Chapter- 7

Integrals

Introduction:-

If a function 'f' is differentiable in an interval I, i.e its derivative f^1 exists at each point of I. Then functions that could have possibly given function as a derivative are called anti-derivative of the function. Further, the formula that gives all these anti-derivatives is called the indefinite integral of the function, and such process of finding anti-derivatives is called integration. The two forms of integral are indefinite and definite integral which together constitute integral calculus.

Integration as an inverse process of differentiation:-

Let $F(x)$ and $f(x)$ be two functions connected such that $\frac{d}{dx}F(x) = f(x)$, then $F(x)$ is called integral $f(x)$ or indefinite integral or anti-derivatives.

If $\frac{d}{dx}F(x) = f(x)$ then for any constant 'C' $\frac{d}{dx}[F(x) + C] = f(x)$. Thus $F(x) + C$ is an anti-derivatives of $f(x)$.

i.e $\int f(x)dx = F(x) + C$

Where C is an arbitrary constant known as the constant of integration

The process of finding the integral of a function is called integration.

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration.

Fundamental integration formulae:-

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(b) $\int dx = x + c$

(c) $\int \frac{1}{x} dx = log|x| + C$

(d) $\int sin\, x\, dx = -cos\, x + C$

(e) $\int cos\, x\, dx = sin\, x + C$

(f) $\int sec^2 x\, dx = tan\, x + C$

(g) $\int cos\, e\, c^2 x dx = -cot\, x + C$

(h) $\int sec\, x\, .tan\, x\, dx = sec\, x + C$

(i) $\int cos\, e\, cx. cot\, x\, dx = -cos\, e\, cx + C$

(j) $\int a^x dx = \frac{a^x}{log\, a} + C, a > 0, a \neq 1$

(k) $\int e^x dx = e^x + C$

(l) $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1} x + Cor - cos^{-1} x + C$

(m) $\int \frac{1}{1+x^2} dx = tan^{-1} x + C\ or - tan^{-1} x + C$

(n$\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1} x + C\ \ or - cos\, e\, c^{-1} x + C$

(o) $\int 0 dx =$ constant

Example:-

Evaluate the following integrals

(a) $\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$

(b) $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c$

(c) $\int 5^x dx = \frac{5^x}{log\, 5} + c$

(d) $\int e^{3\, ln\, x} dx = \int e^{ln\, x^3} dx = \int x^3 dx = \frac{x^4}{4} + C$

Algebra of indefinite integral:-

(a) The process of differentiation and integration are inverse of each other.

i.e $\frac{d}{dx}\int f(x)dx = f(x)\ and\ \int f^1(x)dx = f(x) + C$

(b) $\int\left(f(x) \pm g(x)\right)dx = \int f(x)dx \pm \int g(x)dx$

(c) $\int K.f(x)dx = K\int f(x)dx, K$ is any constant

Note:- If more than one constant of integration is used while solving the integral, then at the end of the solution write only one constant of integration.

Example:- Evaluate the integrals

(a) $\int(\sin x + \cos x)dx = \int \sin x\, dx + \int \cos x\, dx$

$= -\cos x + c_1 + \sin x + c_2 = -\cos x + \sin x + c$

(b) $\int(x+1)dx = \int xdx + \int 1dx = \frac{x^2}{2} + x + c$

(c) $\int \frac{x^3-1}{x^2}dx = \int xdx - \int x^{-2}dx = \frac{x^2}{2} - \frac{x^{-1}}{-1} + c = \frac{x^2}{2} + \frac{1}{x} + c$

(d) $\int\left(e^{x\log a} + e^{a\log x} + e^{a\log a}\right)dx = \int a^x dx + \int x^a dx + \int a^a dx = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a x + c$

(e) $\int \frac{1}{1+\sin x}dx = \int \frac{1-\sin x}{\cos^2 x}dx = \int(\sec^2 x - \sec x.\tan x)dx = \tan x - \sec x + C$

Integration by the method of inspection:-

We can find an anti-derivative of a given function by searching intuitively a function whose derivative is the given function.

Example:- Write an anti-derivative of $3x^2 + 4x^3$ by the method of inspection.

Solution:- $\frac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$ $\therefore$ An anti-derivative of $3x^2 + 4x^3$ is $x^3 + x^4$ or $x^3 + x^4 + C$

Integration by substitution:- In the previous topic, we discussed the integrates of those functions which are in standard forms. But integrals of certain functions cannot be obtained directly but they may be reduced to standard forms by proper substitution. The method of evaluating an integral by reducing it to standard form by a proper substitution is called integration by substitution.

Method of substitution:-

To evaluate an integral of the type $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ and $\phi'(x)dx = dt$

The above integral is $\int f(t)dt$

After evaluating this integral we substitute back the value of t.

Note:- It is often important to guess what will be a useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integral.

Theorem:- Prove that

(a) $\int \tan x \, dx = \log|\sec x| + C$

(b) $\int \cot x \, dx = \log|\sin x| + C$

(c) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

(d) $\int \cos e\, c x dx = \log|\cos e\, c x - \cot x| + C$

Integrals of the form $\int f(ax + b)dx$:-

If $\int f(x)dx = g(x)$, then $\int f(ax + b)dx = \frac{1}{a}g(ax + b)$+C

Some Important deductions:-

(a) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

(b) $\int \frac{1}{ax+b} dx = \frac{1}{a} log|ax+b| + C$

(c) $\int a^{bx+c} dx = \frac{a^{bx+c}}{b \log a} + C, a > 0 \text{ and } a \neq 1$

(d) $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$

(e) $\int sin(ax+b)\, dx = -\frac{1}{a} cos(ax+b) + C$

(f) $\int cos(ax+b)\, dx = \frac{1}{a} sin(ax+b) + C$

(g) $\int sec^2(ax+b)\, dx = \frac{1}{a} tan(ax+b) + C$

(h) $\int cosec^2(ax+b) dx = -\frac{1}{a} cot(ax+b) + C$

(i) $\int sec(ax+b).tan(ax+b)\, dx = \frac{1}{a} sec(ax+b) + C$

(j) $\int cosec(ax+b).cot(ax+b)\, dx = -\frac{1}{a} cosec(ax+b) + c$

(k) $\int tan(ax+b)\, dx = \frac{1}{a} log|sec(ax+b) + C|$

(l) $\int cot(ax+b)\, dx = \frac{1}{a} log|sin(ax+b)| + C$

(m) $\int sec(ax+b)\, dx = \frac{1}{a} log|sec(ax+b) + tan(ax+b)| + C$

(n) $\int cosec(ax+b) dx = \frac{1}{a} log|cosec(ax+b) - cot(ax+b)| + C$

Type - 1

Evaluate:-

(a) $\int (2x-3)^5 dx = \frac{(2x-3)^6}{2\times 6} + C = \frac{(2x-3)^6}{12} + C$

(b) $\int \sqrt{3x+2} dx = \frac{(3x+2)^{\frac{3}{2}}}{3\times\frac{3}{2}} + C = \frac{2}{9}(3x+2)^{\frac{3}{2}} + C$

(c) $\int \frac{1}{2-3x} dx = -\frac{1}{3} log|2-3x| + C$

(d) $\int \frac{1}{\sqrt{5x-4}} dx = \int (5x-4)^{-\frac{1}{2}} dx = \frac{2}{5}\sqrt{5x-4} + C$

(e) $\int e^{2x-3} dx = \frac{e^{2x-3}}{2} + c$

(f) $\int a^{3x+2} dx = \frac{a^{3x+2}}{3 \log a} + C$

Type – II

Examples:-

(a) $\int sec^2(7-4x)\, dx = -\frac{1}{4} tan(7-4x) + C$

(b) $\int cos\, e\, c^2(3x+2) dx = \frac{-1}{3} cot(3x+2) + C$

(c) $\int sin(ax+b) . cos(ax+b)\, dx = \frac{1}{2}\int 2\, sin(ax+b) . cos(ax+b)\, dx$

$= \frac{1}{2}\int sin\, 2\,(ax+b) dx = -\frac{1}{4a} cos(2ax+2b) + C$

(d) $\int \frac{sin\, 4x}{sin\, 2x} dx = \int \frac{2\, sin\, 2x . cos\, 2x}{sin\, 2x}$

$= 2\int cos\, 2\, x dx = 2 \times \frac{sin\, 2x}{2} + C = sin\, 2\, x + C$

(e) $\int tan^2(2x-3)\, dx$

$= \int (sec^2(2x-3) - 1) dx = \int sec^2(2x-3)\, dx - \int dx = \frac{tan(2x-3)}{2} - x + C$

(f) $\int \frac{e^{tan^{-1} x}}{1+x^2} dx$ Put $tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$= \int e^t dt \quad = e^t + C \quad = e^{tan^{-1} x} + C$

(g) $\int cos\, 6\, x\sqrt{1 + sin\, 6\, x} dx$ Put $1 + sin\, 6\, x = t \Rightarrow cos\, 6\, x dx = \frac{dt}{6}$

$= \frac{1}{6}\int \sqrt{t} dt = \frac{1}{6} \times \frac{2}{3} t^{\frac{3}{2}} + C = \frac{1}{9}(1 + sin\, 6\, x)^{\frac{3}{2}} + C$

(h) $\int \frac{tan^4 \sqrt{x}.sec^2 \sqrt{x}}{\sqrt{x}} dx$ Put $tan \sqrt{x} = t$ $\Rightarrow \frac{sec^2 \sqrt{x}}{\sqrt{x}} dx = 2dt$

$$= \int t^4(2dt) = 2\int t^4 dt = \frac{2}{5}t^5 + c = \frac{2}{5} tan^5 \sqrt{x} + C$$

(i) $\int \frac{\sin x}{\sin(x+a)} dx$ Put $x + a = t \Rightarrow dx = dt$

$$\int \frac{\sin(t-a)}{\sin t} dt$$

$$= \int \frac{\sin t.\cos a - \cos t.\sin a}{\sin t} dt = \cos a \int dt - \sin a \int \cot t \, dt$$

$$= t \cos a - \sin a . log|\sin t| + C = x \cos a - \sin a . log|\sin(x+a)| + C$$

(j) $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} dx \quad = \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right] + C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

Integration using Trigonometric Identities:-

When the integral involves some trigonometric functions then some known trigonometric identities are used to evaluate integral easily.

Integrals of the form $\int sin^m x \, dx or \int cos^m x \, dx, \ m \in N$

To evaluate we express $sin^m x \ (or \ cos^m x)$ in terms of sines and cosines of multiples of x. For which we use the following trigonometrical identities.

(a) $sin^2 x = \frac{1-\cos 2x}{2}$ (b) $sin^3 x = \frac{3\sin x - \sin 3x}{4}$

(c) $cos^2 x = \frac{1+\cos 2x}{2}$ (d) $cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

Example:-

(a) $\int sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) + C$

(b) $\int cos^4 x\, dx = \int (cos^2 x)^2 dx = \int \left(\frac{1+cos\,2x}{2}\right)^2 dx$

$$= \frac{1}{8}\int (3 + 4\,cos\,2\,x + cos\,4\,x)dx \quad = \frac{3}{8}x + \frac{sin\,2x}{4} + \frac{sin\,4x}{32} + C$$

Integrals of the form $\int sin^m x\,.\,cos^n x\, dx,\ m, n \in N$

To evaluate the integrals of the form $\int sin^m x\,.\,cos^n x\, dx$ we may use the following algorithm

Algorithm:-

Step – I:- Obtain the integral say $\int sin^m x\,.\,cos^n x\, dx$

Step – II:- Check the exponents of $sin\,x$ and $cos\,x$

Step – III:- If the exponent of $sin\,x$ is an odd positive integer put $cos\,x = t$

If the exponent of $cos\,x$ is an odd positive integer put $sin\,x = t$

If the exponents of $sin\,x$ and $cos\,x$ both are odd positive integers put either $sin\,x = t$ or $cos\,x = t$. If the exponents of $sin\,x$ and $cos\,x$ both are positive even integers then express $sin^m x\,.\,cos^n x$ in terms of sines and cosines of multiples of x by using trigonometric results.

Example:-

$$\int sin^3 x\,.\,cos^4 x\, dx = -dt \int sin^2 x\,.\,cos^4 x\,.\,sin\,x\, dx$$

$= \int (1 - cos^2 x)\, cos^4 x\,.\,sin\,x\, dx$ Put $cos\,x = t \Rightarrow sin\,x\, dx = -dt$

$$= \int (t^2 - 1)t^4 dt = \int (t^6 - t^4)dt = \frac{(cos\,x)^7}{7} - \frac{(cos\,x)^5}{5} + C$$

Integrals of the form $\int\ sin\,p\,x.\,cos\,q\,x dx$or $\int\ sin\,p\,x.\,sin\,q\,x dx$or $\int\ cos\,p\,x.\,cos\,qx\, dx$

To evaluate these types of integrals firstly multiply and divide by 2 then use the following trigonometric identities.

$$2\,sin\,A\,.\,cos\,B = sin(A + B) + sin(A - B)$$

$$2\,cos\,A\,.\,sin\,B = sin(A + B) - sin(A - B)$$

$$2\cos A.\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A.\sin B = \cos(A-B) - \cos(A+B)$$

Example:-

$$\int \cos 2x.\cos 4x.\cos 6x dx = \frac{1}{2}\int (2\cos 4x.\cos 2x)\cos 6x dx$$

$$= \frac{1}{2}\int (\cos 6x + \cos 2x).\cos 6x dx = \frac{1}{2}\int \quad (\cos^2 6x + \cos 6x.\cos 2x)\, dx$$

$$= \frac{1}{4}\int \quad (2\cos^2 6x + 2\cos 6x.\cos 2x)dx = \frac{1}{4}\int (1 + \cos 12x + \cos 8x + \cos 4x)dx$$

$$= \frac{1}{4}\left(x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right) + C$$

$$= \frac{x}{4} + \frac{1}{48}\sin 12x + \frac{1}{32}\sin 8x + \frac{1}{16}\sin 4x + C$$

Example:-

$$\int \quad \sin 2x.\cos 3x\, dx$$

$$= \frac{1}{2}\int \; 2\sin 2x.\cos 3x dx = \frac{1}{2}\int (\sin 5x - \cos x)dx$$

$$= \frac{1}{2}\left(\frac{-\cos 5x}{5} + \cos x\right) + C = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

Integrals of type $\int \tan^m x.\sec^n x\, dx$ & $\int \cot^n x.\cos e\, c^m x dx$

(a) $\int \tan^2 x\, dx$ (b) $\int \tan^3 x\, dx$ (c) $\int \tan^4 x\, dx$ (d) $\int \tan^5 x\, dx$

(e) $\int \tan^6 x\, dx$ (f) $\int \cot^2 x\, dx$ (g) $\int \cot^4 x\, dx$

Integrals of some Particular functions:- Here we discussed some standard formulae with their proof and the methods to solve some other standard integrals with the help of these formulae.

(a) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} log\left|\frac{x-a}{x+a}\right| + C$

(b) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} log\left|\frac{a+x}{a-x}\right| + C$

(c) $\int \frac{dx}{x^2+a^2} = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$

(d) $\int \frac{dx}{\sqrt{x^2-a^2}} = log\left|x + \sqrt{x^2 - a^2}\right| + C$

(e) $\int \frac{dx}{\sqrt{x^2+a^2}} = log\left|x + \sqrt{x^2 + a^2}\right| + C$

(f) $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right) + C$

Poof:-

(a) Let $I = \int \frac{dx}{x^2-a^2} = \frac{2a}{2a}\int \frac{1}{(x-a)(x+a)} dx$

$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)} dx = \frac{1}{2a}\left[\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx\right]$

$= \frac{1}{2a}[log|x-a| - log|x+a|] + C = \frac{1}{2a} log\left|\frac{x-a}{x+a}\right| + C$

(b) Let $I = \int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\int \frac{(a+x)+(a-x)}{(a+x)(a-x)} dx$

$= \frac{1}{2a}\left[\int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx\right]$

$= \frac{1}{2a}[-log|a-x| + log|a+x|] + c = \frac{1}{2a} log\left|\frac{a+x}{a-x}\right| + C$

(c) Put $x = a\tan\theta$ then $dx = a\sec^2\theta\, d\theta$

Therefore $\int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2} = \frac{1}{a}\int d\theta = \frac{\theta}{a} + C = \frac{1}{a} tan^{-1}\frac{x}{a} + C$

(d) Put $x = a\sec\theta$

(e) Put $x = a\tan\theta$

(f) Put $x = a\sin\theta$

Example:-

Find the following integrals

(a) $\int \frac{1}{x^2-16} dx = \int \frac{1}{x^2-4^2} dx = \frac{1}{2\times 4} log\left|\frac{x-4}{x+4}\right| + C = \frac{1}{8} log\left|\frac{x-4}{x+4}\right| + C$

(b) $\int \frac{1}{\sqrt{9-25x^2}} dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}} dx = \frac{1}{5} sin^{-1}\left(\frac{x}{3/5}\right) + C$

$= \frac{1}{5} sin^{-1}\left(\frac{5x}{3}\right) + C$

Integral of the type

$$\int \frac{1}{ax^2 + bx + c} dx \quad or \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

We write $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right\}$$

$$= a(t^2 \pm k^2), t = x + \frac{b}{2a}, \frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$$

Hence $\int \frac{1}{ax^2+bx+c} dx = \frac{1}{a}\int \frac{1}{t^2 \pm k^2} dt$ and $\int \frac{1}{\sqrt{ax^2+bx+c}} dx = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{t^2 \pm k^2}} dt$ which can be integrated by using suitable formulae.

Example:-

(a) Evaluate $\int \frac{1}{3x^2+13x-10} dx$

Let $I = \int \frac{1}{3x^2+13x-10} dx = \frac{1}{3}\int \frac{1}{x^2+\frac{13}{3}x-\frac{10}{3}} dx$

$= \frac{1}{3}\int \frac{1}{\left(x+\frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2} = \frac{1}{3\times 2\times\frac{17}{6}} log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + C_1$

$= \frac{1}{17} log\left|\frac{3x-2}{x+5}\right| + C_1 + \frac{1}{17} log\frac{1}{3} = \frac{1}{17} log\left|\frac{3x-2}{x+5}\right| + C$

(b) Evaluate $\int \frac{1}{\sqrt{3-x+x^2}} dx$

$= \int \frac{1}{\sqrt{x^2-x+3}} dx = \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{11}{4}}} dx = \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}} dx$

$= log\left|x - \frac{1}{2} + \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{11}{4}}\right| + C$

Integral of the type $\int \frac{px+q}{ax^2+bx+c} dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

To evaluate such integrals we first write the numerator as

$$px + q = A\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + B$$

$$= A(2ax + b) + B$$

Then find A and B by comparing the coefficients of like powers of x from both sides. Now put the value of A and B, then given integral reduced to one of the known forms which can be integrated easily.

Examples:-

Find the following integrals

(a) $\int \frac{x+2}{2x^2+6x+5} dx$ (b) $\int \frac{x+3}{\sqrt{x^2-4x+5}} dx$

Solution

Let $x + 2 = A.\frac{d}{dx}(2x^2 + 6x + 5) + B$

$\Rightarrow x + 2 = A(4x + 6) + B$

Equating coefficient, $A = \frac{1}{4}, B = \frac{1}{2}$

$$\therefore x + 2 = \frac{1}{4}(4x + 6) + \frac{1}{2}$$

$$I = \int \frac{x + 2}{2x^2 + 6x + 5} dx = \int \frac{\frac{1}{4}(4x + 6) + \frac{1}{2}}{2x^2 + 6x + 5} dx$$

$$= \frac{1}{4}\int \frac{4x + 6}{2x^2 + 6x + 5} dx + \frac{1}{2}\int \frac{1}{2x^2 + 6x + 5} dx = \frac{1}{4}I_1 + \frac{1}{2}I_2$$

Now $I_1 = \int \frac{4x+6}{2x^2+6x+5} dx = log|2x^2 + 6x + 5| + C_1$

$$I_2 = \int \frac{1}{2x^2 + 6x + 5} dx = \frac{1}{2} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = tan^{-1}(2x + 3) + C_2$$

$$I = \frac{1}{4} log|2x^2 + 6x + 5| + \frac{1}{2} tan^{-1}(2x + 3) + C$$

Integration by Partial Fractions

Partial fraction decomposition:- It is always possible to write the integrand of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$ as a sum of simpler rational functions by a method which is known as partial fraction decomposition. Each such fraction is called a partial fraction and it has the simplest factor $Q(x)$.

Working Rule:-

Step – I, Suppose the given integral is in the form $\frac{P(x)}{Q(x)}$ then the first check $P(x)$ and $Q(x)$ are polynomials $Q(x) \neq 0$. Also for proper and improper.

Step – II If $\frac{P(x)}{Q(x)}$ is a proper fraction, then we go to the next step directly.

If $\frac{P(x)}{Q(x)}$ is an improper fraction then we divide $P(x)$ by $Q(x)$, then $\frac{P(x)}{Q(x)}$ is expressed in the form of $T(x) + \frac{P_1(x)}{Q(x)}$, $T(x)$ are a polynomial in x and $\frac{P_1(x}{Q(x)}$ proper factional function using division Algorithm.

Step – 3 Now the decomposition of the proper fraction $\frac{P(x)}{Q(x)}$ or $\frac{P_1(x)}{Q(x)}$ into partial fractions depends mainly upon the nature of the factors $Q(x)$.

Form of the rational Function	Form of the partial fraction
(a) $\frac{Px+q}{(x-a)(x-b)}; a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
(b) $\frac{Px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
(c) $\frac{Px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
(d) $\frac{Px^2+qx+r}{(x-a)^2.(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
(e) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Note:- If a rational function contains only even powers of x (i.e. 2 or 4) in both the numerator and denominator then resolve it into partial fractions we proceed as follows.

Step – I, Put $x^2 = y$

Step – II, $x^2 = y$ Resolved the reduced rational function into partial fractions.

Step – III, Replace y by x^2

Example:-

(a) $\int \frac{1}{(x+1)(x+2)} dx$

$$= \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Solving these equations we get

A = 1, B = 1

$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{1}{x+1} dx - \int \frac{1}{x+2} dx = log \left|\frac{x+1}{x+2}\right| + C$$

(b) $\int \frac{x^2+1}{x^2-5x+6} dx$

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{(x-2)(x-3)}$$

$$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Solving this equation we get $A = -5, B = 10$

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{-5}{x-2} + \frac{10}{x-3}$$

Therefore $\int \frac{x^2+1}{x^2-5x+6} dx = \int dx - 5\int \frac{1}{x-2} dx + 10\int \frac{1}{x-3} dx$

$= x - 5\,log|x-2| + 10\,log|x-3| + C$

(c) $\int \frac{3x-2}{(x+1)^2.(x+3)} dx,$

We write $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\Rightarrow 3x - 2 = A(x^2 + 4x + 3) + B(x+3) + C(x^2 + 2x + 1)$

Solving $A = \frac{11}{4}, B = \frac{-5}{2}, C = \frac{-11}{4}$

Thus the integral and is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} - \frac{5a}{2(x+1)^2} - \frac{11}{4(x+3)}$$

$$\int \frac{3x-2}{(x+1)^2(x+3)} dx = \frac{11}{4} log|x+1| + \frac{5}{2(x+1)^2} - \frac{11}{4} log|x+3| + C$$

$$= \frac{11}{4} log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)^2} + C$$

(d) $\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$

$$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

Solving $\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)} + \frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$

$$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}log|x+2| + \frac{1}{5}log|x^2+1| + \frac{1}{5}tan^{-1}x + C$$

(e) $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}dx$

$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{y+1}{(y+2)(2y+1)}$ let $x^2 = y$

$$\frac{y+1}{(y+2)(2y+1)} = \frac{A}{y+2} + \frac{B}{2y+1}$$

Solve it

Integration by parts:- let u and v be two differentiate functions of a single variable x, then the integral of the product of these two functions denoted as $\int u.vdx$ and defined as $\int u.vdx = u\int vdx = \int\left(\frac{d}{dx}(u)\int vdx\right)dx$. If in the product two functions are of different types, then take that function as first function (i.e u) which comes first in the word ILATE, where,

I: Inverse trigonometric function

L: Logarithmic function

A: Algebraic function

T : Trigonometric function

E: Exponential function

Note:- Integration by parts is not applicable in all cases for instance the method does not work for $\int \sqrt{x}\sin x\,dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\sin x$.

Example:- $\int x\cos x\,dx$

$$= x\int \cos x\,dx - \int\left(\frac{d}{dx}(x)\int \cos x\,dx\right)dx$$

$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

Note:- Write the constant of integration in the last part of the integral while integrating.

Example:- $\int \log x\,dx$

$$= \int (1\times \log x)dx$$

$$= \log x\int 1dx - \int\left(\frac{d}{dx}(\log x)\int 1dx\right)dx = (\log x).x - \int \frac{1}{x}.xdx = x(\log x) - x + C$$

Example:-

Evaluate $\int e^x.\cos x\,dx$

Let $I = \int e^x.\cos x\,dx$

$$= \cos x\int e^x dx - \int\left(\frac{d}{dx}(\cos x)\int e^x dx\right)dx$$

$$= e^x.\cos x + \int \sin x\,.e^x dx$$

$$= e^x\cos x + \sin x\int e^x dx - \int\left(\frac{d}{dx}(\sin x)\int e^x dx\right)dx$$

$$= e^x\cos x + \sin x\,.e^x - \int e^x.\cos x\,dx$$

$$I = e^x\cos x + e^x\sin x - I + C_1$$

$$\Rightarrow 2I = e^x(\cos x + \sin x) + C_1 \Rightarrow I = \frac{e^x}{2}(\cos x + \sin x) + C$$

Note:- Above integral can also be determined by taking $\cos x$ as the first function and e^x the second function.

Integral of the type $\int e^x(f(x)+f'(x))dx$

We have $I=\int e^x(f(x)+f'(x))dx$

$$=\int e^x f(x)dx+\int e^x.f'(x)dx$$

$$=f(x)e^x-\int f'(x)e^x dx+\int e^x.f'(x)dx+C=e^x.f(x)+C$$

Thus $\int e^x(f(x)+f'(x))dx=e^x.f(x)+C$

Example:-

(a) $\int e^x(\sin x+\cos x)dx=e^x\sin x+C$

(b) $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx=e^x.\tan^{-1}x+C$

(c) $\int e^x\left(\frac{x^2+1}{(x+1)^2}\right)dx \quad =\int e^x\left(\frac{(x^2-1)+2}{(x+1)^2}\right)dx$

$=\int e^x\left(\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right)dx \quad =\int e^x\left(\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right)dx$

$=e^x.\left(\frac{x-1}{x+1}\right)+C$ here $f(x)=\frac{x-1}{x+1}$,$f'(x)=\frac{2}{(x+1)^2}$

(d) $\int\frac{x.e^x}{(1+x)^2}dx=\int e^x\left(\frac{x+1-1}{(1+x)^2}\right)dx \quad =\int e^x\left(\frac{x+1}{(1+x)^2}+\frac{-1}{(1+x)^2}\right)dx$

$=\int e^x\left(\frac{1}{1+x}+\frac{-1}{(1+x)^2}\right)dx=e^x\times\frac{1}{1+x}+C$

Integrals of some more types based on integration by parts method:-

Here we will discuss some more special types of integrals which can be proved by using integration by parts and directly used to evaluate given integrals.

(a) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right| + C$

(b) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}log\left|x+\sqrt{x^2+a^2}\right| + C$

(c) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right) + C$

Proof:-

(a) Let $I = \int 1 \times \sqrt{x^2-a^2}dx$

$$= \sqrt{x^2-a^2}\int dx - \int\left(\frac{d}{dx}\sqrt{x^2-a^2}\int 1dx\right)dx$$

$$= x.\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$= x.\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$$

$$= x.\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}dx - \int \frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - I - a^2\int \frac{1}{\sqrt{x^2-a^2}}dx$$

$$2I = x\sqrt{x^2-a^2} - a^2\, log\left|x+\sqrt{x^2-a^2}\right| + C_1$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right| + C$$

Similarly, we can prove other formulae given above

Example:-

(a) $\int \sqrt{1-4x^2}dx = \int \sqrt{1-(2x)^2}dx$ Put $2x = t$

$\Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

$$= \int \sqrt{1-t^2}\frac{dt}{2} = \frac{1}{2}\int \sqrt{1-t^2}dt = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}sin^{-1}\left(\frac{t}{2}\right)\right] + C$$

$$= \frac{1}{2}\left[x\sqrt{1-4x^2} + \frac{1}{2}sin^{-1}2\,x\right] + C \quad = \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}sin^{-1}2\,x + C$$

(b) $\int \sqrt{1+\frac{x^2}{9}}\,dx$

$=\int \sqrt{1+\left(\frac{x}{3}\right)^2}\,dx = \frac{x}{6}\sqrt{9+x^2}+\frac{3}{2}log\left|x+\sqrt{9+x^2}\right|+C$

Method to evaluate integrals of the form $\int \sqrt{ax^2+bx+c}\,dx$

Let $I = \int \sqrt{ax^2+bx+c}dx$

$= \int \sqrt{t^2 \pm k^2}dt or \int \sqrt{k^2-t^2}dt$ (As discussed earlier)

Then apply a suitable formula to integrate.

Example:-

(a) $\int \sqrt{x^2-8x+7}\,dx$

Let $I = \int \sqrt{x^2-8x+7}\,dx = \int \sqrt{x^2-8x+16-9}\,dx = \int \sqrt{(x-4)^2-3^2}dx$

$=\frac{x-4}{2}\sqrt{x^2-8x+7}-\frac{9}{2}log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

(b) $\int \sqrt{1+3x-x^2}dx$

$= \int \sqrt{-(x^2-3x-1)}\,dx = \int \sqrt{-\left\{\left(x-\frac{3}{2}\right)^2-\frac{13}{4}\right\}}\,dx$

$= \int \sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2}\,dx = \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$

Method to Evaluate integrals of the form $\int (px+q)\sqrt{ax^2+bx+c}dx$

To evaluate such integrals we firstly write $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

$$= A(2ax+b)+B$$

Then find A and B by comparing the coefficient of like powers of x from both sides

Example:- Evaluate $\int(x-3)\sqrt{x^2+3x-18}dx$

Solution:-

Let $I=\int(x-3)\sqrt{x^2+3x-18}dx$

Now $x-3=A(2x+3)+B$

$=2Ax+(3A+B)$

Equating the coefficient of x and constant term from both sides we get

$$2A=1,3A+B=-3$$

$$\Rightarrow A=\frac{1}{2},B=-\frac{9}{2}$$

Thus the given integral reduces in the following form

$$I=\int\left(\frac{1}{2}(2x+3)-\frac{9}{2}\right)\sqrt{x^2+3x-18}dx$$

$=\frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}dx-\frac{9}{2}\int\sqrt{x^2+3x-18}dx$

$=\frac{1}{2}I_1-\frac{9}{2}I_2$

$I_1=\frac{2}{3}(x^2+3x-18)^{3/2}+C_1$

$I_2=\frac{2x+3}{4}\sqrt{x^2+3x-18}-\frac{81}{8}log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C_2$

$\therefore I=\frac{1}{3}(x^2+3x-18)^{3/2}-\frac{9}{8}(2x+3)\sqrt{x^2+3x-18}+\frac{729}{16}log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|$
$+C$

Definite Integral:-

An integral of the form of $\int_a^b f(x)dx$ is known as definite integral, 'a' and 'b' are called the lower and upper limits of a definite integral.

The value of the definite integral is given by either as the limit of a sum or if it has an anti-derivative F, then its value is the difference between the values of F at the endpoints i.e $F(b) - F(a)$

Definite Integral as the limit of a sum:-

Suppose $f(x)$ be a continuous function in $[a, b]$ divide interval $[a, b]$ into n equal subintervals each of length h so that $h = \frac{b-a}{n}$

Then $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f(a+(n-1)h)]$

The above expression is known as the definition of definite integral as the limit of a sum

Note:- The above expression is also written as

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + \ldots.. + f(a+(n-1)h)]$$

Where $h = \frac{b-a}{n} \to 0 \; as \; n \to \infty$

Working Rule:-

Suppose the given definite integral is $\int_a^b f(x)dx$. Then to find its value as the limit of a sum we use the following steps.

Step – I

Compare the given integral with standard form and find the values of $f(x), a, b,$ and

$nh = b - a$

Step – II

Find the values of $f(x)$ at $x = a, a+h, a+2h.\dots, x = a+(n-1)h$

Step – III

Put the values obtained in the formula

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h)+\dots+f(a+(n-1)h)]$ and simplify it.

Step – IV

While simplifying step – III we make a collection of constants $h, h^2, \dots$..etc and simplify it.

Step – V

Finally, put the values of "nh" and simplify the limit to get the required result.

Example – 1

Evaluate $\int_0^2 (x^2+1)dx$ as the limit of a sum.

Solution:-

By definition

$$\int_a^b f(x)dx = (b-a)\lim_{h\to\infty}\frac{1}{n}[f(z) + f(a+h)+\dots+f(a+(n-1)h)]$$

Where $h = \frac{b-a}{n}$

Here $a = 0, b = 2, h = \frac{2-0}{n} = \frac{2}{n}$

$f(x) = x^2 + 1$

Therefore $\int_0^2 (x^2+1)dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(0) + f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right)+\dots+f\left(\frac{2(n-1)}{n}\right)\right]$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{2^2}{n^2}+1\right)+\left(\frac{4^2}{n^2}+1\right)+\ldots.+\left(\frac{(2n-2)^2}{n^2}+1\right)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[(1+1+1+\ldots.+1)+\frac{1}{n^2}(2^2+4^2+\ldots\ldots+(2n-2)^2)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}(1^2+2^2+\ldots..+(n-1)^2)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{4}{n^2}\frac{(n-1).n.(2n-1)}{6}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2}{3}\frac{(n-1).(2n-1)}{n}\right]$$

$$= 2\lim_{n\to\infty}\left[1+\frac{2}{3}\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)\right] = 2\left(1+\frac{4}{3}\right) = \frac{14}{3}$$

Example – 2

Evaluate $\int_0^4 (x+e^{2x})dx$

Solution:-

By comparing with its standard form $a=0, b=4, f(x)=x+e^{2x}, nh=4$

$$f(a)=f(0)=1$$

$$f(a+h)=f(0+h)=h+e^{2h}$$

$$f(a+2h)=2h+e^{4h}$$

$$f(a+(n-1)h)=(n-1)h+e^{2(n-1)h}$$

Now by using the formula

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+\ldots.+f(a+(n-1)h)]$$

We get $\int_0^4 (x+e^{2x})dx = \lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\ldots.+f(0+(n-1)h)]$

$$= \lim_{h\to 0} h\left[1+(h+e^{2h})+(2h+e^{4h})+\ldots.+(n-1)h+e^{2(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[(h+2h+\ldots+(n-1)h)+\left(1+e^{2h}+e^{4h}+\ldots.+e^{2(n-1)h}\right)\right]$$

$$= \lim_{h\to 0} h\left[h\left(1+2+\ldots.+(n-1)\right)+\frac{(e^{2h})^n-1}{e^{2h}-1}\right]$$

$$= \lim_{h\to 0} h\left[h\times\frac{n(n-1)}{2}+\frac{(e^{2h})^n-1}{e^{2h}-1}\right]$$

$$= \lim_{h\to 0}\frac{nh(nh-h)}{2}+\lim_{h\to 0}\frac{h(e^{2nh}-1)}{e^{2h}-1}$$

$$= \lim_{h\to 0}\frac{4(4-h)}{2}+\lim_{h\to 0}\frac{h(e^8-1)}{e^{2h}-1} = \frac{4(4-0)}{2}+(e^8-1)\frac{1}{\lim\limits_{h\to 0}\frac{e^{2h}-1}{h}}$$

$$= 8+(e^8-1)\frac{1}{2\lim\limits_{h\to 0}\frac{e^{2h}-1}{2h}} = 8+\frac{e^8-1}{2} = \frac{15+e^8}{2}$$

Evaluate the following definite integrals as a limit of a sum

(a) $\int_0^2 e^x dx$ (b) $\int_0^5 (x+1)dx$ (c) $\int_1^4 (x^2-x)dx$

Fundamental Theorem of Integral Calculus:-

The fundamental theorem of integral calculus is a connection between indefinite integral and the definite integral.

The first fundamental theorem of integral calculus

Let 'f' be a continuous function defined on the closed interval [a, b], then

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$ for all $x \in [a,b]$

Second Fundamental Theorem of Integral Calculus:-

Let 'f' be a continuous function defined on the closed interval $[a,b]$ and F be an anti-derivative of 'f' then,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Note:- There is no need to keep integration constant C because

$$\int_a^b f(x)dx = [F(x) + C]_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

Question:- Evaluate

(a) $\int_1^2 (4x^3 + 6x + 9)dx$ (b) $\int_0^{\pi/2} cos^2 x\, dx$ (c) $\int_0^{\pi/2} e^x (sin\, x - cos\, x)dx$

(d) $f(x) = \int_0^x e^t . sin\, t\, dt$, write $f'(x)$ (e) $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

(f) $\int_1^2 \frac{x}{(x+1)(x+2)} dx$ (g) $\int_0^{\pi/4} tan\, x\, dx$ (h) $\int_2^3 \frac{x}{x^2+1} dx$

(i) $\int_0^{2/3} \frac{1}{9x^2+4} dx =$ ____

Evaluation of Definite Integral by substitution:-

There are several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution

To evaluate $\int_a^b f(x)dx$ by substitution, we use the following steps

Working Rule:-

Step – 1, Substitute some part of integral as another variable (say t) such that its differentiation exists in the integral so that given integral reduces to a known form.

Step – 2, Change the upper and lower limits corresponding to the new variable.

Step – 3, Integrate the new integral w.r.t the new variable.

Step – 4, Find the difference of the values of the answer obtained in step – III at new upper and lower limits.

Example:-

(a) Evaluate $\int_{-1}^1 5x^4\sqrt{x^5 + 1}dx$ (b) Evaluate $\int_0^1 \frac{tan^{-1} x}{1+x^2} dx$

(c) Evaluate $\int_1^3 \frac{1}{x(1+log\, x)} dx$ (d) Evaluate $\int_0^{\pi/2} \sqrt{sin\, \theta}\, cos^5 \theta\, d\theta$

(e) Evaluate $\int_0^2 \frac{1}{x+4-x^2} dx$ (f) Evaluate $\int_0^2 x\sqrt{x + 2}dx$

(g) Evaluate $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$ (h) Evaluate $\int_0^1 sin^{-1} \left(\frac{2x}{1+x^2}\right) dx$

(i) $\frac{d}{dx} \int_2^3 e^x\, sin\, x\, dx =$ _____ (j) $\int_0^{\pi/2} sin^2\, x . cos\, x\, dx$

Basic Properties of definite Integral:- In this section, we will study some fundamental properties of definite integrals which are very useful in evaluating integrals.

Properties:-

(a) $\int_a^b f(x)dx = \int_a^b f(t)dt$ (b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

In general $\int_a^b f(x)dx = \int_a^{C_1} f(x)dx + \int_{C_1}^{C_2} f(x)dx + \ldots + \int_{C_h}^{b} f(x)dx$

Where $a < c_1 < c_2 < c_3 < \ldots < c_n < b$

Example:-

(a) $\int_{-1}^{1} f(x)dx$, where $f(x) = \begin{cases} 1-2x, & x \le 0 \\ 1+2x, & x \ge 0 \end{cases}$

(b) $\int_1^4 f(x)dx$, where $f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$

(c) $\int_0^1 |5x-3|dx$ (d) $\int_0^{\pi} |\cos x|dx$ (e) $\int_{-5}^{5} |x-2|dx$

(f) $\int_1^4 (|x-1| + |x-2| + |x-3|)dx$

(g) $\int_{-1}^{3/2} |x \sin \pi x|dx$

Hints here $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x \text{ for } & -1 \le x \le 1 \\ -x \sin \pi x \text{ for } & 1 \le x \le \frac{3}{2} \end{cases}$

(h) $\int_{-1}^{2} |x^3 - x|dx$

Hints $f(x) = |x^3 - x| = \begin{vmatrix} x^3 - x & , & -1 \le x \le 0 \\ -(x^3 - x) & , & 0 \le x \le 1 \\ x^3 - x & , & 1 \le x \le 2 \end{vmatrix}$

(i) $\int_{-1}^{1} e^{|x|}dx$ (j) $\int_0^{\pi/2} \sin^2 x\, dx + \int_0^{\pi/2} \cos^2 y\, dy - \int_0^{\pi/2} dt$

Properties of definite integral

Some important properties which will be useful in evaluating the definite integrals are given below.

(a) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(b) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (Particular case of property 4)

(c) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$

(d) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ \ f(x) \text{ is even function} \\ 0, & if\ \ f(x) \text{ is odd function} \end{cases}$

By using properties of definite integral evaluate the following integrals

(a) $\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x+\cos^4 x} dx$

(b) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

(c) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

(d) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}} dx$

(e) $\int_0^1 x(1-x)^n dx$

(f) $\int_0^{\pi/4} \log(1+\tan x)\, dx$

(g) $\int_0^{\pi/2} \log(\sin x)\, dx$

(h) $\int_{-\pi}^{\pi} \sin^3 x\, dx$

(i) $\int_{-\pi}^{\pi} \cos^3 x\, dx$

(j) $\int_0^{2\pi} \cos^5 x\, dx$

Problems For Practice

Evaluate the following integrals

(a) $\int \cos 6x\sqrt{1+\sin 6x}dx$

(b) $\int \frac{x^4}{(x-1)(x^2+1)} dx$

(c) $\int \left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$

(d) $\int \sec^3 x\, dx$

(e) $\int \frac{1}{2+3\sin x} dx$

(f) $\int \frac{1}{2\cos^2 x+3\sin^2 x} dx$

(g) $\int_{-1}^{3/2} |x \sin(\pi x)| dx$

(h) $\int_0^{\pi} \frac{x}{a^2\cos^2 x+b^2\sin^2 x} dx$

(i) $\int \frac{x^3}{\sqrt{1-x^8}} dx$

(j) $\int_{-1}^{1} x^{17}\cos^4 x\, dx$