg onion, 3 kg wheat, and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat, and 2 kg rice is Rs. 90. The cost of 6 kg onion, 2 kg wheat, and 3 kg rice is Rs. 70. Find the cost of each item kg by matrix method.

**Solution:-**

Let the cost of onion per kg = Rs. $x$

Cost of wheat per kg = Rs. $y$

And the cost of rice per kg = Rs.$z$

According to question $4x+3y+2z=60;\ 2x+4y+6z=90 \text{ and } 6x+2y+3z=70$

The giver system of equations can be expressed as AX = B where

$$A=\begin{bmatrix}4&3&2\\2&4&6\\6&2&3\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}60\\90\\70\end{bmatrix}$$

$$|A|=\begin{vmatrix}4&3&2\\2&4&6\\6&2&3\end{vmatrix}$$

$$|A|=4(12-12)-3(6-36)+2(4-24)=0+90-40=50\neq 0$$

$\Rightarrow$ The system has a unique solution

$$C_{11}=0, C_{12}=30, C_{13}=-20, C_{21}=-5, C_{22}=0, C_{23}=10, C_{31}=10, C_{32}=-20, C_{33}=10$$

$$\text{adj } A=\begin{bmatrix}0&30&-20\\-5&0&10\\10&-20&10\end{bmatrix}^T=\begin{bmatrix}0&-5&10\\30&0&-20\\-20&10&10\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{50}\begin{bmatrix}0&-5&10\\30&0&-20\\-20&10&10\end{bmatrix}$$

$$AX=B\Rightarrow X=A^{-1}B$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0&-5&10\\30&0&-20\\-20&10&10\end{bmatrix}\begin{bmatrix}60\\90\\70\end{bmatrix}$$

$$=\frac{1}{50}\begin{bmatrix}0-450-700\\1800+0-1400\\-1200+900+700\end{bmatrix}=\frac{1}{50}\begin{bmatrix}250\\400\\400\end{bmatrix}=\begin{bmatrix}5\\8\\8\end{bmatrix}$$

X = 5, y = 8 and z = 8

Thus, cost of 1 kg onion = Rs. 5

Cost of 1 kg Wheat = Rs. 8

Cost of 1 kg Rice = Rs. 8

**Problem – 4**

If $A=\begin{bmatrix}1&-1&1\\2&1&-3\\1&1&1\end{bmatrix}$ then find $A^{-1}$ and hence solve the system of linear equations

$$x+2y+z=4,\ -x+y+z=0,\ x-3y+z=2$$

**Solution:-** We have $A=\begin{bmatrix}1&-1&1\\2&1&-3\\1&1&1\end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3)+1(2+3)+1(2-1) = 10 \neq 0$$

So A is invertible

Let $C_{ij}$ be the co-factors of element $a_{ij}$ in A, then

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5 \qquad C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \qquad C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5 \qquad C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$

$$\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{10}\begin{bmatrix} -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Now, the given system of equations is expressible as

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Or, $A^T X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Now, $|A^T| = |A| = 10 \neq 0$, so the given system of equations is consistent with a unique solution given by

$$X = (A^T)^{-1} B = (A^{-1})^T B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}^T \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+2 \\ 8+0-4 \\ 8+0+6 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 2/5 \\ 7/5 \end{bmatrix}$$

$\Rightarrow$ x = 9/5, y = 2/5 and z = 7/5

Hence, x = 9/5, y = 2/5 and z = 7/5 is the required solution.

**Problem – 5** Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x-y+z=4, x-2y-2z=9, 2x+y+3z=1$

**Solution:-** Let $A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and $C=\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$.

Then the given product is $CA=\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$\Rightarrow CA=\begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}=8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=8\,I_3$$

$$\Rightarrow \frac{1}{8}CA=I_3$$

$$\Rightarrow \left(\frac{1}{8}C\right)A=I_3$$

$$\Rightarrow A^{-1}=\frac{1}{8}C$$

$$\Rightarrow A^{-1}=\frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \text{.................................... (1)}$$

The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

Or $AX=B$, where $A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

The solution to this system of equations is given by

$$X=A^{-1}B$$

$$\Rightarrow X=\frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}=\frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}=\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x=3,\ y=-2 \text{ and } z=-1$$

**Problem:- 06** Solve the following system of equations $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4;\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1;\ \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

**Solution:-** Putting $\frac{1}{x}=a,\ \frac{1}{y}=b,\ \frac{1}{z}=c$ in the given system of equations we get

$2a+3b+10c=4 \qquad 4a-6b+5c=1 \qquad 6a+9b-20c=2$

The system of the equation can be expressed as

$AX=B$ where $A=\begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}a\\b\\c\end{bmatrix}$ and $B=\begin{bmatrix}4\\1\\2\end{bmatrix}$

Now, $|A|=\begin{vmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{vmatrix}=2(120-45)-3(-80-30)+10(36+36)=1200\neq 0$

$\therefore$ The system has a unique solution

$C_{11}=75, C_{12}=110, C_{13}=72, C_{21}=150, C_{22}=-100, C_{23}=0, C_{31}=75, C_{32}=30, C_{33}=-24$

$$\text{Adj } A=\begin{bmatrix}75 & 110 & 72\\150 & -100 & 0\\75 & 30 & -24\end{bmatrix}^T=\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\text{Adj } A=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}a\\b\\c\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}300+150+150\\440-100+60\\288+0-48\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}600\\400\\240\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1/2\\1/3\\1/5\end{bmatrix}$$

$\Rightarrow a=\frac{1}{2}\Rightarrow x=2 \qquad b=\frac{1}{3}\Rightarrow y=3 \qquad c=\frac{1}{5}\Rightarrow z=5$

**Questions:-**

01. If $\Delta=\begin{vmatrix}4 & -1 & 0\\2 & 1 & 4\\1 & 0 & 3\end{vmatrix}$ find (i) $M_{23}$ (ii) $C_{32}$

02. Area of a triangle with vertices (k, 0), (1, 1), and (0, 3) is 5 sq units. Find the value of k

03. Find the area of a triangle, whose vertices are 90,3), (-1, 4), (2, 6)

04. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, find the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{33}$

05. Find the value of p such that the matrix $\begin{bmatrix} -1 & 2 \\ 4 & p \end{bmatrix}$ is singular

06. Write the value of $\begin{vmatrix} \sin 20^0 & -\cos 20^0 \\ \sin 70^0 & \cos 70^0 \end{vmatrix}$

07. If A is a square matrix satisfying $A^T A = I$, write the value of $|A|$

08. If A and B are square matrices of the same order such that |A| = 3 and AB = I, then write the value of |B|.

09. If A is a skew-symmetric matrix of order 3, write the value of |A|.

10. If $A = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find the value of $|A| + |B|$

11. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ find |AB|

12. Find x, if $\begin{vmatrix} -1 & 2 \\ 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & x \\ x & -4 \end{vmatrix}$

13. If matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find |A|

14. Write down the adjoint of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

15. If $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find $(AB)^{-1}$

16. Is the following system of equations consistent? $2x - 3y = 4$ $3x + 2y = 1$

17. Find the number of solutions to the system of equations $3x + 2y = 5$ $6x + 4y = 3$

18. If A is a scalar matrix of order $3 \times 3$ such that $a_{22} = 5$ then find det A.

19. Find the value of x, such that the points (0, 2), (1, x), (3, 1) are collinear

20. For two given square matrices A and B of the same order, such that |A|=20 and |B| = -20, find |AB|

21. For the adjoint of the matrix $A = \begin{bmatrix} 3 & 1 \\ -5 & 4 \end{bmatrix}$

22. Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ -6 & -18 \end{bmatrix}$, if possible

23. If A is of order $5\times5$ and |A| = I, find |A|

24. If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $adj(adj\ A)$

25. Write the value of the determinant $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

26. Find the value of the determinant $\begin{vmatrix} cosec^2\theta & cot^2\theta & 1 \\ cot^2\theta & cosec^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

27. Find the value of the determinant $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

28. Find the value of $\begin{vmatrix} 3 & 1 & 8 \\ -4 & 2 & 6 \\ -5 & 3 & 24 \end{vmatrix}$

**Problems based on Determinants:-**

**Problem – 1** Prove, using properties of determinants $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$

**Solution:-** LHS = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$=\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ (On applying $R_2 \to R_2-R_1$ $R_3 \to R_3-R_1$)

$=(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$ (On taking common $(b-a)$ from $R_2$ and $(c-a)$ from $R_3$)

$=(b-a)(c-a)\left[1\{(c+a)-(b+a)\}-0+0\right]$ (On expanding along $C_1$)

$=(b-a)(c-a)(c-b)$

$=(a-b)(b-c)(c-a)=RHS$

**Problem – 2** Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

**Solution:-** $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$=\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$ (On applying $C_2 \to C_2 - C_1$ $C_3 \to C_3 - C_1$)

$$=(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+a^2+ab & c^2+a^2+ac \end{vmatrix}$$ (On taking common $(b-a)$ from $C_2$ and $(c-a)$ from $C_3$)

$$=(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & b^2-c^2+ab-ac & c^2+a^2+ac \end{vmatrix}$$ (On applying $C_2 \to C_2 - C_3$)

$$=(b-a)(c-a)(b^2-c^2+ab-ac)\begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & 1 & c^2+a^2+ac \end{vmatrix}$$ (On taking common $(b^2-c^2+ab-ac)$ from $C_2$)

$$=(b-a)(c-a)(b-c)(b+c+a)\left[1\{0-1\}-0+0\right]$$ (On expanding along $R_1$)

$$=(a-b)(b-c)(c-a)(a+b+c)=RHS$$

**Problem – 3**

Prove that $\begin{vmatrix} x=y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}=3(x+y+z)^3$

**Solution:-**

$$\begin{vmatrix} x=y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$=\begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$ (On applying $C_1 \to C_1 + C_2 + C_3$)

$$=(2x+2y+2z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$ (On taking common $(2x+2y+2z)$ from $C_1$)

$$=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$ (On applying $R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$)

$$=2(x+y+z)\left[1\{(x+y+z)^2-0\}-0+0\right]$$ (On expanding along $C_1$)

= RHS

**Problem – 4** Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

**Solutions:-** $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ (On applying $R_1 \to R_1 + R_2 + R_3$)

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ (On taking common $(a+b+c)$ from $C_1$)

$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$ (On applying $C_2 \to C_2 - C_1$ $C_3 \to C_3 - C_1$)

$= (a+b+c)\left[1\left\{(a+b+c)^2 - 0\right\} - 0 + 0\right]$ (On expanding along $R_1$)

$= (a+b+c)^3 = RHS$

**Problem – 5** Prove, using properties of determinates $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

**Solution:-** LHS = $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$ (On applying $R_1 \to R_1 - R_2 - R_3$)

$= -2x\begin{vmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$ (On taking common $(-2x)$ from $R_1$)

$= -2x\begin{vmatrix} 0 & 0 & 1 \\ z & z & x \\ y & -y & x+y \end{vmatrix}$ (On applying $C_2 \to C_2 - C_3$)

$= -2x\left[0 - 0 + 1(-yz - yz)\right]$ (On expanding along $R_1$)

$= 4xyz = RHS$

**Homework:-** NCERT Exercise – 4.2 Question number 10 (ii) and question number 11

**Problem – 6** Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

**Solution:-** LHS = $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ $R_1 \to R_1 - R_2 - R_3$

$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (Expanding along $R_1$, we obtain)

$\Delta = 0-(-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= 2c(ab+b^2-bc)-2b(bc-a^2-ac)$

$= 2abc+2cb^2-2bc^2-2b^2c+2bc^2+2abc \quad = 4abc = RHS$

**Problem – 5** If x, y, and z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then show that $1+xyz=0$

**Solution:-** $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$ (By property -5)

$\Rightarrow (-1)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$

For $1^{st}$ determinant using $(C_2 \Leftrightarrow C_2 \text{ and then } C_1 \Leftrightarrow C_2)$

For the $2^{nd}$ determinant taking x, y, and z common from $R_1, R_2 \, and \, R_3$ respectively

$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz) = 0$ $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$

Taking out a common factor $(y-x),(z-x)$ common from $R_2$ and $R_3$ respectively.

$\Rightarrow (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} = 0$

$\Rightarrow (1+xyz)(y-x)(z-x)(z-y) = 0$, Expanding by $C_1$

Since $\Delta = 0$ and x, y, z are all different i.e $x-y \neq 0$, $y-z \neq 0, z \neq x = 0$

$\therefore \Delta = (1+xyz) = o$

**Problem – 6** Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$

**Solution:-** LHS $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ (Taking a, b, c common from $R_1$, $R_2$, and $R_3$ respectively)

$$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ (Applying $R_1 \to R_1 + R_2 + R_3$)

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ (Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix}$$ Expanding by $R_1$ by we get

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab = RHS$$



**Home Work:-** NCERT Exercise 4.2 question number 12

**Problem – 07**

Prove by using properties of determinants $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2+b^2\right)^3$

**Solution:-** From LHS, taking b and a common from $C_1$ and $C_2$ respectively we get

$$=b.a\begin{vmatrix}\frac{1+a^2-b^2}{b} & 2b & -2b\\ 2a & \frac{1-a^2+b^2}{a} & 2-a\\ 2 & -2 & 1-a^2-b^2\end{vmatrix}$$

Now multiplying $b$ and $a$ in $R_1$ and $R_2$ respectively we get

$$=\begin{vmatrix}1+a^2-b^2 & 2b^2 & -2b^2\\ 2a^2 & 1-a^2+b^2 & 2a^2\\ 2 & -2 & 1-a^2-b^2\end{vmatrix}$$ Applying $C_1\to C_1+C_2,\ C_2\to C_2+C_3$

$$=\begin{vmatrix}1+a^2+b^2 & 0 & -2b^2\\ 1+a^2+b^2 & 1+a^2+b^2 & 2a^2\\ 0 & -(1+a^2+b^2) & 1-a^2-b^2\end{vmatrix}$$ Taking common $(1+a^2+b^2)$ from both $C_1$ and $C_2$ we get

$$=(1+a^2+b^2)^2\begin{vmatrix}1 & 0 & -2b^2\\ 1 & 1 & 2a^2\\ 0 & -1 & 1-a^2-b^2\end{vmatrix}$$

Now expanding by $C_1$ we get

$$=(1+a^2+b^2)\{1[(1-a^2-b^2)+2a^2]-1(0-2b^2)+0\}$$

$$=(1+a^2+b^2)^2(1+a^2+b^2)=(1+a^2+b^2)^3$$

**Problem – 8** Prove by using properties of determinant that $\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$

**Solution:-** From LHS taking a, b, and c common from $R_1$, $R_2$, and $R_3$ respectively

$$=abc\begin{vmatrix}\frac{a^2+1}{a} & b & c\\ a & \frac{b^2+1}{b} & c\\ a & b & \frac{c^2+1}{c}\end{vmatrix}$$ Again a, b, and $c$ multiplying in $C_1, C_2$ and $C_3$ respectively

$$=\begin{vmatrix}a^2+1 & b^2 & c^2\\ a^2 & b^2+1 & c^2\\ a^2 & b^2 & c^2+1\end{vmatrix}$$ $C_1\to C_1+C_2+C_3$

$$=\begin{vmatrix}1+a^2+b^2+c^2 & b^2 & c^2\\ 1+a^2+b^2+c^2 & b^2+1 & c^2\\ 1+a^2+b^2+c^2 & b^2 & c^2+1\end{vmatrix}$$ Taking $(1+a^2+b^2+c^2)$ common from $C_1$

$$=\left(1+a^2+b^2+c^2\right)\begin{vmatrix}1 & b^2 & c^2\\1 & b^2+1 & c^2\\1 & b^2 & c^2+1\end{vmatrix} \qquad R_1\to R_1-R_2, R_2\to R_2-R_3$$

$$=\left(1+a^2+b^2+c^2\right)\begin{vmatrix}0 & -1 & 0\\0 & 1 & -1\\1 & b^2 & c^2+1\end{vmatrix}=\left(1+a^2+b^2+c^2\right)\times 1=1+a^2+b^2+c^2 \qquad \text{(By expanding by } R_1\text{)}$$

**Problem – 9** If a, b, and c are positive and unequal, show that the value of the determinant

$$\Delta=\begin{vmatrix}a & b & c\\b & c & a\\c & a & b\end{vmatrix} \text{ is } -\text{ve}$$

**Solution:-** Applying $C_1\to C_1+C_2+C_3$

$$\Rightarrow \Delta=\begin{vmatrix}a+b+c & b & c\\a+b+c & c & a\\a+b+c & a & b\end{vmatrix} \qquad \text{Taking } (a+b+c) \text{ common from } C_1$$

$$=(a+b+c)\begin{vmatrix}1 & b & c\\1 & c & a\\1 & a & b\end{vmatrix} \qquad R_2\to R_2-R_1, R_3\to R_3-R_1$$

$$=(a+b+c)\begin{vmatrix}1 & b & c\\0 & c-b & a-c\\0 & a-b & b-c\end{vmatrix} \qquad \text{Expanding by } C_1 \text{ we get}$$

$$=(a+b+c)\left[(c-b)(b-c)-(a-c)(a-b)\right]$$

$$=(a+b+c)\left(-a^2-b^2-c^2+ab+bc+ca\right)$$

$$=-\frac{1}{2}(a+b+c)\left(2a^2+2b^2+2c^2-2ab-2bc-2ca\right)$$

$$=-\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

Which is –ve (since a + b + c > 0 and $(a-b)^2+(b-c)^2+(c-a)^2>0$

**Problem – 10** If a, b, and c are in AP find the value of $\begin{vmatrix}2y+4 & 5y+7 & 8y+a\\3y+5 & 6y+8 & 9y+b\\4y+6 & 7y+9 & 10y+c\end{vmatrix}$

**Solution:-** Given a, b, and c are in AP so $2b=a+c$ ........................ (1)

From LHS applying $R_1\to R_1+R_3-2R_2$

$$=\begin{vmatrix}6y+10-6y-10 & 12y+16-12y-16 & 18y+a+c-18y-2b\\3y+5 & 6y+8 & 9y+b\\4y+6 & 7y+9 & 10y+c\end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{From (10)}$$

= 0

**Problem – 11** Show that $\Delta=\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+y)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix}=2xyz(x+y+z)^3$

**Solution:-** From LHS $R_1\to xR_1,\ R_2\to yR_2, R_3\to zR_3$ and dividing by $xyz$ we get

$$\frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common x, y, and z from $C_1$, $C_2$, and $C_3$ respectively we get

$$=\frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} \quad C_2\to C_2-C_1,\ C_3\to C_3-C_1$$

$$=\begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+y)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix} \quad \text{Taking common factors } (x+y+z) \text{ from } C_2 \text{ and } C_3$$

$$=(x+y+z)^2\begin{vmatrix} (y+z)^2 & x-(y+z) & x-(y+z) \\ y^2 & (x+z)-y & 0 \\ z^2 & 0 & (x+y)-z \end{vmatrix} \quad \text{Applying } R_1\to R_1-(R_2+R_3)$$

$$=(x+y+z)^2\begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x-y+z & 0 \\ z^2 & 0 & x+y-z \end{vmatrix} \quad \text{Applying } C_2\to\left(C_2+\frac{1}{y}C_1\right) \text{and } C_3\to\left(C_2+\frac{1}{z}C_1\right)$$

$$=(x+y+z)^2\begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+y & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x+y \end{vmatrix} \quad \text{Expanding by } R_1 \text{ we get}$$

$$=(x+y+z)^2\,2yz\big((x+z)(x+y)-yz\big)=(x+y+z)^2\,2yz\left(x^2+xy+xz\right)=(x+y+z)^3\,2xyz$$

**Problem – 12** Use properties of determinants to solve for x: $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$ and $x \neq 0$

**Solution:-** Let $\Delta = \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}$

$= \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix}$ On applying $C_1 \to C_1 + C_2 + C_3$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}$ On taking common $(x+a+b+c)$ from $C_1$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & a-c \\ 0 & 0 & x \end{vmatrix}$ On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$= (x+a+b+c)\left[1(x^2-0)-0+0\right]$

$= x^2(x+a+b+c)$

Given that $\Delta = 0$

$\Rightarrow x^2(x+a+b+c) = 0$

$\Rightarrow x+a+b+c = 0$

$\Rightarrow x = -a-b-c$

**Problem – 13** If $x = -4$ is the root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$, then find the other two roots

**Solution:-** Given $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = \begin{vmatrix} x-3 & 0 & 3-x \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$ On applying $R_1 \to R_1 - R_3$

$= (x-3)\begin{vmatrix} 1 & 0 & -1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$ On taking common $(x-3)$ from $R_1$

$= (x-3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 2 \\ 3 & 2 & x+3 \end{vmatrix}$ On applying $C_2 \to C_3 + C_1$

$= (x-3)\left[1\{x(x+3)-4\}-0+0\right]$

$= (x-3)(x^2+3x-4)$

$=(x-3)(x-1)(x+4)$

Given that $\Delta=0$

$\Rightarrow(x-3)(x-1)(x+4)=0$

$\Rightarrow x=3,1,-1$

Hence, the other two required roots are 3 and 1

**Problem – 14**

Prove the following using properties of determinants: $\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}=(x^4-1)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$

**Solution:-** We have $\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$

$\begin{vmatrix} a-ax^4 & c-cx^4 & p-px^4 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$ On operating $R_1\to R_1-x^4$

$=\begin{vmatrix} a(1-x^4) & c(1-x^4) & p(1-x^4) \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$

$=(1-x^4)\begin{vmatrix} a & c & p \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$ On taking $(1-x^4)$ common from $R_1$)

$=(1-x^4)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$ On performing $R_2\to R_2-x^2R_1$

$=-(1-x^4)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$

$=(x^4-1)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$ $[R_1\Leftrightarrow R_2]$

**Problem – 15**

Prove the following using properties of determinants: $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=2(3abc-a^3-b^3-c^3)$

**Solution:-** Consider $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ By performing $R_1 \to R_1 + (R_2 + R_3)$

$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ By taking $2(a+b+c)$ common from $R_1$

$= 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & a-c & b+c \\ b-c & b-a & c+a \end{vmatrix}$ By performing $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$= 2(a+b+c)\left[0-0+1\{(a-b)(b-a)-(a-c)(b-c)\}\right]$

$= 2(a+b+c)\left[-a^2 - b^2 + 2ab - ab + ac + bc - c^2\right]$

$= 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2) = 2(3abc - a^3 - b^3 - c^3)$

**Problem – 16** Using properties of determinants, prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

**Solution:-** Consider $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$ By performing $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$= \left[a\{a(7a+3b) - 3a(2a+b)\} - 0 + 0\right]$

$= a\{7a^2 + 3ab - 6a^2 - 3ab\} = a\{a^2\} = a^3$

**Problem – 17** Find the value of x satisfying the determinant equation $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

**Solution:-** Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -4 & -18 & -48 \end{vmatrix} = 0$

$$\Rightarrow(-2)(-2)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 2 & 9 & 24 \end{vmatrix}=0$$

On operating $C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1$

$$\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 5 & 18 \end{vmatrix}=0$$

$$\Rightarrow(x-2)(18-15)-1(18-6)+2(5-2)=0$$

$$\Rightarrow 3(x-2)-12+6=0$$

$$\Rightarrow 3(x-2)=6$$

$$\Rightarrow x-2=2$$

$$\Rightarrow x=4$$

**Problem – 18** Using properties of determinants prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}=0$

**Solution:-** LHS = $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

$$=\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$$

$$=\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix}-\begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\sin\delta \end{vmatrix}$$

$$=\cos\delta\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix}-\sin\delta\begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta & \sin\beta \\ \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

(On taking common $\cos\delta$ from $C_3$ in the first determinant and $\sin\delta$ from $C_3$ in the second determinant)

$$\Rightarrow \cos\delta(0)+\sin\delta(0)=0=RHS$$

**Problem – 19** Using properties of the determinant prove that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}=(a-1)^3$

**Solution:-** LHS apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$LHS = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & 1-a & 0 \\ 3-2a-a^2 & 2-2a & 0 \end{vmatrix}$$

$$= (1-a)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ (1-a)(3+a) & 2(1-a) & 0 \end{vmatrix}$$

$$= (1-a)^2\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1+a & 1 & 0 \\ 3+a & 2 & 0 \end{vmatrix}$$

$= (1-a)^2\{1(2+2a-3-a)-0+0\}$ Expanding in $C_3$

$= (1-a)^2(a-1)$

$= (a-1)^2(a-1) = (a-1)^3 = RHS$

**Problem – 20** Show that the $\Delta ABC$ is an isosceles triangle if the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

**Solution:-** We have $\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2 A+\cos A & \cos^2 B-\cos^2 A+\cos B-\cos A & \cos^2 C-\cos^2 A+\cos C-\cos A \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\text{coa}A \\ \cos^2 A+\cos A & (\cos B-\cos A)(\cos B+\cos A+1) & (\cos C-\cos A)(\cos C+\cos A+1) \end{vmatrix}$$

$$= (\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A+\cos A & \cos B+\cos A+1 & \cos C+\cos A+1 \end{vmatrix}$$

On taking common $(\cos B-\cos A)$ from $C_2$ and $(\cos C-\cos A)$ from $C_3$

$$= (\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 0 & 1 \\ \cos^2 A+\cos A & \cos B-\cos C & \cos C+\cos A+1 \end{vmatrix}$$

On applying $C_2 \to C_2 - C_3$

$$=(\cos B-\cos A)(\cos C-\cos A)(\cos B-\cos C)\begin{vmatrix}1 & 0 & 0\\ 1+\cos A & 0 & 1\\ \cos^2 A+\cos A & 1 & \cos C+\text{coa}A+1\end{vmatrix}$$

On taking common $(\cos B - \cos C)$ from $C_2$

$$=(\cos B-\cos A)(\cos C-\cos A)(\cos B-\cos C)\left[\{0-1\}-0+0\right]$$

$$=(\cos A-\cos B)(\cos B-\cos C)(\cos C-\cos A)$$

Suppose that $\begin{vmatrix}1 & 1 & 1\\ 1+\cos A & 1+\cos B & 1+\cos C\\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C\end{vmatrix}=0$

$$(\cos A-\cos B)(\cos B-\cos C)(\cos C-\cos A)=0$$

$\cos A=\cos B$ or $\cos B=\cos C$ or $\cos C=\cos A$

$A=B$ or $B=C$ or $C=A$

Since $\Delta ABC$ is isosceles.

**Problem – 21** Prove $\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}=2abc(a+b+c)^3$

**Solution:-** $\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}$

$$=\begin{vmatrix}(b+c)^2-a^2 & 0 & a^2\\ 0 & (c+a)^2-b^2 & b^2\\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2\end{vmatrix}$$ On applying $C_1\to C_1-C_3, C_2\to C_2-C_3$

$$=\begin{vmatrix}(b+c+a)(b+c-a) & 0 & a^2\\ 0 & (c+a+b)(c+a-b) & b^2\\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2\end{vmatrix}$$

$$=(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ c-a-b & c-a-b & (a+b)^2\end{vmatrix}$$ On taking common $(a+b+c)$ from $C_1$ and $C_2$

$$=(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ -2b & -2a & 2ab\end{vmatrix}$$ On applying $R_3\to R_3-R_2-R_1$

$$=(a+b+c)^2\begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

On applying $C_1 \to C_1+\frac{1}{a}C_3, C_2 \to C_2+\frac{1}{b}C_3$

$$=(a+b+c)^2\left[0+0+2ab\{(b+c)(c+a)-ab\}\right]$$

$$=(a+b+c)^2\left[2abc\{a+b+c\}\right]=2abc(a+b+c)^3=RHS$$

**Multiple Choice Questions:-**

01. The value of $\begin{vmatrix} 4^2 & 4^3 & 4^4 \\ 4^3 & 4^4 & 4^5 \\ 4^4 & 4^5 & 4^6 \end{vmatrix}$

(a) $4^2$ (b) $4^9$ (c) $4^{12}$ (d) 0

02. If $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix}=0$ then the values of x are

(a) $\frac{2}{3},\frac{11}{3}$ (b) $\frac{7}{3},\frac{9}{3}$ (c) $\frac{1}{2},\frac{11}{4}$ (d) $\frac{2}{5},\frac{9}{5}$

03. The value of determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ is

(a) 1 (b) -1 (c) 0 (d) 2

04. The value of $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ is

(a) 0 (b) 2 (c) 7 (d) -2

05. The value of $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$ is

(a) 102 (b) 0 (c) 18 (d) 4

**Answer :-**

1. (d) 02. (a) 03. (c) 04. (a) 05. (b)