Chapter- 12

Linear Programming Problems

Previous knowledge:

- In Linear Inequalities, we learned how to make a graph of equations with Inequalities.
- Here, we will try to find the minimum or maximum value of an equation within a given set of conditions.
- For example: Maximise profit, if the Number of hours worked < 10, etc.
- These types of problems, where we need to find the minimum or maximum values, based on some conditions are known as **Optimization Problems.**

Linear Programming

Linear programming is an optimization technique for a system of linear constraints and a linear objective function. For obtaining the maximum or minimum values as required of a linear expression to satisfy a certain number of given linear restrictions.

Linear programming deals with that class of problems for which all relations among the variables are linear.

Linear Programming Problem (LPP)

The linear programming problem in general calls for optimizing a linear function of variables called the objective function subject to a set of linear equations and/or linear inequations called the constraints or restrictions.

Objective Function

The function which is to be optimized (maximized/minimized) is called an objective function.

$e.g.$ Maximum $Z = 250\,x + 120\,y$

Constraints

The system of linear inequations (or equations) under which the objective function is to be optimized is called constraints.

$$e.g. -2x + 5y \geq 10, \qquad 5x - y \geq 0$$

Non-negative Restrictions

All the variables considered for making decisions assume non-negative values.

Mathematical Description of a General Linear Programming Problem

- A general LPP can be stated as (Max/Min) $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ (Objective function)

subject to constraints

$$\begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_1 (\leq = \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n (\leq = \geq) b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{Bmatrix}$$

$$a_{m_1}x_1 + a_{m_2}x_2 + \ldots + a_{m_n}x_n (\leq = \geq) b_m$$

and the non-negative restrictions $x_l, x_2, \ldots, x_n \geq 0$

where all $a_{11}, a_{12}, \ldots, a_{mn}$; $b_l, b_2, \ldots, b_m$ are constants; $c_l, c_2, \ldots, c_n$ are constants known as cost coefficients and $x_1, x_2, \ldots, x_n$ are known as decision variables.

Different Type of Linear Programming Problems

The various types of problems in linear programming problems are included. They are

- **Manufacturing problem-** Here we maximize the profit with the help of minimum utilization of the resource.
- **Diet Problem-** We determine the number of different nutrients in a diet to minimize the cost of manufacturing resource
- **Transportation problem-** Here we determine the schedule to find the cheapest way of transporting a product at minimum time.
- **Assignment Problem –**It deals with the allocation of the various resources to the various activities on one to one basis.

Mathematical Formulation of LPP

Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form. There is not any set procedure to formulate linear programming problems. The following algorithm will be helpful in the formulation of linear programming problems.

ALGORITHM

STEP I	In every LPP certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denote them by $x_1, x_2, \ldots\ldots\ldots$
STEP-II	Identify the objective function and express it as a linear function of the variables introduced in Step I.
STEP III	In an LPP, the objective function may be in the form of maximizing profits or minimizing costs. So, after expressing the objective function as a linear function of the decision variables, we must find the type of optimization $i.e.$ maximization or minimization. Identify the type of objective function.
STEP IV	Identify the set of constraints, stated in terms of decision variables, and express them as linear inequations or equations as the case may be.

Example 1

Two tailors A and B earn ₹ 300 and ₹ 400per days respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

Answer:

To minimize labour cost means to assume minimized earnings

i.e. Min(Z) = $150x + 200y$

subject to constraints

$$6x + 10y \geq 60$$
$$4x + 4y \geq 32$$
$$x \geq 0. y \geq 0$$

Example 2

A toy company manufactures two types of dolls; a basic version of doll A and a deluxe version of doll B. Each doll of type B takes twice as long to produce a one of type A and the company would have time to make a maximum of 2000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day(both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of ₹ 3 and ₹ 5 per doll respectively on doll A and doll B; how many of each should be produced per day to maximize profit?

Answer:

Maximize $Z = 3x + 5y$

Subject to constraints

$$x + 2y \leq 2000$$
$$x + y \leq 1500$$
$$y \leq 600$$
$$x \geq 0. y \geq 0$$

Example 3

A dietician wishes to mix two types of foods in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C. Food II contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs ₹ 50 per kg to purchase food I and ₹ 70 per kg to purchase food II. Formulate the problem as a linear programming problem to minimize the cost of such a mixture.

Example 4

A firm manufactures 3 products A, B, and C. The profits are ₹ 3, ₹ 2, and ₹ 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.

Machine	Products		
	A	B	C
M_1	4	3	5
M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 A's, 200 B's, and 50 C's but not more than 150 A's. Set up an LPP to maximize the profit.

Answer:

Maximize $U = 3x + 2y + 4z$

Subject to constraints

$$4x + 3y + 5z \leq 2000$$
$$2x + 2y + 4z \leq 2500$$
$$100 \leq x \leq 150$$
$$y \geq 200$$
$$z \geq 50$$
$$x, y, z \geq 0$$

Example 5

An airline agrees to charter planes for a group. The group needs at least 160 first-class seats and at least 300 tourist class seats. The airline must use at least two of its model 314 planes which have 20 first-class seats and 30 tourist class seats. The airline will also use some of its model 535 planes which have 20 first-class seats and 60 tourist class seats. Each flight of a model 314 plane costs the company ₹ 100,000 and each flight of a model 535 plane costs ₹ 150, 000. How many of each type of plane should be used to minimize the flight cost? Formulate this as an LPP.

Answer:

	Model 314		Model	Limit
Variable	X	+	Y	
F class	20x	+	20y	≥ 160
T class	30x	+	60y	≥ 300
Cost	1.x lakh	+	1.5y lakh	Z

The above LPP can be presented in the table above.

The flight cost is to be minimized i.e $MinZ = X + 1.5y$ s.t the constraints.

$x \geq 2$ At least 2 planes of model 314 must be used

$y \geq 0$ At least 1 plane of model 535 must be used

$20x + 20y \geq 160$ required at least 160 F class seats

$30x + 60y \geq 300$ require at least 300 T class seats

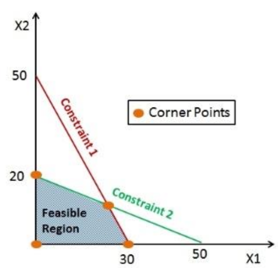
Solving the above inequalities as equations we get,

When x = 0, y = 8 and when y=0, x = 8, when x = 0, y=5 and when y=0, x=10

We get an unbounded region 8-E-10 as a feasible solution. Plotting the corner points and evaluating we have.

Important Definitions and Results

- **The solution of an LPP:** A set of values of the variables $x_1, x_2, \ldots, x_n$ satisfying the constraints of an LPP is called a solution of the LPP.
- **Feasible Solution of an LPP:** A set of values of the variables $x_1, x_2, \ldots, x_n$ satisfying the constraints and non-negative restrictions of an LPP is called a feasible solution of the LPP.
- **Feasible Region:** The common region determined by all the constraints of an LPP is called the feasible region and every point in this region is a feasible solution of the given LPP.



Infeasible Solution: A solution of an LPP is an infeasible solution if it does not satisfy the non-negativity restrictions.

- **Infeasible Region:** The region other than the feasible region is called the infeasible region.
- **Optimal Solution of an LPP**: A feasible solution of an LPP is said to, be optimal (or optimum) if it also optimizes the objective function of the problem.
- **Convex set**

A set S is convex if any point on the line segment connecting any two points in the set is also in S.

Remember:

The set of all feasible solutions of an LPP is a convex set.

- **Bounded Feasible Region:** A feasible region is said to be bounded if it can be enclosed within a circle.
- **Unbounded Feasible Region:** A feasible region is said to be unbounded if it cannot be enclosed within any circle $i.e.$, if it extends indefinitely in any direction.
- **Fundamental Extreme Point Theorem:** An optimal solution of an LPP, if it exists, occurs at one of the extreme (corner) points of the convex polygon of the set of all feasible solutions.

Example 1:

Choose the correct answer from the given options.

1. The inequation $2x + 3y \leq 6$ represents ______
   (a) The open half-plane containing the origin
   (b) A closed half-plane containing the origin
   (c) Open half-plane not containing the origin
   (d) Closed half-plane not containing the origin
2. The objective function of an LPP is
   (a) a constant (b) a function to be optimized
   (c) a relation between the variables (d) None of these
3. The optimal value of the objective function is attained at the points
   (a) On the x-axis
   (b) On the y-axis
   (c) which are corner points of the feasible region
   (d) None of these
4. The value of the objective function is maximum under linear constraints:
   (a) At the center of the feasible region
   (b) At $(0, 0)$
   (c) At any vertex of the feasible region
   (d) The vertex which is the maximum distance from $(0, 0)$
5. Which of the following statement is correct?
   (a) Every LPP admits an optimal solution
   (b) An LPP admits a unique optimal solution
   (c) If an LPP admits two optimal solutions if it has an infinite number of optimal solutions
   (d) The set of all feasible solutions of an LPP is not a convex set.

Example 2:

Fill in the blanks:

6. Any solution to an LPP which satisfies the non-negative restrictions are called _____.
7. If the half-planes include the straight-line $ax + by + c = 0$, then these are represented by _____.
8. In LPP, the function is to be optimized is called _____.

Example 3:

9. What happens when the objective function attains optimum value at more than one point?
10. Check whether the region given by $2x + 5y \geq 1$ is a bounded region.

Graphical Method of solution for problems in two variables.

There are two techniques of solving an LPP by graphical method

1. Corner point method and 2. Iso-profit or Iso-cost method

CORNER-POINT METHOD

This method of solving an LPP graphically is based on the principle of the extreme point theorem.

The following algorithm can be used to solve an LPP in two variables graphically by using the corner–point method.

ALGORITHM

STEP I Formulate the given LPP in the mathematical form if it is not so.

STEP-II Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, put $y = 0$ in it and obtain a point on $x - axis$. Similarly, by putting $x = 0$ obtain a point on $y - axis$. Join these two points to obtain the graph representing the equation.

STEP III Determine the region represented by each inequation. To determine the region represented by an inequation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation.

STEP IV Obtain the region in $xy -$ plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the LPP.

STEP V Determine the coordinates of the vertices (corner points) of the convex polygon obtained in Step II. These vertices are known as the extreme points of the set of all feasible solutions of the LPP.

STEP VI Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LPP.

REMARK 1: If the feasible region of an LPP is bounded $i.e.$, it is a convex polygon. Then, the objective function $Z = ax + by$ has both a maximum value M and a minimum value m and each of these values occurs at a corner point of the convex polygon.

REMARK 2: Sometimes the feasible region of an LPP is not a bounded convex polygon. That is, it extends indefinitely in any direction. In such cases, we say that the feasible region is unbounded.

If the feasible region is unbounded, then we find the values of the objective function $Z = ax + by$ at each corner point of the feasible region. Let M and m respectively denote the largest and smallest values of Z at these points. To check whether Z has maximum and minimum values as M and m respectively, we proceed as follows:

i. Draw the line $ax + by = M$ and find the open half-plane $ax + by > M$. If the open half-plane represented by $ax + by > M$ has no point common with the unbounded feasible region, then M is the maximum value of Z. Otherwise, Z has no maximum value.

ii. Draw the line $ax + by = m$ and find the open half-plane $ax + by < m$. If the open half-plane represented by $ax + by < m$ has no point common with the unbounded feasible region, then m is the minimum value of Z. Otherwise, Z has no minimum value.

Example 1:

Solve the following LPP graphically.

Maximize: $Z = 5x + 3y$ Subject to $3x + 5y \le 15$

$5x + 2y \le 10$ and $x \ge 0,\ y \ge 0$

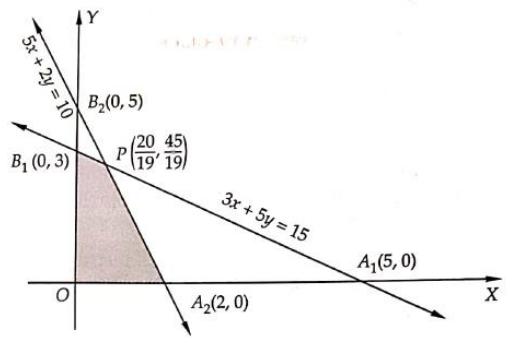
Solution:- converting the given inequations into equations, we obtain the following equations:

$$3x + 5y = 15, 5x + 2y = 10, x = 0 and y = 0$$

The region represented by $3x + 5y \le 15$: The line $3x + 5y = 15$ meets the coordinate axes at $A_1(5,0) and B_1(0,3)$ respectively. Join these points to obtain the line $3x + 5y = 15$. Clearly (0, 0) satisfies the inequation $3x + 5y \le 15$. So, the region containing the origin represents the solution set of the inequation $3x + 5y \le 15$.

Region Represented by $5x + 2y \le 10$: The line $5x + 2y = 10$meets the coordinate axes at $A_2(2,0)$ and $B_2(0,5)$ respectively. Join these points to obtain the graph of line $5x + 2y = 10$. Clearly, (0, 0) satisfies the inequation $5x + 2y \le 10$. So, the region containing the origin represents the solution set of this inequation.

The region represented by $x \ge 0 and y \ge 0$: Since every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the inequations $x \ge 0 and y \ge 0$. The shaded region OA_2PB_1 in Fig. 30.6 represents the common region of the above inequations. This region is the feasible region of the given LPP. The coordinates of the vertices (corner points) of the shaded feasible region are O(0, 0), A_2(2, 0), P(20/19, 45/19), and B_1(0, 3).



These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table

Point (x, y)	Value of the objective function $Z = 5x + 3y$
$O(0,0)$	$Z = 5 \times 0 + 3 \times 0 = 0$
$A_2(2,0)$	$Z = 5 \times 2 + 3 \times 0 = 10$
$P\left(\frac{20}{19}, \frac{45}{19}\right)$	$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{235}{19}$
$B_1(0,3)$	$Z = 5 \times 0 + 3 \times 3 = 9$

Z is maximum at $P(20/19, 45/19)$. Hence, $x = 20/19, y = 45/19$ is the optimal solution of the given LPP and the optimal value of Z is 235/19.

Example – 2

Solve the following LPP by the graphical method:

Minimize $Z = 20x + 10y$

Subject to

$x + 2y \leq 40$ $3x + y \geq 30$ $4x + 3y \geq 60$ and $x, y \geq 0$

Solution: Converting the given inequations into equations, we obtain the following equations:

$$x + 2y = 40, 3x + y = 30, 4x + 3y = 60, x = 0 and y = 0$$

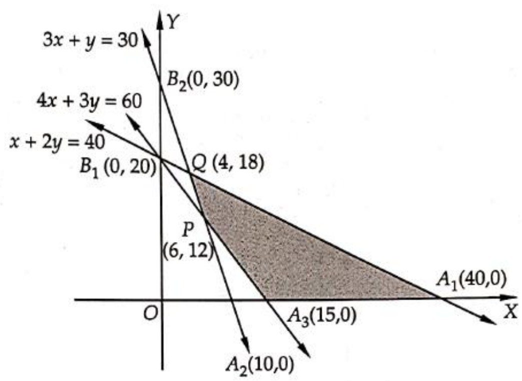
The region represented by $x + 2y \leq 40$: The line $x + 2y = 40$ meets the coordinate axes at $A_1(40,0) and B_1(0,20)$respectively. Join these points to obtain the line $x + 2y = 40$. Clearly $(0,0)$ satisfies the inequation $x + 2y \leq 40$. So, the region in XY-plane that contains the origin represents the solution set of the given inequation.

The region represented by $3x + y \geq 30$: The line $3x + y = 30$ meets x and y axes at $A_2(10, 0)$ and $B_2(0, 30)$ respectively. Join these points to obtain this line. We find that the point O(0, 0) does not satisfy the inequation $3x + y \geq 30$. So, that region in the xy-plane that does not contain the origin is the solution set of this inequation.

The region represented by $4x + 3y \geq 60$: The line $4x + 3y = 60$ meets the x and y-axis at $A_3(15, 0)$ and $B_1(0, 20)$ respectively. Join these points to obtain the line $4x + 3y = 60$. We observe that the point O(0, 0) does not satisfy the inequation $4x + 3y \geq 60$. So, the region not containing the origin in xy-plane represents the solution set of the given inequation.

The region represented by $x \geq 0, y \geq 0$: Clearly, the region represented by the non-negativity restriction $x \geq 0\ and y \geq 0$ is the first quadrant in xy-plane.

The shaded region A_3 A_1 QP in Fig. 30.7 represents the common region of the regions represented by the above inequations. This region represents the feasible region of the given LPP.



The coordinates of the corner points of the shaded feasible region are $A_3(15, 0)$, $A_1(40, 0)$, Q(4, 18), and P(6, 12). These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously. The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 20x + 10y$
$A_3(15,0)$	$Z = 20 \times 15 + 10 \times 0 = 300$
$A_1(40,0)$	$Z = 20 \times 40 + 10 \times 0 = 800$
$Q(4,18)$	$Z = 20 \times 4 + 10 \times 18 = 260$
$P(6,12)$	$Z = 20 \times 6 + 10 \times 12 = 240$

Out of these values of Z, the minimum value is 240 which is attained at point P(6, 12). Hence, x = 6, y = 12 is the optimal solution of the given LPP and the optimal value of Z is 240.

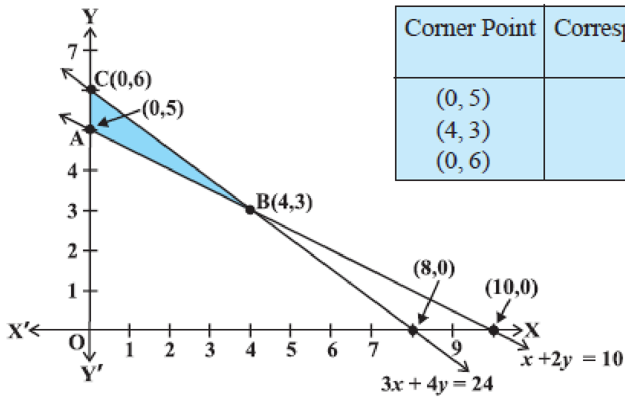
Example 3:

Solve the following LPP graphically.

Minimize: $Z = 200x + 500y$

Subject to $x + 2y \geq 10 \quad 3x + 4y \leq 24$ and $x \geq 0,\ y \geq 0$

Solution:- the shaded region in Fig. 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is bounded. The coordinates of corner points.



Corner Point	Corresponding value of Z
(0, 5)	2500
(4, 3)	**2300** ← Minimum
(0, 6)	3000

A, B and C are (0, 5), (4, 3) and (0, 6) respectively. Now we evaluate $Z = 200x + 500y$ at these points.

Hence, the minimum value of Z is 2300 attained at the point (4, 3)

Example 4:

Determine graphically the minimum value of the objective function

$$Z = -50x + 20y$$

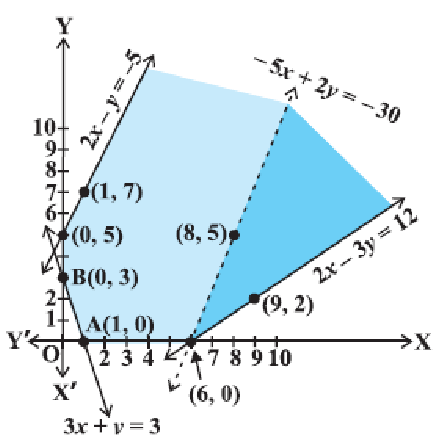
Subject to the constraints: $2x - y \geq -5$

$$3x + y \geq 3$$

$$2x - 3y \leq 12$$

and $x \geq 0,\ y \geq 0$

Solution: First of all, let us graph the feasible region of the system of inequalities (2) to (5). The feasible region (shaded) is shown in Fig. 12.5. Observe that the feasible region is unbounded. We now evaluate Z at the corner points.



Corner Point	$Z = -50x + 20y$
(0, 5)	100
(0, 3)	60
(1, 0)	−50
(6, 0)	−300 ← smallest

From this table, we find that -300 is the smallest value of Z at the corner point (6, 0). Can we say that the minimum value of Z is -300? Note that if the region would have been bounded, this smallest value of Z is the minimum value of Z 9Theorem 2). But here we see that the feasible region is unbounded. Therefore, -300 may or may not be the minimum value of Z. To decide this issue, we graph the inequality.

$-50x + 20y < -300$ (see step 3(ii) of corner Point Method.)

i.e $-5x + 2y < -30$and check whether the resulting open half-plane has points in common with a feasible region or not. If it has common points, then -300 will not be the minimum value of Z. Otherwise, -300 will be the minimum value of Z.

As shown in Fig. 12.5, it has common points. Therefore, $Z = -50x + 20y$ has no minimum value subject to the given constraints.

In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0, 5): For this, check whether the graph of $-50x + 20y > 100$ has points in common with the feasible region.

Example 4: A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain at least 8 units of vitamin a and 10 units of vitamin C. Food 'I' contains 2 units/ kg of vitamin and 1 unit/kg of vitamin c. Food 'II' contains 1 unit/kg of vitamin a and 2 units/ kg of vitamin C. It costs Rs. 50 per kg to purchase Food 'I' and Rs. 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimize the cost of such a mixture.

Solution: Let the mixture contain x kg of Food 'I' and y kg of Food 'II'. $x \geq 0, y \geq 0$. We make the following table from the given data.

Resources	Food I (x)	Food II (y)	Requirement
Vitamin A (units/kg)	2	1	8
Vitamin C (units/kg)	1	2	10
Cost (Rs/kg)	50	70	

Since the mixture must contain at least 8 units of vitamin a and 10 units of vitamin c, we have the constraints:

$2x + y \geq 8 \qquad x + 2y \geq 10$

Total cost Z of purchasing x kg of food 'I' and y kg of Food 'II' is

$$Z = 50x + 70y$$

Hence, the mathematical formulation of the problem is:

Minimize $Z = 50x + 70y$................................. (1)

Subject to the constraints:

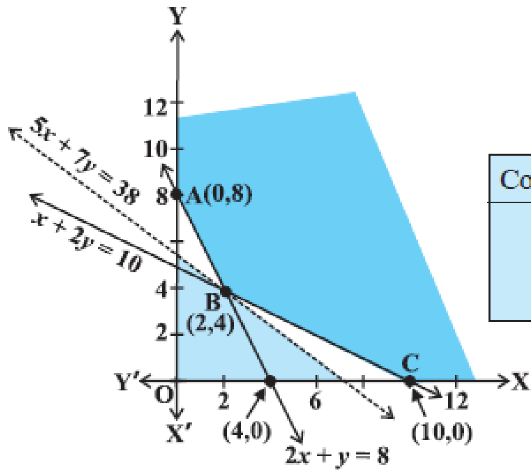
$2x + y \geq 8$............................... (2)

$x + 2y \geq 10$ .... (3)

$x, y \geq 0$ .... (4)

Let us graph the inequalities (2) to (4). The feasible region determined by the system is shown in Fig. 12.7. Here again, observe that the feasible region is unbounded.

Let us evaluate Z at the corner points A(0, 8), B(2, 4), and C(10, 0)



Corner Point	$Z = 50x + 70y$	
(0,8)	560	
(2,4)	**380**	← Minimum
(10,0)	500	

In the table, we find that the smallest value of Z is 380 at the point (2, 4). Can we say that the minimum value of Z is 380? Remember that the feasible region is unbounded. Therefore, we have to draw the graph of the inequality.

$$50x + 70y < 380 \text{ i.e } 5x + 7y < 38$$

To check whether the resulting open half-plane has any point common with the feasible region. From fig. 12.7, we see that it has no points in common.

Thus, the minimum value of Z is 380 attained at the point (2, 4). Hence, the optimal mixing strategy for the dietician would be to mix 2 kg of Food 'I' and 4 kg of Food 'II', and with this strategy, the minimum cost of the mixture will be Rs. 380.

Example 5:

A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol, and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol, and 3 units of vitamin A. The diet requires at least 240 units of calcium, at least 460 units of iron, and at most 300 units of cholesterol. How many packets of each food should be used to minimize the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution:- Let x and y be the number of packets of food P and Q respectively. $x \geq 0, y \geq 0$. The mathematical formulation of the given problem is as follows:

Minimise $Z = 6x + 3y$ (vitamin A)

Subject to the constraints

$12x + 3y \geq 240$ (a constraint on calcium), i.e $4x + y \geq 80$ .... (1)

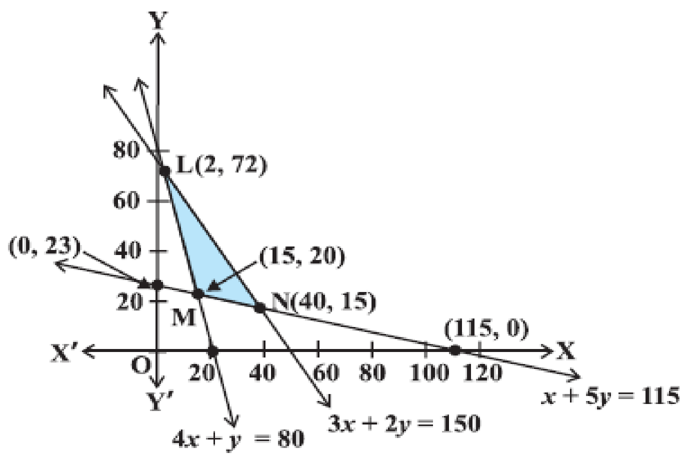
$4x + 20y \geq 460$ (a constraint on iron) i.e $x + 5y \geq 115$ .... (2)

$6x + 4y \leq 300$ (a constraint on cholesterol) i.e $3x + 2y \leq 150$ .... (3)

$x \geq 0, y \geq 0$ .... (4)

Let us graph the inequalities (1) to (4)

The feasible region (shaded) determined by the constraints (1) to (4) is shown in fig. 12.10 and note that it is bounded.



The coordinates of the corner points L, M, and N are (2, 72), (15, 20), and (40, 15) respectively. Let us evaluate Z at these points.

From the table, we find that Z is the minimum at the point (15, 20). Hence, the amount of vitamins under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin a will be 150 units.

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** ←	Minimum
(40, 15)	285	

Example-6(Transportation Problem)

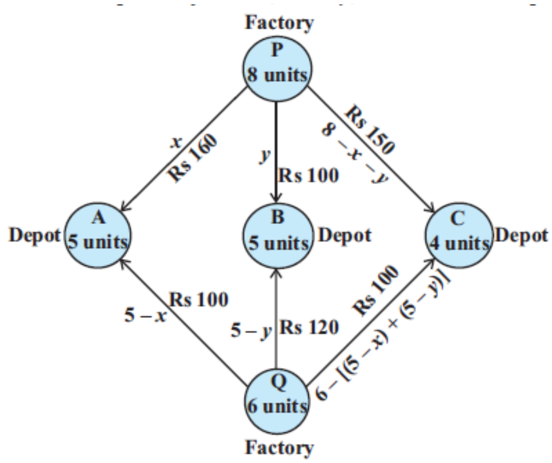
There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situations at A, B, and C. The weekly requirements of the depots are respectively 5, 5, and 4 units of the commodity while the production capacity of the factories at P and q are respectively 8 and 6 units. The cost of transportation per unit is given below.

From/To	Cost (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot so that the transportation cost is minimum? What will be the minimum transportation cost?

Solution:- The problem can be explained diagrammatically as follows.

Let x units and y units of the commodity being transported from the factory at P to the depots at A and B respectively. Then $(8-x-y)$ units will be transported to the depot at C (Why?)



Hence, we have $x \geq 0, y \geq 0 and 8-x-y \geq 0$

i.e $x \geq 0, y \geq 0 and x+y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5-x)$ units need to be transported from the factory at Q. Obviously, $5-x \geq 0, i.e x \leq 5$.

Similarly, $(5-y) and 6-(5-x+5-y) = x+y-4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5-y \geq 0, x+y-4 \geq 0$ i.e $y \leq 5, x+y \geq 4$

Total transportation cost Z is given by

$$Z = 160x + 100y + 100(5 - x) + 120(5 - y) + 100(x + y - 4) + 150(8 - x - y)$$
$$= 10(x - 7y + 190)$$

Therefore, the problem reduces to minimize $Z = 10(x - 7y + 190)$ subject to the constraints:

$x \geq 0, y \geq 0$ .... (1)

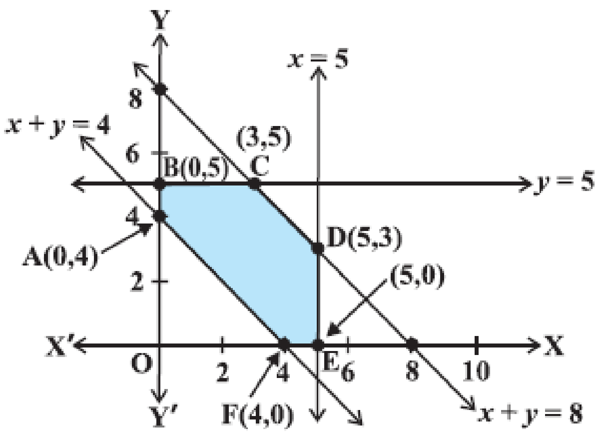
$x + y \leq 8$ .... (2)

$x \leq 5$ .... (3)

$y \leq 5$ .... (4)

And $x + y \geq 4$ .... (5)

The shaded region ABCDEF represented by the constraints (1) to (5) is the feasible region:



Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

From the table, we see that the minimum value of z is 1550 at the point (0, 5). Hence, the optimal transportation strategy will be to deliver 0, 5, and 3 units from the factory at P and 5,0 and 1 units from the factory at ! to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e Rs. 1550.

Corner Point	$Z = 10(x - 7y + 190)$	
(0, 4)	1620	
(0, 5)	1550 ←	Minimum
(3, 5)	1580	
(5, 3)	1740	
(5, 0)	1950	
(4, 0)	1940	