Range, Domain, Principal value Branches of Trigonometric Functions

SUBJECT : MATHEMATICS
CHAPTER NUMBER:2
CHAPTER NAME :INVERSE TRIGONOMETRIC FUNCTIONS

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What we expect to learn?

- Learn about finding the principal value branch of an Inverse Trigonometric Function
- Learn about drawing the graph of Inverse Trigonometric Functions
- Learn about properties of Inverse Trigonometric Functions and their limitations.
- Learn about reducing the Inverse Trigonometric Functions in the simplest form.
- Learn about the use of properties of inverse trigonometric functions in different questions

Introduction

In previous chapter, we have studied about the existence of inverse of a function. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions, which ensure the existence of their inverse and to study their properties.

Inverse of Trigonometric Functions

We know that trigonometric functions are periodic functions and hence in general all trigonometric functions are not bijections, consequently their inverse do not exist. However if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverses.

Domain and Range of Trigonometric functions

Function	Domain	Range
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R-\left\{x : x=(2n+1)\frac{\pi}{2}, n \in z\right\}$	R
$\cot x$	$R-\{x : x=n\pi, n \in z\}$	R
$\sec x$	$R-\left\{x : x=(2n+1)\frac{\pi}{2}, n \in z\right\}$	$R-(-1,1)$
$\operatorname{cosec} x$	$R-\{x : x=n\pi, n \in z\}$	$R-(-1,1)$

Domain, Principal value branch (Range) of inverse Trigonometric Functions

Function	Domain	Principal value (Range)
$(i) y = \sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$(ii) y = \cos^{-1} x$	$[-1,1]$	$[0, \pi]$
$(iii) y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$(iv) y = \cot^{-1} x$	R	$(0, \pi)$
$(v) y = \sec^{-1} x$	$R - (-1,1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$(vi) y = \operatorname{cosec}^{-1} x$	$R - (-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Important Points

- $sin^{-1} x \neq (sin\, x)^{-1} or\, sin^{-1} x \neq \frac{1}{sin\, x}$. These relations also hold for other inverse trigonometric functions.
- Whenever no branch of an inverse trigonometric function is mentioned we consider the principal value branch of that function.
- If $sin^{-1} x = y$, then x and y are the elements of domain and range of principal vale branch of $sin^{-1} x$ respectively.
- If $sin\, y = x$ then $sin^{-1} x = y$.

For example, $sin \frac{\pi}{6} = \frac{1}{2} \Rightarrow sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Graphs of Inverse Trigonometric Functions

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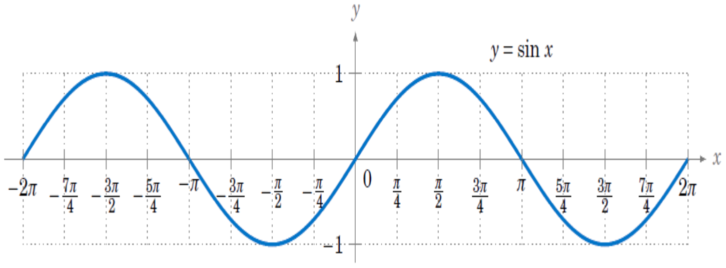
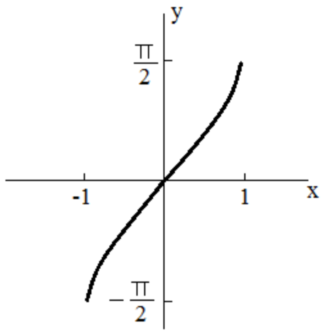
Graph of $sin\,x\; and\; sin^{-1}\,x$





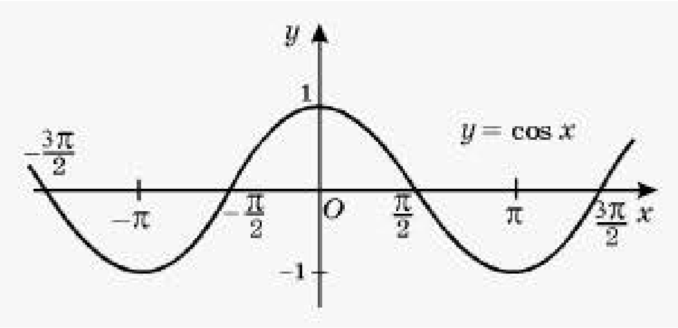
$y = sin\,x$

$y = sin^{-1}\,x$

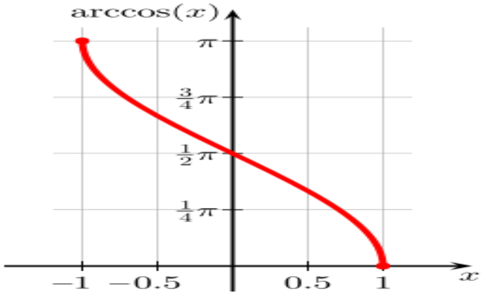
Remember:

The graphs of $y = sin\,x\; and\; y = sin^{-1}\,x$ are mirror images of each other in the line mirror $y\; =\; x$

Graph of $cos\ x$ and $cos^{-1}\ x$

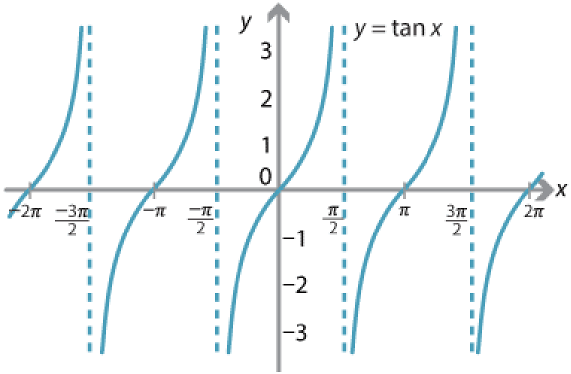


$y = cos\ x$

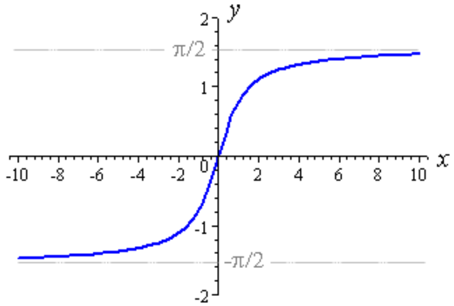


$y = cos^{-1}\ x$

Graph of $tan\,x$ and $tan^{-1}\,x$

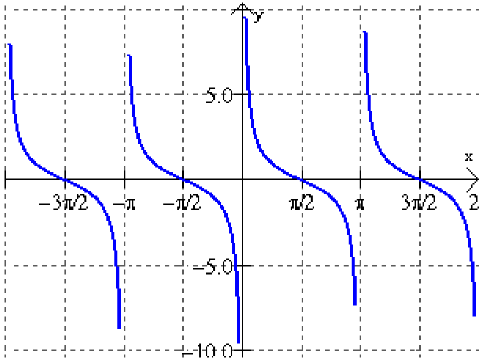


$y = tan\,x$

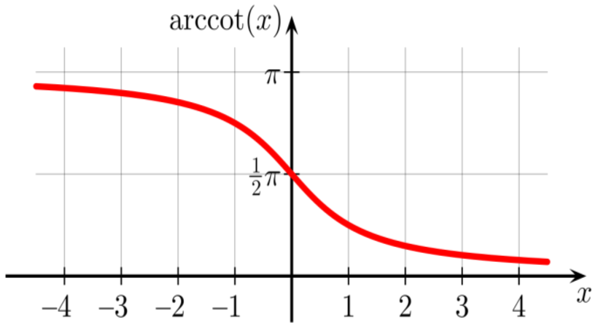


$y = tan^{-1}\,x$

Graph of $cot\, x$ and $cot^{-1}\, x$

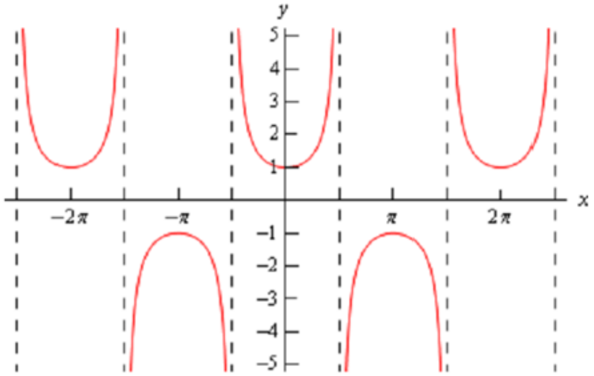


$y = cot\, x$

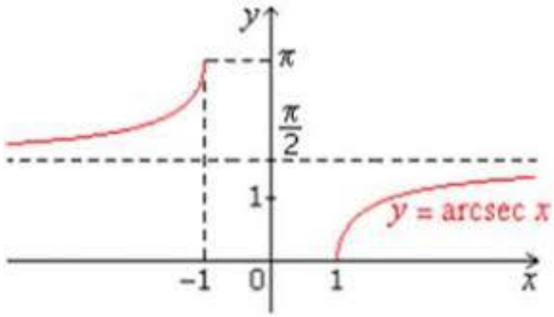


$y = cot^{-1}\, x$

Graph of $sec\, x\; and\; sec^{-1} x$

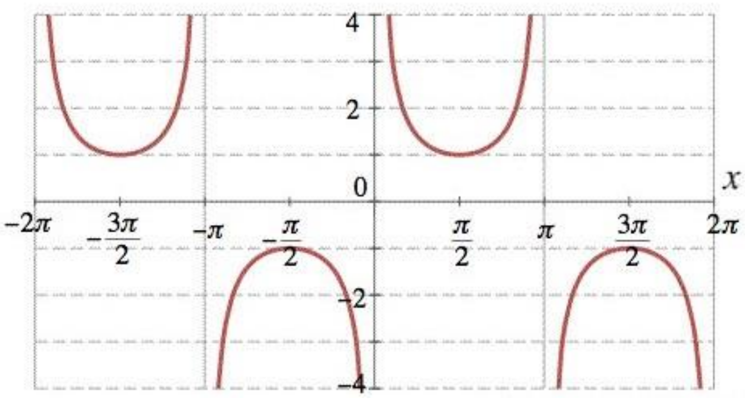


$y = sec\, x$

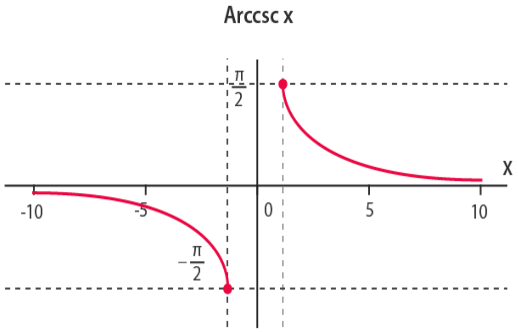


$y = sec^{-1} x$

Graph of $cosec\, x$ and $cosec^{-1}\, x$



$y = cosec\, x$



$y = cosec^{-1}\, x$

Note:-

The graph of an inverse trigonometric function can be obtained from the graph of the original function by interchanging the coordinate axes.

Assignments

1. The domain of $sin^{-1} 2x$ is

(a) $[0,1]$ (b) $[-1,1]$ (c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) $[-2,2]$

2. The domain of $cos^{-1}(x^2 - 4)$ is

(a) $[3,5]$ (b) $[0, \pi]$

(c) $[-\sqrt{5}, -\sqrt{3}] \cap [-\sqrt{5}, \sqrt{3}]$ (d) $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$

3. Find the principal values of the following

(a) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $sin\left[cos^{-1}\left(\frac{1}{2}\right)\right]$ (c) $cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Elementary Properties of Inverse Trigonometric Functions

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Properties of Inverse Trigonometric Functions

Property – 1

- $sin^{-1}(sin\,\theta) = \theta$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $cos^{-1}(cos\,\theta) = \theta$ where $\theta \in [0, \pi]$
- $tan^{-1}(tan\,\theta) = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $cot^{-1}(cot\,\theta) = \theta$ where $\theta \in (0, \pi)$
- $sec^{-1}(sec\,\theta) = \theta$ where $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $cos\,e\,c^{-1}(cos\,e\,c\theta) = \theta$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

EXAMPLE

Evaluate the following

(a) $cos^{-1}\left(cos\frac{\pi}{3}\right)$

(b) $sin^{-1}\left(sin\frac{2\pi}{3}\right)$

(c) $tan^{-1}\left(tan\frac{5\pi}{4}\right)$

(d) $sin^{-1}(sin\,3)$

Property – 2

- $sin(sin^{-1} x) = x$ where $x \in [-1,1]$
- $cos(cos^{-1} x) = x$ where $x \in [-1,1]$
- $tan(tan^{-1} x) = x$ where $x \in R$
- $cot(cot^{-1} x) = x$ where $x \in R$
- $sec(sec^{-1} x) = x$ where $x \in R - (-1,1)$
- $cosec(cosec^{-1} x) = x$ where $x \in R - (-1,1)$

EXAMPLE

Evaluate:-

(a) $tan\left(tan^{-1}\frac{3}{4}\right)$ (b) $sin\left(sin^{-1}\frac{5}{13}\right)$

(c) $sin\left(cos^{-1}\frac{4}{5}\right)$ (d) $cos\left(cot^{-1}\frac{15}{8}\right)$

EXAMPLE

Find the value of the expression $sin[cot^{-1}\{cos(tan^{-1} 1)\}]$

Property – 3

- $sin^{-1}(-x) = -\,sin^{-1} x\,, x \in [-1,1]$
- $tan^{-1}(-x) = -\,tan^{-1} x\,, x \in R$
- $cos\,e\,c^{-1}(-x) = -\,cos\,e\,c^{-1}x, x \in (-\infty,-1] \cup [1,\infty)$
- $cos^{-1}(-x) = \pi - cos^{-1} x\,, x \in [-1,1]$
- $sec^{-1}(-x) = \pi - sec^{-1} x\,, x \in (-\infty,-1] \cup [1,\infty)$
- $cot^{-1}(-x) = \pi - cot^{-1} x\,, x \in R$

EXAMPLE

Find the principal value of

(a) $sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(b) $cot^{-1}(-\sqrt{3})$

Property – 4

- $sin^{-1}\left(\frac{1}{x}\right) = cos\,e\,c^{-1}x, x \in (-\infty, -1] \cup [1, \infty)$
- $cos^{-1}\left(\frac{1}{x}\right) = sec^{-1}x\,, x \in (-\infty, -1] \cup [1, \infty)$
- $tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} cot^{-1}x\,, & x > 0 \\ -\pi + cot^{-1}x\,, & x < 0 \end{cases}$

Property – 5

- $sin^{-1} x + cos^{-1} x = \frac{\pi}{2}, x \in [-1,1]$
- $tan^{-1} x + cot^{-1} x = \frac{\pi}{2}, x \in R$
- $sec^{-1} x + cos\,e\,c^{-1} x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

EXAMPLE

If $-1 \leq x, y \leq 1$, such that $sin^{-1} x + sin^{-1} y = \frac{\pi}{2}$, find the value of $cos^{-1} x + cos^{-1} y$

Assignments

1. If $sin^{-1} x - cos^{-1} x = \frac{\pi}{6}$, then $x =$ _____

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{2}$ (d) None of these

2. If $4\, cos^{-1} x + sin^{-1} x = \pi$, then the value of x is

(a) $\frac{3}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{3}}$

3. The value of $cos^{-1}\left(cos\frac{5\pi}{3}\right) + sin^{-1}\left(sin\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) Zero

More Properties of Inverse Trigonometric Functions

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Property - 6

- $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ xy<1\\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ x>0,y>0\ and\ xy>1\\ -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ x<0,y<0\ and\ xy>1\end{cases}$$

- $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ xy>-1\\ \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ x>0,y<0\ and\ xy<-1\\ -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ x<0,y>0\ and\ xy<-1\end{cases}$$

EXAMPLE

Prove that $tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{13} = tan^{-1}\frac{2}{9}$

EXAMPLE

Solve $cot^{-1} x - cot^{-1}(x + 2) = \frac{\pi}{12}$

Property - 7

- $$sin^{-1} x + sin^{-1} y = sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$
- $$sin^{-1} x - sin^{-1} y = sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$

EXAMPLE

Prove that $sin^{-1}\frac{3}{5} + sin^{-1}\frac{8}{17} = sin^{-1}\frac{77}{85}$

Property - 8

- $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{(1 - x^2)(1 - y^2)}\right)$

- $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{(1 - x^2)(1 - y^2)}\right)$

Property - 9

- $2\,sin^{-1}x = sin^{-1}\left(2x\sqrt{1-x^2}\right), if\ -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

- $3\,sin^{-1}x = sin^{-1}(3x - 4x^3), if\ -\frac{1}{2} \leq x \leq \frac{1}{2}$

Property - 10

(a) $2\, cos^{-1} x = cos^{-1}(2x^2 - 1), if\ 0 \le x \le 1$

(b) $3\, cos^{-1} x = cos^{-1}(4x^3 - 3x),\ if\ \frac{1}{2} \le x \le 1$

Property - 11

(a) $2\,tan^{-1}x = tan^{-1}\left(\frac{2x}{1-x^2}\right)\, if -1 < x < 1$

(b) $3\,tan^{-1}x = tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\, if -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

(c) $2\,tan^{-1}x = sin^{-1}\left(\frac{2x}{1+x^2}\right) = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Property - 12

(a) $sin^{-1} x = cos^{-1}\sqrt{1-x^2} = tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = cot^{-1}\frac{\sqrt{1-x^2}}{x} = sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = cos\,e\,c^{-1}\left(\frac{1}{x}\right)$

(b) $cos^{-1} x = sin^{-1}\sqrt{1-x^2} = tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = sec^{-1}\left(\frac{1}{x}\right) = cos\,e\,c^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(c) $tan^{-1} x = sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = cot^{-1}\left(\frac{1}{x}\right) = sec^{-1}(\sqrt{1+x^2}) = cos\,e\,c^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

EXAMPLE

Prove that

(a) $2\, sin^{-1}\frac{3}{5} - tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

(b) $4\, tan^{-1}\frac{1}{5} - tan^{-1}\frac{1}{239} = \frac{\pi}{4}$

EXAMPLE

If $sin^{-1}\frac{2a}{1+a^2} - cos^{-1}\frac{1-b^2}{1+b^2} = tan^{-1}\frac{2x}{1-x^2}$, then prove that $x = \frac{a-b}{1+ab}$

EXAMPLE

Prove that $tan\left(sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5}\right) = \frac{63}{16}$

Assignments

1. Simplify $tan^{-1} 2\,x + tan^{-1} 3\,x = \frac{\pi}{4}$

2. Express $tan^{-1}\left(\frac{cos\,x}{1-sin\,x}\right), \frac{-\pi}{2} < x < \frac{\pi}{2}$ in the simplest form

3. Find the value of $cot\left(\frac{\pi}{2} - 2\,cot^{-1}\sqrt{3}\right)$

Practice of Problems Based on Properties

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Problem-1

Express $tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$ in the simplest form

NOTE:

It is advised you to substitute

(i) $x = sin\theta/\cos\theta$ if $\sqrt{1-x^2}$ term exists

(ii) $x = sec\theta/cosec\theta$ if $\sqrt{x^2-1}$ term exists

(ii) $x = tan\theta/cot\theta$ if $\sqrt{x^2+1}$ term exists

Problem-2

Simplify $tan^{-1}\left(\frac{a\ cos\ x-b\ sin\ x}{b\ cos\ x+a\ sin\ x}\right)$, if $\frac{a}{b}tan\,x > -1$

Problem-3

Prove that $tan^{-1}\sqrt{x} = \frac{1}{2}cos^{-1}\left(\frac{1-x}{1+x}\right)$

Problem-4

Prove that $cos[tan^{-1}\{sin(cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$

Problem-5

Prove that $cot^{-1}7 + cot^{-1}8 + cot^{-1}18 = cot^{-1}3$

Problem-6

Solve for x: $tan^{-1}\left(\frac{x-2}{x-3}\right) + tan^{-1}\left(\frac{x+2}{x+3}\right) = \frac{\pi}{4}, |x| < 1$

Problem-7

Prove that $tan\left[\frac{\pi}{4}+\frac{1}{2}cos^{-1}\frac{a}{b}\right]+tan\left[\frac{\pi}{4}-\frac{1}{2}cos^{-1}\frac{a}{b}\right]=\frac{2b}{a}$

Assignment

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1. Prove that $tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x - \frac{1}{\sqrt{2}} \leq x \leq 1$

2. Prove that $tan^{-1}\frac{1}{4} + tan^{-1}\frac{2}{9} = \frac{1}{2}tan^{-1}\frac{4}{3}$

3. If $tan^{-1}x + tan^{-1}y = \frac{\pi}{4}; xy < 1$, then write the value of $x + y + xy$

4. Solve for x, $cos(2\,sin^{-1}x) = \frac{1}{9}, x > 0$

5. Simplify $tan^{-1}\sqrt{\frac{1-cos\,x}{1+cos\,x}}, x < \pi$

6. Solve for x, $tan^{-1}(x+1) + tan^{-1}(x-1) = tan^{-1}\frac{8}{31}, 0 < x < 1$

THANKING YOU

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