Types of Relations

SUBJECT : MATHEMATICS
CHAPTER NUMBER: 01
CHAPTER NAME :RELATIONS AND FUNCTIONS

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What we expect to learn?

- Students will be able to learn different types of relations.
- Students will be able to learn about finding equivalence class related to a given relation.
- Students will be able to learn about different types of functions.
- Students will be able to learn about composite functions and invertible functions.
- Students will able to find the inverse of a function.

Introduction

Relation:

Let A and B be two be two non -empty sets. A relation R from a set A to set B is a subset of A x B.

i.e., $R \subseteq A \times B$.

Relation on a set A:

Let A be any non empty set. Then a set R is said to be a relation on A if R is a subset of $A \times A$.

Types of Relations:-

1. **Empty or Void relation:-** A relation R on the set A is called an empty relation if no elements of A is related to any elements of A, i.e., R = ϕ.

Example: Let A = { 1, 2, 3 }, define R ={(a, b) : a - b = 12 }. Show that R is an empty relation on set A.

2. **Universal Relation:-** A relation R on a set A is called universal relation if each element of A is related to every element of A. i.e. if $R = A \times A$.

Example: Let A = {1, 2} and define R = { (a, b) : a + b > 0 }. Show that R is a universal relation on set A.

3. Identity Relation:- A relation R on set A is called identity relation if every element of A is related to itself only. i.e. if R = {(a, a) : a ∈ A }.

The identity relation on set A is denoted by I_A.

Example: Let A = {1, 2, 3} and the relation R defined by R = {(a, b) : a - b = 0 : a, b ∈ A }. Show that R is an identity relation.

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Assignments

1. Let R be the relation in the set N given by R = {(a, b) : a = b - 2 , b > 6}. Choose the correct answer.

   a) (2, 7) ∈ R   b) (3, 8) ∈ R   c) (6, 8) ∈ R   d) (8, 7) ∈ R

1. Let A and B be finite sets containing m and n elements respectively. Find the number of relations that can be defined from A to B is

   a) 2^{mn} b) 2^{m+n} c) mn   d) 0

1. If R = {(a, a^3) : a is a prime number less than 5} be a relation, then find the range of R.
2. If R = {(x, y) : x + 2y = 8} is a relation on N, then write the range of R.

Reflexive, Symmetric, and Transitive Relations

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Types of Relations (contd.)

4. Reflexive Relation:

A relation R on the set A is called reflexive relation if **a R a** for every a ∈ A i.e., if (a, a) ∈ R for every a ∈ A.

Points to Remember:

- ✓ Identity and Universal relations are reflexive, but empty relation is not reflexive.
- ✓ All reflexive relations are not identity relation.

5. Symmetric Relation:

A relation R on the set A is called symmetric relation if **a R b** implies **b R a**.

i.e., for every a, b ∈ A , (a, b) ∈ R ⇒ (b, a) ∈ R.

Points to Remember:

- ✓ Identity and Universal relations are symmetric relations.
- ✓ Empty relation is also symmetric as there is no situation in which (a, b) ∈ R. .

6. Transitive Relation:

A relation R on the set A is called transitive relation if **a R b** and **b R c** implies **a R c**, for every a, b, c ∈ A.

i.e., if (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R. .

Point to Remember:

✓ If there is no situation in which (a, b) ∈ R and (b, c) ∈ R, then relation is transitive.

Examples:

1. Determine whether the relation $R = \{(x, y) : y = x + 5 , x < 4 \}$ reflexive, symmetric and transitive on the set of natural numbers.
2. Find whether the relation R' on the set R of all real numbers defined as $R' = \{(a, b) : a, b \in R$ and $a - b + \sqrt{3} \in S \}$ where S is the set of all irrational numbers, is reflexive, symmetric and transitive.
3. Let $A = \{a, b, c\}$ and the relation R be defined on A as follows: $R = \{(a, a), (b, c), (a, b)\}$. Then write the minimum number of ordered pairs added to R to make it reflexive and transitive.
4. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2) , (2, 1)\}$ not to be transitive.

Assignments

1. Show that the relation S = {(a, b) : a, b ∈ R and $a \leq b^3$} defined in the set of real numbers, is neither reflexive nor symmetric nor transitive .
2. Let A = {1, 2, 3}. Define a relation on A which have properties being

   (i) reflexive, symmetric but not transitive.

   (ii) symmetric but neither reflexive nor transitive.

   (iii) reflexive, symmetric and transitive.
3. Show that the relation S on R the set of real numbers given as S = {(x, y) : $x \leq y$ } is reflexive but not symmetric.

Assignments

4. Check whether the relation R in the set **R** of real numbers, defined by R = {(a, b) : 1 + ab > 0 } is reflexive, symmetric or transitive.
5. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 3) , (3, 1)} not to be transitive.
6. Let W denotes the words in the English dictionary. Define the relation R by R = {(x,y) ∈ W × W : the words x and y have at least one letter in common}. Prove that R is reflexive and symmetric.

Equivalence Relations & Equivalence Classes

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Equivalence Relation and Equivalence Class

Equivalence Relation:- A relation R on a set A is called equivalence relation if R is reflexive, symmetric and transitive.

Equivalence Class: - Let R be an equivalence relation on A set a and let $a \in A$. Then we define the equivalence class of 'a' as $[a] = \{b \in A : bisrelatedtoa\}$ $= \{b \in A : (b, a) \in R\}$

Example:-

$A = \{1,2,3\}$. Define the relations R_1, R_2, R_3 and R_4 on A as.

(i) $R_1 = \{(1,1), (1,2), (2,1), (2,2)\}$ (ii) $R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$

(iii) $R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$

(iv) $R_4 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

Check whether R_1, R_2, R_3 and R_4 are equivalence relations or not, if yes, then find the equivalence classes of all elements of set A.

EXAMPLE:

$A = \{1,2,3\}$. Define the relations R_1, R_2, $R_{3,}$ and R_4 on A as.

(i) $R_1 = \{(1,1),(1,2),(2,1),(2,2)\}$

(ii) $R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$

(iii) $R_3 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$

(iv) $R_4 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

Check whether R_1, R_2, $R_{3,}$ and R_4 are equivalence relations or not, if yes, then find the equivalence classes of all elements of set A.

EXAMPLE

Prove that the relation R on Z, defined by $(a, b) \in R \Leftrightarrow a - b$ is divisible by 5 is an equivalence relation on Z.

EXAMPLE

Show that the relation R on IR defined as $R = \{(a, b): a \leq b^2\}$ is neither reflexive not symmetric nor transitive.

EXAMPLE

Show that the relation R defined by $(a,b)R(c,d) \Rightarrow a+b=b+c$, in $A \times A$ where $A=\{1,2,3,.....,10\}$

is an equivalence relation. Hence write the equivalence class $[(3,4)]$; $a,b,c,d \in A$.

Assignments

1. Write the smallest and largest equivalence relation on the set $A = \{1,2,3\}$.

2. For the set $A = \{1,2,3\}$,define a relation R on the set A as follows.

   $R = \{(1,1), (2,2), (3,3), (1,3)\}$ write the ordered pair(s) to be added to R to make it the smallest equivalence relation.

3. Show that the relation R on the set Z, given by $R = \{(a, b){:}\ |a - b|\ is\ a\ \ multiple\ of\ 4\ \}$ is an equivalence relation.

Types of Functions

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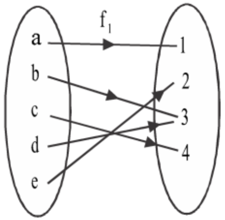
Function

Function from set A to set B:- Let A and B be two non empty sets, then a function f from set A to set B is a rule (or map or correspondence) which associates each element of set A to exactly one element of set B. If f is a function from set A to set B, then we denote it by $f: A \to B$.

Example:-

Check whether the maps in following diagram are functions or not.

(i)



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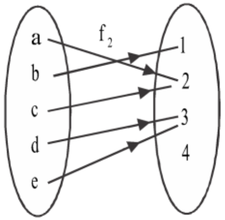

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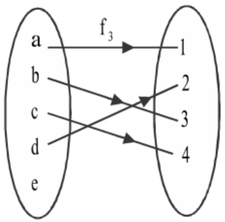

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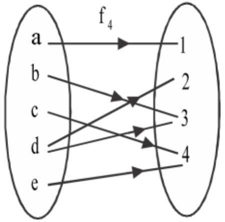
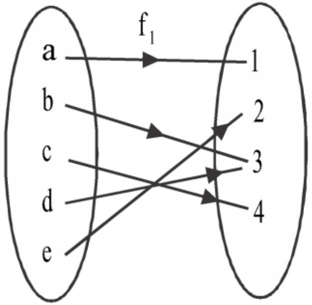
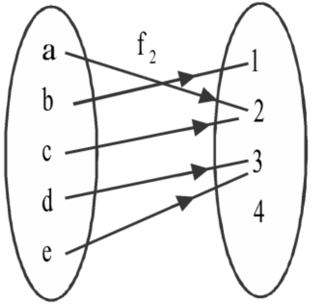
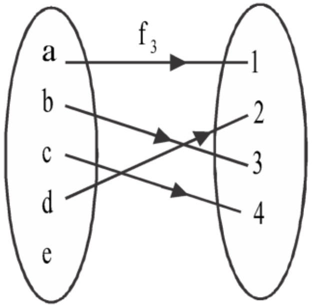


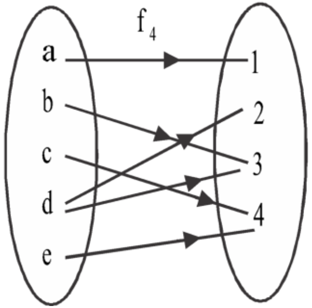
Example:-

Check whether the maps in the following diagrams are functions or not.









Domain, Co-domain and Range of a Function:-

Let $f: A \rightarrow B$ be function, then

(i) The set A is called domain of function 'f'.

(ii) The set B is called co-domain of 'f'.

(iii) The set of all images of elements of set A under f is called range under f.

Notes:-

(1) The range of A under f is denoted by $f(A)$.

(2) If $f(a) = b$ then, b is called as image of f under f and a is called pre-image of b.

(3) Range is always subset of the co-domain.

(4) If $n(A) = p, n(B) = q$, then number of functions from A to B is $(q)^p$

Types of Functions

(1) One-one function or Injective function:- A function $f: A \to B$ is said to be one-one if no two elements of A have same image. i.e. if $a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

Or, $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

Note:-

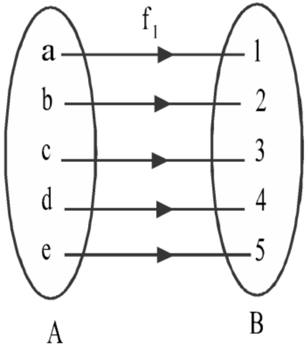
(i) If a function $f: A \to B$ is not one-one then it is called many-one function

(ii) if a function $f: A \to B$ is one-one then $n(A) \leq n(B)$

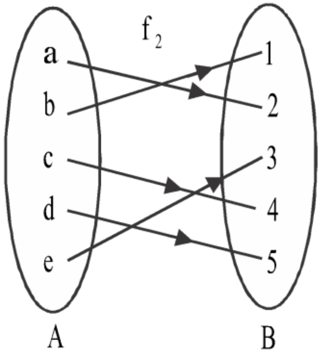
(iii) If $n(A) = p, n(B) = q$, then no of one-one functions from A to B $= \begin{cases} 0, & if \quad p > q \\ qP_p = \frac{q!}{(q-p)}, & if \quad p \leq q \end{cases}$

EXAMPLE:

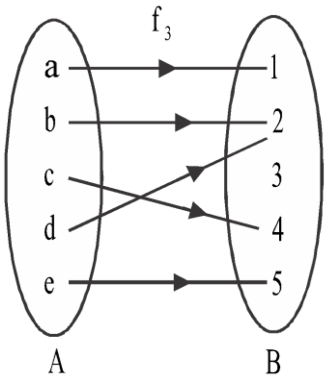
Check whether the functions in the diagram are one-one or not.



(i)



(ii)



(iii)

(2) Onto Function or Surjective Function

A function $f: A \rightarrow B$ is said to be onto if, for each $b \in B$, there exists $a \in A$ such that $f(a) = b$, we say that 'a' is pre-image of 'b'.

In other words, f is onto if Range of f = Co-domain of f

i.e if every element in B has a pre-image in A.

Note:-

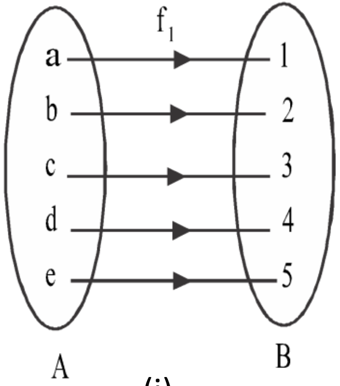
(i) If a function $f: A \rightarrow B$ is not onto then it is called into function.

(ii) If a function $f: A \rightarrow B$ is onto then $n(A) \geq n(B)$

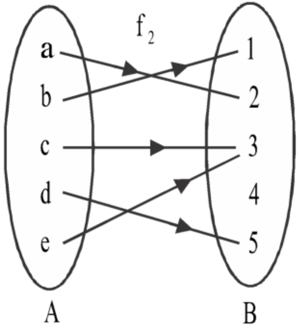
(iii) Let A be any finite set $n(A) = p$ then no of onto function from A to A is p!

EXAMPLE:

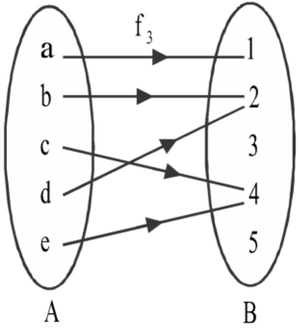
Check whether functions in the figure are onto?



(i)



(ii)



(iii)

(3) Bijective Function

A function $f: A \rightarrow B$ is said to be bijective if it is both one-one and onto

Example:-

Let $A = \{1,2,3,4\}$ and $B = \{5,6,7,8\}$ let $f: A \rightarrow B$ be defined by $f = \{(1,5), (2,6), (3,7), (4,8)\}$. Show that f is one-one and onto (bijective).

EXAMPLE:

Show that the function $f : Z \to Z$ defined by $f(x) = |x|$ is not a Bijective function?

EXAMPLE:

Let the function $f: R - \{3\} \rightarrow R - \{1\}$ defined by $f(x) = \frac{x-2}{x-3}$. Is f one-one and onto?

Justify your answer.

Assignments

1. Show that the greatest integer function $f: R \to R,$ given by $f(x) = [x]$ is neither one-one nor onto.

2. Show that the function $f: N \to N$ defined by $f(n) = 2n + 3$, for all $n \in N$ is not Bijective.

Inverse of a Function

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Inverse of a Function

DEFINITION:

Let f be a one-one and onto function from A to B. Let y be an arbitrary element of B. Then f being onto, there exists an element $x \in A$. Such that $f(x) = y$, Also f being one-one this x must be unique. Thus for each $y \in B$, there exists a unique element $x \in A$ such that $f(x) = y$. So we may define a function denoted by f^{-1} as $f^{-1} : B \to A$. Such that $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

The function f^{-1} is called the inverse of f.

Properties of Inverse of a function

- If the inverse of a function f exists then f is called an invertible function.
- A function f is invertible if and only if f is one-one and onto.
- The domain of f^{-1} =Range of f and range of f^{-1} =domain of f.
- $(f^{-1})^{-1} = f$
- If f is one-one and onto then f^{-1} is also one-one and onto.

Algorithm to find Inverse of a function

Let $f : A \to B$ be a bijective function. To find the inverse f of we the following steps:

Step-I

Put $f(x) = y$, where $y \in B$ and $x \in A$.

Step-II

Solve $f(x) = y$ to obtain x in terms of y.

Step-III

In the relation obtained in the step-II replace x by $f^{-1}(\mathrm{y})$ to obtain the required inverse of f.

EXAMPLE:

Consider $f: R \rightarrow R$ given by $f(x) = 4x + 3$. Show that f is invertible, find the inverse of f.

EXAMPLE:

Consider $f:\{1,2,3\} \to \{a, b, c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that f is invertible, find f^{-1} and show that $(f^{-1})^{-1} = f$

EXAMPLE:

Consider $f: \{1,2,3\} \rightarrow \{a, b, c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that f is invertible, find f^{-1} and show that $(f^{-1})^{-1} = f$

Assignments

1. Prove that the function $f: [0, \infty] \to [-5\infty]$, given by $f(x) = 9x^2 + 6x - 5$ is invertible and find f^{-1}.

2. Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}$, for all $x \in R$ is neither one-one nor onto.

3. Write the total no of functions from $\{1,2\}$ to $\{x, y, z\}$.

4. Write the total no of one-one functions from $\{1,2\}$ to $\{x, y, z\}$.

5. Write the total no of onto functions from $\{x, y, z\}$ to $\{1,5,25,125\}$.

6. Write the total no of bijective functions from $\{1,2,3\}$ to $\{x, y, z\}$.

Composition of Functions

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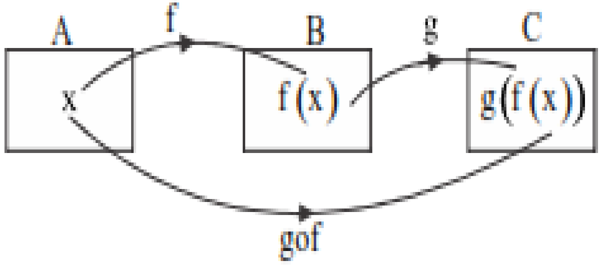
Composition of Function

The composition of two functions is a chain process in which the output of the first function becomes the input of the 2^{nd} function.

Let $f: A \to B$ and $g: B \to C$ be two functions. Exactly one element for every $x \in A$, there is exactly one element $f(x) \in B$. For $f(x) \in B$, there is exactly one element $g(f(x)) \in C$. This result is a new function from A to C as shown in the figure.



Definition:-

Let $f: A \to B$ and $g: B \to C$ be any two functions. Then the composition of f and g is a function $gof: A \to C$ defined as $(gof)(x) = g(f(x))$.

Some Useful Observations

- The composition gof exists if the range of f ⊆domain of g.
- The composition fog exists if the range of g ⊆ domain of f.
- gof and fog may or may not be equal.
- It may be possible gof exists but fog does not exist.

EXAMPLE

Find gof and fog when $f: R \to R$ and g: $R \to R$ are defined by $f(x) = 5x + 2$ and $g(x) = x^2 + 6$.

Note:-

Is fog defined?

Are fog and gof are equal you observe from this example.

EXAMPLE

Let $f:\{2,3,4,5\} \to \{3,4,5,9\}$ and $g:\{3,4,5,9\} \to \{7,11,15\}$ defined as

$f = \{(2,3),(3.4),(4,5),(5,5)\}$ and $g = \{(3,7),(4,7),(5,11),(9,11)\}$,

Find gof. Is fog defined?

Properties of Composition

- If $f: A \to B$ and $g: B \to C$ are one-one, then $gof: A \to C$ is also one-one.
- If $f: A \to B$ and $g: B \to C$ are onto, then $gof: A \to C$ is also onto.
- If $f: A \to B$ and $g: B \to C$ are invertible, then $gof: A \to C$ is also invertible.
- Composition of functions is associative Let $f: A \to B, g: B \to C and g: C \to D$ then $(hog)of = ho(gof)$

Inverse of a function with help of composition

Let $f: A \to B$ be one-one and onto function, then the function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$, where $I_A and I_B$ are identity functions on A and B respectively, g is called the inverse of f, i.e. $g = f^{-1}$.

EXAMPLE

Find fog and gof if $f(x) = 8x^3$ and $g(x) = x^{1/3}$

EXAMPLE

If $f(x) = \frac{4x+3}{6x-4}$; $x \neq \frac{2}{3}$, Show that $fof(x) = x$ for all $x \neq \frac{2}{3}$, What is the inverse of f?

Assignments

1. Let $f, g: R \to R$ be two functions defined as $f(x) = |x| + x and g(x) = |x| - x, \forall x \in R$ then find $fog(-3) and gof(-2)$.

2. If the mapping f and g are given by $f = \{(1,2), (3,5), (4,1)\}, g = \{(2,3), (5,1), (1,3)\}$ write fog and gof.

3. Let $f: R \to R$ be signum function as $f(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x = 0 \\ -1 & if \quad x < 0 \end{cases}$ and $g: R \to R$, be the greatest integer function given by $g(x) = [x]$. Do fog and gof coincide in $(0,1]$?

Practice of Problems Based on Functions

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EXAMPLE

Let $f: N \rightarrow N$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & if\ n\ is\ odd \\ \frac{n}{2}, & if\ n\ is\ even \end{cases}$ for all $n \in N$.

State whether the function is bijective justify your answer.

EXAMPLE

Let f and g be two bijective functions defined as $f: A \to B$ and $g: B \to C$, then prove that composition function $gof: A \to C$ is bijective.

Equal and Unequal Functions

Two functions $f: A \to B$ and g: $A \to B$ are said to be equal if $f(x) = g(x)$ for every $x \in A$, in this case we write $f = g$.

Two functions $f: A \to B$ and g: $A \to B$ are said to be unequal if $f(x) \neq g(x)$ for every $x \in A$, in this case we write $f \neq g$.

Example:

Let $A = \{1,2,3,4\}$ and $B = \{5,6,7,8\}$. Let $f: A \to B$ and g: $A \to B$ be defined by

$f = \{(1,5), (2,6), (3,7), (4,8)\}$ and $g = \{(2,6), (4,8), (1,5), (3,7), (4,8)\}$

So, $f(1) = 5 = g(1), f(2) = 6 = g(2), f(3) = 7 = g(3), f(4) = 8 = g(4)$

EXAMPLE:-

Let $A = \{1,2,3\}$ and $B = \{5,6,7\}$ and Let $f: A \rightarrow B$ and $g: A \rightarrow B$ be defined by

$f = \{(1,5), (2,6), (3,7)\}$ and $g = \{(2,5), (1,6), (3,7)\}$. Show that $f \neq g$.

EXAMPLE

Let $f: R \to R$ be signum function as $f(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x = 0 \\ -1 & if \quad x < 0 \end{cases}$ and $g: R \to R$, be the greatest integer function given by $g(x) = [x]$. Do fog and gof coincide in $(0,1]$?

EXAMPLE

Consider $f: N \to N$, $g: N \to R$ and $h: N \to R$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = sinz$ for all $x, y, z \in N$. Show that $ho(gof) = (hof)of$.

EXAMPLE

If the function $f(x) = \sqrt{2x-3}$ is invertible, then find f^{-1}. Hence prove that $(f o f^{-1})(x) = x$

EXAMPLE

If the function $f: R \to R$ be defined by $f(x) = 2x - 3$ and $g: R \to R$ by $g(x) = x^2 + 5$.

Then find fog and show that fog is invertible. Also find $(fog)^{-1}$, Hence find $(fog)^{-1}(9)$.

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