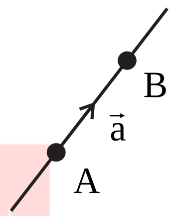
Chapter- 10

Vector Algebra

Some Basic Concept of Vector and Scalar

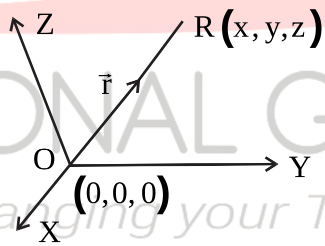
- A quantity that has magnitude, as well as direction, is called a vector.
- A quantity that involves only one value (magnitude) which is a real number called a scalar.

A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The magnitude of a vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$; which is the distance of A from B.



Here $\overrightarrow{AB} = \vec{a}$ and $|\overrightarrow{AB}| = \vec{a}$ or a

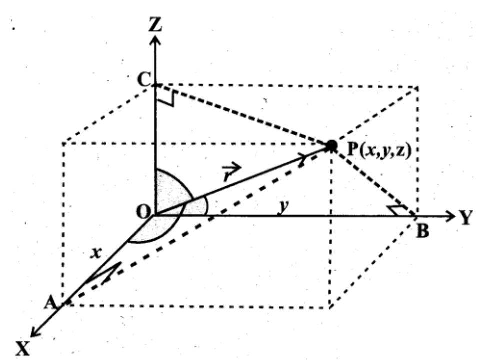
Position Vector: Let R be any point in space having co-ordinates (x, y, z) with reference to the origin $(0,0,0,)$. Then vector $\overrightarrow{OR}$ is called the position vector of the point R, with reference to the origin. The position vector of R is denoted by $\vec{r}$.



i.e. $\overrightarrow{OR} = \vec{r}$. The magnitude of $\overrightarrow{OP}$ is $|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ in practice, the position vectors of the points A, B, C, etc with respect to origin O one denoted by $\vec{a}, \vec{b}, \vec{c}$, etc respectively.

Direction cosines and direction ratio

Let us consider a point $R(x, y, z)$ whose position vector as $\overrightarrow{OR}$ or $\vec{r}$. The angle α, β, γ made by the vector $\vec{r}$ with the positive directions of x, y and z-axes respectively are called its direction angles. The cosine values of these angles, i.e. $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines (dcs) of $\vec{r}$. They are denoted by l, m, n respectively from the figure.



$\cos\alpha = \frac{x}{|r|} = \frac{x}{r} \Rightarrow l = \frac{x}{r}$ $\quad \cos\beta = \frac{y}{r} \Rightarrow m = \frac{y}{r}$ $\quad \cos\gamma = \frac{z}{r} \Rightarrow$

$n = \frac{z}{r}$, Thus coordinates of point P can be expressed as $(\ell r, mr, nr)$. The number $\ell r, mr$ and nr proportional to the direction cosines are called direction ratios (drs) of $\vec{r}$. They are denoted by a, b and c respectively.

Note : (1) $\ell^2 + m^2 + n^2 = 1$ But $a^2 + b^2 + c^2 \neq 1$ in general

(2) $cos^2\,\alpha + cos^2\,\beta + cos^2\,\gamma = 1$ But $sin^2\,\alpha + sin^2\,\beta + sin^2\,\gamma = 2$ (Prove this)

Types of Vector

Zero Vector: A vector of zero magnitudes is a zero vector. i.e. which has the same initial and terminal point, is called a zero vector. It is denoted by $\vec{0}$. The direction of the zero vector is indeterminate.

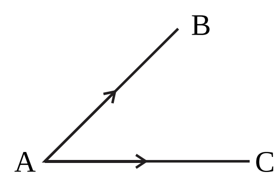
Unit Vector: A vector of unit magnitude in the direction of $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$; where $\left[\hat{a} = \frac{\vec{a}}{|\vec{a}|}\right]$

Equal vectors: Two vectors are said to be equal if they have the same magnitude and direction.

Collinear vector: Two vectors are said to be collinear if their directed line segments are parallel disregarding their direction. Collinear vectors are also called parallel vectors. If they have the same direction they are called the like vectors otherwise called as, unlike vectors.

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a} = k\vec{b}$, where $k \in R - \{0\}$.

Coinitial Vectors: Two or more vectors having the same initial point called coinitial vector hence $\overrightarrow{AB}$ & $\overrightarrow{AC}$ are coinitial vector.

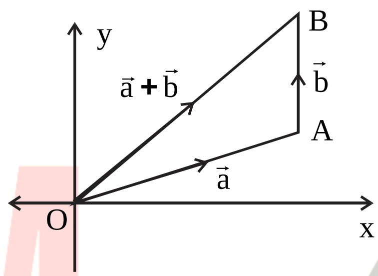


Negative of a vector :

A vector having the same magnitude as that of a given vector (say $\overrightarrow{PQ}$) but the opposite direction is called negative of the given vector. For example, $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are negative to each other and written as $\overrightarrow{PQ} = -\overrightarrow{QP}$.

Addition of Vectors: The sum or resultant of more than two vectors is called the composition of vectors.

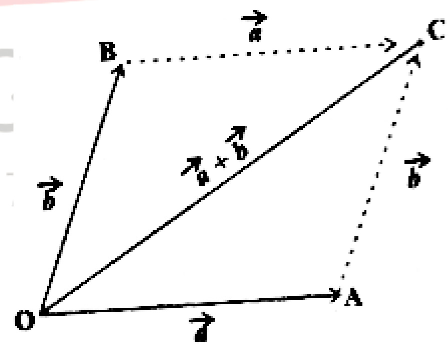
Triangle law of addition If $\vec{a}$ & $\vec{b}$ are two vectors represented by directed line segments $\overrightarrow{OA}$ and $\overrightarrow{AB}$, i.e., $\overrightarrow{OA} = \vec{a}\ and\ \overrightarrow{AB} = \vec{b}$.



Then the sum or resultant of $\vec{a}$ and $\vec{b}$ is defined as the vector represented by the line segment $\overrightarrow{OB}$. i.e. $\overrightarrow{OB} = \vec{a} + \vec{b}$ or $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

- **Parallelogram Law of Addition :**

If two vector $\vec{a} and \vec{b}$ is represented by the two adjacent sides of a parallelogram in magnitude and direction, then the sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point.



Hence $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ and $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$

This leads us to the conclusion that triangle law and parallelogram law are equivalent to each other.

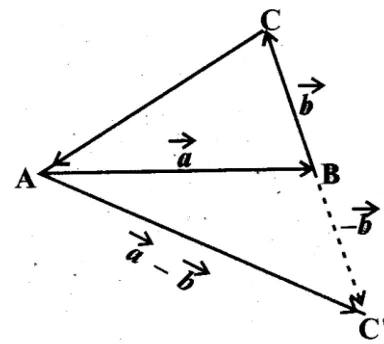
Properties : (i) For any two vectors $\vec{a}$ and$\vec{b}$ $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)

(ii) For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$ (Associative)

(iii) For any vector $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$

(iv)For any vector, $\vec{a}$ there exists another vector - $\vec{a}$ such that $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$.

Subtraction of two vectors :



If $\vec{a}$ and $\vec{b}$ are any two vectors then subtraction of $\vec{b}$ from $\vec{a}$ is defined as the sum of $\vec{a}$ and $-\vec{b}$. It is written as $\vec{a}-\vec{b}$.

In process of subtracting $\vec{b}$ from $\vec{a}$, we find $-\vec{b}$ (by reversing the direction) and add to $\vec{a}$.

Multiplication of a vector by a scalar

Let $\vec{a}$ be a given vector and k be a scalar. Then multiplication of the vector $\vec{a}$ by a scalar, k is defined as a vector denoted by $k\vec{a}$. Such that.

(i) The magnitude of $k\vec{a}$ is $|k|$ times of the magnitude of $\vec{a}$. i.e., $|k\vec{a}|=|k||\vec{a}|$ and

(ii) Direction of $k\vec{a}$ is the same as that of $\vec{a}$ if k is positive and direction of $k\vec{a}$ is opposite to that of $\vec{a}$ if k is negative.

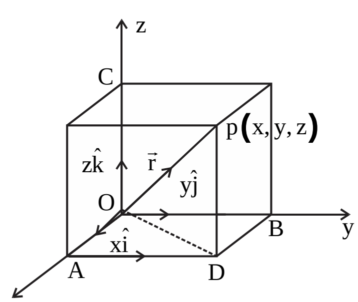
Note: Two vectors $\vec{a}$ & $\vec{b}$ are parallel or collinear if there exists a nonzero scalar k such that $\vec{a}=k\vec{b}$ if $k>0$, then $\vec{a}$ and $\vec{b}$ are like vectors if $k<0$ then $\vec{a}$ and $\vec{b}$ are unlike vectors.

Properties: Let $\vec{a}$ and $\vec{b}$ be any two vectors and k, m be any two scalars. Then,

(i) $(k+m)\vec{a}=k\vec{a}+m\vec{a}$ (ii) $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$

Components of a vector :

Let us consider a point $P(x,y,z)$ in space with position vector OP as shown in the figure. $\hat{\imath},\hat{\jmath},\hat{k}$ are unit vectors along the x-axis, y-axis, and z-axis respectively. Let D be the foot of the perpendicular from P on the XOY plane. Thus PD is parallel to the z-axis.



So $\overrightarrow{DP}=\overrightarrow{OC}=z\hat{K}$ similarly $\overrightarrow{AD}=\overrightarrow{OB}=y\hat{\jmath}$ and $\overrightarrow{OA}=x\hat{\imath}$.

Now $\overrightarrow{OP}=\overrightarrow{OD}+\overrightarrow{DP} \Rightarrow \overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AD}+\overrightarrow{DP} \quad =x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

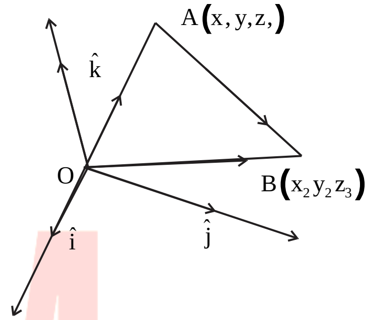
This representation of any vector is called the component form.

Thus position vector of the point $P(x, y, z)$ is $\vec{r} = \vec{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Here x, y, and z are called scalar components and $x\hat{\imath}, y\hat{\jmath}$ and $z\hat{k}$ are called vector components of $\vec{r}$.

The magnitude of the vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

Vector Joining two points: If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are any two points then vector joining A and B is the vector $\vec{AB}$. In ΔOAB. We have



$\vec{OA} + \vec{AB} = \vec{OB}$

$$\Rightarrow \vec{AB} = \vec{OB} - \vec{OA}$$
$$= (x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$$
$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

$\vec{AB} = P.V$ of B – P.V. of A and $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example: Find the unit vector in the directing $\vec{AB}$ when A and B are points $(-2,4,3)$ and $(-1, -4,6)$ respectively.

Ans: $\vec{AB}$ = P.V. of B – P.V. of A

$$= (-1 + 2)\hat{\imath} + (-4 - 4)\hat{\jmath} + (6 - 3)\hat{k}$$
$$= 1\hat{\imath} - 8\hat{\jmath} + 3\hat{k}$$
$$|\vec{AB}| = \sqrt{1^2 + 8^2 + 3^2} = \sqrt{74}$$

Unit vector along $\vec{AB} = \frac{\vec{AB}}{|\vec{AB}|}. = \frac{1}{\sqrt{74}}(\hat{\imath} - 8\hat{\jmath} + 3\hat{k})$

Example: If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$, then find a vector of magnitude 6 units in the direction of $\vec{a} - \vec{b}$.

Ans : $\vec{a} - \vec{b} = -\hat{\imath} - 2\hat{\jmath} + 8\hat{k}$

Unit vector $\vec{a}-\vec{b}$ is $\frac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|}=\frac{-\hat{\imath}-2\hat{\jmath}+8\hat{k}}{\sqrt{69}}$

Vector of magnitude 6 in direction of $\vec{a}-\vec{b}$ is $\frac{6}{\sqrt{69}}(-\hat{\imath}-2\hat{\jmath}+8\hat{k})$.

Note: If $\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ $\quad\quad$ $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$

Then $\vec{a}=\vec{b}$ iff $a_1=b_1, a_2=b_2 \& a_3=b_3$

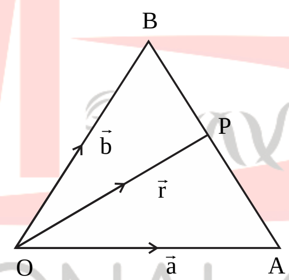
Example: Find the value of x, y, z so that the vector $3\hat{\imath}+y\hat{\jmath}-2\hat{k}$ and $x\hat{\imath}+5\hat{\jmath}+z\hat{k}$ are equal.

Section formulae :

Case I (Internal Division)

Let A and B be two points represented by position vectors $\vec{a}$ and $\vec{b}$ with O as origin than the position vector of point P which divides AB internally in the ratio in $m:n$ is given by

$$\vec{r}=\overrightarrow{OP}=\frac{m\vec{b}+n\vec{a}}{m+n}$$



Proof: Let $\overrightarrow{OP}=\vec{r}$, P divide $\overrightarrow{AB}$ in ratio $m:n$ internally. Then

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}}=\frac{m}{n}\Rightarrow n\overrightarrow{AP}=m\overrightarrow{PB}$$

$$\Rightarrow n(\overrightarrow{OP}-\overrightarrow{OA})=m(\overrightarrow{OB}-\overrightarrow{OP})$$

$$\Rightarrow n(\vec{r}-\vec{a})=m(\vec{b}-\vec{r})$$

$$\Rightarrow (m+n)\vec{r}=m\vec{b}+n\vec{a}$$

$$\Rightarrow \vec{r}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

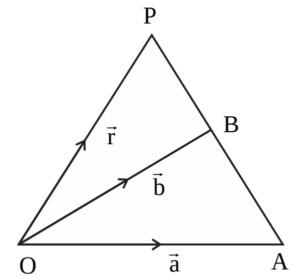
Case II : (External Division)

Here P divides $\overrightarrow{AB}$ in the ratio m:n externally

$$Hence, \frac{\overrightarrow{AP}}{\overrightarrow{BP}}=\frac{m}{n}$$

$$\Rightarrow n\overrightarrow{AP} = -m\overrightarrow{PB}$$

$$\Rightarrow n(\vec{r} - \vec{a}) = -m(\vec{b} - \vec{r})$$



$$\Rightarrow (m-n)\vec{r} = m\vec{b} - n\vec{a}$$

$$\Rightarrow \vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Midpoint formula: If P is the midpoint of $\overrightarrow{AB}$ then $m:n = 1:1$

Then $P.V.$ of P is $\vec{r} = \frac{\vec{a}+\vec{b}}{2}$

Example: Find the position vector of point R which divides the line segment joining P and Q whose position vectors are $2\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ externally in the ratio 1: 2.

Also, show that P is the midpoint of the line RQ.

Solution:

The position vector of the point R dividing the join of P and Q externally in the ratio 1:2 is

$$\overrightarrow{OR} = \frac{1(p.v.ofQ) - 2(P.V.ofP)}{1-2} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{-1}$$

$$= \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Now the midpoint of the line RQ is

$$\frac{P.V.ofR + P.V.ofQ}{2} = \frac{3\vec{a} + 5\vec{b} + \vec{a} - 3\vec{b}}{2}$$

$$= \frac{4\vec{a} + 2\vec{b}}{2} = 2\vec{a} + \vec{b} = P.V.ofP$$

Some useful results on DCS and DRS

Let $P(x, y, z)$ be a point in space such that $\overrightarrow{OP} = \vec{r}$ has dcs l,m,n. Then

$$(i) x = \ell|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|$$

(ii) Unit vector along $\overrightarrow{r}$ is $\hat{r} = \ell\hat{\imath} + mj + n\hat{k}$. The scalar components of a unit vector give the dcs of that vector.

(iii) If a,b,c be the drs of the vector whose dcs as l,m,n then $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$. Here $\ell^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ and $\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}} \& n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

(iv) The dcs of $\vec{r}$ are unique but drs of a vector a not unique. If a, b, c are drs of a vector then ka, kb, kc are also drs.

(v) If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, Here x, y, z can be taken as the drs of $\vec{r}$. They are its scalar components.

Example: Find the direction ratio and direction cosines of the $\vec{r} = \hat{\imath} + 2j + 3\hat{k}$.

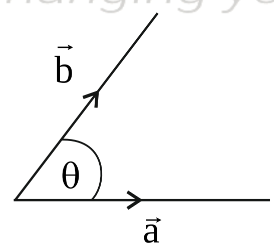
Example: Show that the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Product of vectors

Multiplication of two vectors is defined in two ways, namely scalar (or dot) product where the result is a scalar and vector (or cross) product where the result is a vector.

Scalar (or dot) product of two vectors

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, the scalar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ (read as $\vec{a}$ dot $\vec{b}$) and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ $0 \leq \theta \leq \pi$. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}.\vec{b} = 0$.



Some properties and observations:-

(i) As $\vec{a}.\vec{b}$ is a scalar that is why dot product is called scalar product.

(ii) $\vec{a}.\vec{b}$ can be positive, negative, or zero according to as $\cos\theta$ is positive, negative, or zero

(iii) If $\vec{a}$ & $\vec{b}$ are like vectors $(i.e. \theta = 0)$ then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|$

(iv) If $\vec{a}$ & $\vec{b}$ are unlike vectors $(i.e. \theta = \pi)$ then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos \pi = -|\vec{a}||\vec{b}|$

(v) $(\vec{a})^2 = \vec{a}.\vec{a} = |\vec{a}||\vec{a}| \cos 0 = |\vec{a}|^2$

(vi) If $\vec{a}$ and $\vec{b}$ are perpendicular or orthogonal vectors then $\vec{a}.\vec{b} = 0$

(vii) The dot product of vectors commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

(viii) For unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$

$\hat{\imath}.\hat{\imath} = 1$ $\quad$ $\hat{\jmath}.\hat{\jmath} = 1$ $\quad$ $\hat{k}.\hat{k} = 1$

$\hat{\imath}.\hat{\jmath} = 0$ $\quad$ $\hat{\jmath}.\hat{k} = 0$ $\quad$ $\hat{k}.\hat{\imath} = 0$

(xi) For $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$.

(x) let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(xi) The angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\cos \theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$.

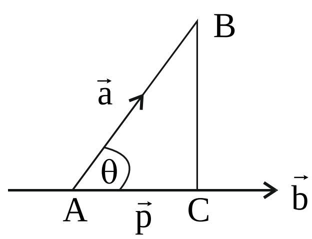
(xii) If $\vec{a}.\vec{b} = 0$ then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

Projection of a vector on another vector

The projection of any object can be obtained by focusing a source of light on the object and obtaining its shadow on a surface. Light is focused on a vector to obtain its shadow on a plane. The shadow so formed is the projection of the vector on the plane.

Let $\vec{a}$ and $\vec{b}$ be any two vectors with $\vec{b} \neq \vec{0}$. Let $\vec{a} = \overrightarrow{AB}$ and θ be the angle between $\vec{a}$ & $\vec{b}$. Draw $\overrightarrow{BC} \perp \vec{b}$ as shown in fig.

Then, projection of $\vec{a}$ on $\vec{b}$ (also known as the scalar projection of $\vec{a}$ on $\vec{b}$)



$$|\vec{p}| = AC = AB \cos \theta$$

$= |\vec{a}|\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$ $\left[\text{as } \cos \theta \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$

$$= \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

Note:- Vector projection $\vec{a}$ on $\vec{b}$ is given by $\left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

Exp:- If $\vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{c} = 5\hat{\imath} - 4\hat{\jmath} + 3\hat{k}$ then find $(\vec{a} + \vec{b}).\vec{c}$

Sol.

$$\vec{a} + \vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 0\hat{k}$$

$$(\vec{a} + \vec{b}).\vec{c} = (3\hat{\imath} + 3\hat{\jmath} + 0\hat{k}).(5\hat{\imath} - 4\hat{\jmath} + 3\hat{k}) = (3)(5) + (3)(-4) + (0)(3) = 3$$

Exp:- If $|\vec{a} + \vec{b}| = 60, |\vec{a} - \vec{b}| = 40$ and $|\vec{a}| = 22$ then find $|\vec{b}|$.

Exp:- If $\vec{a}, \vec{b}$ and $\vec{c}$ three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 5, |\vec{b}| = 12, |\vec{c}| = 13$ then find $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

Exp:- Find λ so that vectors such $\vec{a} = 4\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = \lambda\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$ perpendicular to each other.

Exp:- If $\vec{a} = 7\hat{\imath} + \hat{\jmath} - 4\hat{k}$ and $\vec{b} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

Exp:- If the vertices A, B, C of ΔABC are (1, 2, 3), (-1, 0, 0), (0, 1, 2) respectively, then find $\angle ABC$.

Cauchy Schwartz Inequality:-

For any two vectors $\vec{a}$ and $\vec{b}$; $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$

Proof:-

As $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$ $\quad \left(\begin{array}{l} \because as |\cos\theta| \leq 1 \\ \Rightarrow |\vec{a}||\vec{b}|\cos\theta \leq |\vec{a}||\vec{b}| \end{array}\right)$

$$\Rightarrow |\vec{a}.\vec{b}| = \left||\vec{a}||\vec{b}|\cos\theta\right|$$

$$\Rightarrow |\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$$

Triangle Inequality:-

For any two vectors $\vec{a}$ & $\vec{b}$

Prove that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Solution:- $\quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$

$\leq |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}.\vec{b}|(\text{as x} \leq |x|)$

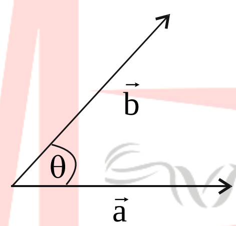
$\leq |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$= (|\vec{a}| + |\vec{b}|)^2$ (by Cauchy Schwartz Inequality)

Then $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Vector (Cross) Product of Vectors

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, vector product (or Cross product) of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ (read as $\vec{a}$ cross $\vec{b}$) and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$



Here θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector perpendicular to the plane containing the vectors $\vec{a}$ and $\vec{b}$.

Some properties and observations :

(i) $\vec{a} \times \vec{b}$ is always a vector. That is why the cross product is called a vector product.

(ii) If $\vec{a}$ and $\vec{b}$ are like vectors (i.e. $\theta = 0$) then $\vec{a} \times \vec{b} = \vec{0}$.

(iii) If $\vec{a}$ and $\vec{b}$ are unlike vectors (i.e. $\theta = \pi$) then $\vec{a} \times \vec{b} = \vec{0}$.

(iv) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. But $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$ and $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

(v) If $\vec{a} \times \vec{b} = \vec{0}$ then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ & $\vec{b}$ is collinear.

(vi) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

(vii) $\vec{a} \times \vec{a} = \vec{0}$

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

$\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}\,,\ \hat{\imath} \times \hat{k} = -\hat{\jmath}$

(viii) Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}, \vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$. then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(ix) If θ be the angle between $\vec{a}$ and $\vec{b}$ then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(x) **Area of triangle and parallelogram**: If $\vec{a}$ and $\vec{b}$ be the adjacent sides of a parallelogram then the area of the parallelogram is $|\vec{a} \times \vec{b}|$. If $\vec{a}$ and $\vec{b}$ be the adjacent sides of a triangle then area of is $\frac{1}{2}|\vec{a} \times \vec{b}|$

(xi) The unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Note: If $\vec{d}_1$and $\vec{d}_2$ be the two diagonals of a parallelogram then the area of the parallelogram is $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$.

Example: If $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ then find $\vec{a} \times \vec{b}$ & $|\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} = -17\hat{\imath} + 13\hat{\jmath} + 7\hat{k}$$

Hence $|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2} = \sqrt{507}$

Example: Find the value of $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + \hat{\jmath}.(\hat{k} \times \hat{\imath}) + \hat{k}.(\hat{\imath} \times \hat{\jmath})$

Example: Prove that $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$

Example: Find the value of λ for which the vectors $\vec{a} = 3\hat{\imath} - 6\hat{\jmath} + \hat{k}$ and $\vec{b} = 2\hat{\imath} - 4\hat{\jmath} + \lambda\hat{k}$ are parallel.

Example: Find a unit vector that is perpendicular to both the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath}$.

Example: If $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$ then find the value of $\vec{a}.\vec{b}$?

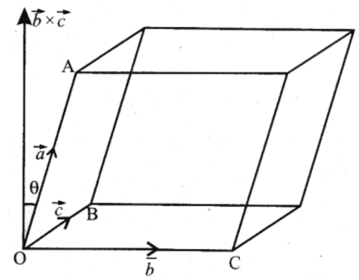
Example: Find the area of the triangle with vertices $A(1,1,2)B(2,3,5)$ and $C(1,5,5)$.

Example: If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of ΔABC, show that the area of ΔABC is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$ sq units.

Scalar Triple Product of Vectors

The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as the number $\vec{a}.(\vec{b} \times \vec{c})$ and is denoted by $[\vec{a} \quad \vec{b} \quad \vec{c}]$.

Geometrical Interpretation



Consider a parallelopiped whose coterminous edges $\vec{a}, \vec{b}$ and $\vec{c}$ from a right-handed system as shown in the figure.

From the geometric definition of the cross product, we know that $|\vec{b} \times \vec{c}|$ is the area of the parallelogram base, and the direction of $\vec{b} \times \vec{c}$ is perpendicular to the base.

Let the direction of $\vec{a}$ making an angle θ with the direction normal to the base i.e with $\vec{b} \times \vec{c}$.

Height of parallelopiped = $|\vec{a}| \, cos\, \theta$.

Then the volume of the parallelopiped = Area of parallelogram base × Height of parallelopiped

= $|\vec{b} \times \vec{c}||\vec{a}| \, cos\, \theta \qquad = |\vec{a}||\vec{b} \times \vec{c}| \, cos\, \theta$

$= \vec{a}.(\vec{b} \times \vec{c}) \qquad = [\vec{a} \quad \vec{b} \quad \vec{c}]$

Conclusion: The scalar triple product $[\vec{a} \quad \vec{b} \quad \vec{c}]$ represents the volume of the parallelepiped whose coterminous edges $\vec{a}, \vec{b}$ and $\vec{c}$.

Properties:-

1. If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}, \vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$, then$[\vec{a} \quad \vec{b} \quad \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
2. If three vectors are cyclically permitted then the scalar triple product remain the same i.e. $[\vec{a} \quad \vec{b} \quad \vec{c}] = [\vec{b} \quad \vec{c} \quad \vec{a}] = [\vec{c} \quad \vec{a} \quad \vec{b}]$.
3. If $\vec{a}, \quad \vec{b}, \quad \vec{c}$ be any three vectors then $[\vec{a} \quad \vec{b} \quad \vec{c}] = -[\vec{a} \quad \vec{c} \quad \vec{b}]$
4. In scalar triple product the position of dot and cross can be interchanged provided cyclic order of vector remains the same. i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$
5. The scalar triple product of three vectors is zero, If any two of the given vector are equal i.e. for any three vectors $\vec{a} \quad \vec{b} \quad \vec{c}$ with $\vec{a} = \vec{b}$ or $\vec{b} = \vec{c}$ or $\vec{c} = \vec{a}$ then $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$.

6. The scalar triple product of three vectors is zero if any two of them are parallel (or collinear)

7. If $\vec{a} \quad \vec{b} \quad \vec{c}$ are coplanar then $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$. As the volume of parallelopiped vanishes.

Example: If $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}, b = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ then find $[\vec{a} \quad \vec{b} \quad \vec{c}]$.

Ans:- -10

Example: Find the volume of the parallelepiped where three coterminous edges are represented by vectors $\hat{\imath} + \hat{\jmath} + \hat{k}, -\hat{\imath} + \hat{\jmath} - \hat{k}$ and $2\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

Ans – 4 cube units.

Example: Find λ if the vectors $\vec{a} = \hat{\imath} + 3\hat{\jmath} + \hat{k}, \vec{b} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{c} = \lambda\hat{\imath} - 3\hat{k}$ are coplanar.

Ans - $\lambda = 7$

Example: Show that the four points $A(4,5,1), B(0,-1,-1), C(3,9,4)$ and $D(-4,4,4)$ are coplanar.

Example: Prove that $[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$. If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then show that $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar.

<u>**The practice of miscellaneous Questions :**</u>

Example: For any three vectors $\vec{a}, \vec{b}, \vec{c}$ show that $\vec{a} - \vec{b}, \vec{b} - \vec{c}$ and $\vec{c} - \vec{a}$ are coplanar.

Example: Find all vectors of magnitude $10\sqrt{3}$ that is perpendicular to both the vector $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $-\hat{\imath} + 3j + 4\hat{k}$.

Ans: $\pm(10i - 10\hat{\jmath} + 10\hat{k})$

<u>**Example**</u>

Let $\vec{a} = \hat{\imath} + 4\hat{\jmath} + 2\hat{k}, \vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$ and $\vec{c} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$, find a vector $\vec{d}$ that is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{c}.\vec{d} = 15$.

Example: If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ to equally inclined to $\vec{a}, \vec{b}, \vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}, \vec{b}$ & $\vec{c}$.

Example: Find the value of λ, if four points with position vectors $3\hat{\imath} + 6\hat{\jmath} + \lambda\hat{k}, \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $4\hat{\imath} + 6\hat{\jmath} + \lambda\hat{k}$ are coplanar.

Ans- $\lambda = 2$.