ODM EDUCATIONAL GROUP
Changing your Tomorrow

CONTINUITY

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERENTIABILITY

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

What we expect to learn?

- Students will be able to learn about the continuity of a function.
- Students will learn algebra of continuous functions.
- Students will learn about differentiability.
- Students will able to find derivative of composite functions, implicit functions.
- Students will able to find derivative of trigonometric functions, inverse trigonometric functions.
- Students will able to find derivative of logarithmic and exponential functions.
- Students will learn about finding derivative of functions in parametric form.
- Students will able to find second order derivative.
- Students will learn about Rolle's theorem and Mean value theorem with verifications.

INTRODUCTION

In this chapter we will discuss two very important concepts of mathematics continuity and differentiability of real functions. Also discuss the relation between them. In order to understand these concepts well one should have the knowledge of the concept of limits which was in Class – XI

Limit

Let $a \in R$ and 'f' be real-valued function in real variable x defined at the points in an open interval containing 'a' except possible at 'a'. Then we say that limit of the function $f(x)$ is a real number l as x tends to 'a'. If the value of $f(x)$ approaches l as x approaches 'a'. Which is denoted by $\lim_{x \to a} f(x) = l$

Here x can approach 'a' on a real number line in two ways, either from the left or from the right of a. This leads to two limits as the left-hand limit (LHL) and the Right-hand limit (RHL).

The left-hand limit is the value of f(x) approaches l as x approaches 'a' from the left of a. It is denoted by

$\lim_{x \to a^-} f(x)$

The right-hand limit is the value of $f(x)$ approaches as x approaches 'a' from the right of 'a'. It is denoted by $\lim_{x \to a^+} f(x)$

Existence of Limit

Whenever $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$

Then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = l$

$LHL = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$

$RHL = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$

ODM EDUCATIONAL GROUP
Changing your Tomorrow

Some Important Results on Limit

a) $\lim_{x\to 0}\frac{\sin x}{x}=1$

b) $\lim_{x\to 0}\frac{\tan x}{x}=1$

c) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

d) $\lim_{x\to 0}\frac{a^x-1}{x}=loga$

e) $\lim_{x\to 0}\frac{\log(1+x)}{x}$

f) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=n.a^{n-1}$

g) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

h) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

Some Important Results on limit

If $a \in R$ and f, g be real valued functions then

a) $\lim_{x \to a} k.f(x) = k \lim_{x \to a} f(x)$

b) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

c) $\lim_{x \to a} [f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

d) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a} g(x) \neq 0$

Intuitive Idea of continuity

Let 'f' be a real valued function in any interval and let $y = f(x)$. Then we can represent the function by a graph in xy –plane. The function 'f' is continuous when we try to draw the graph in one stoke, i.e without lifting pen from the plane of paper. Roughly, a function is continuous if its graph is a single unbroken curve with no holes or jumps.

Intuitive Idea of Continuity

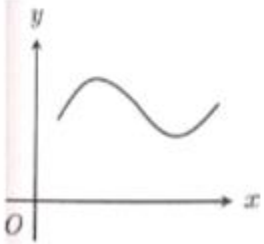


Figure 10.1

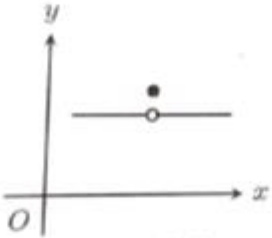


Figure 10.2

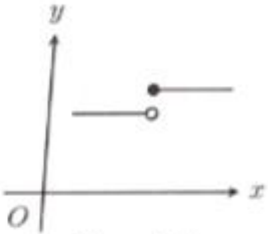


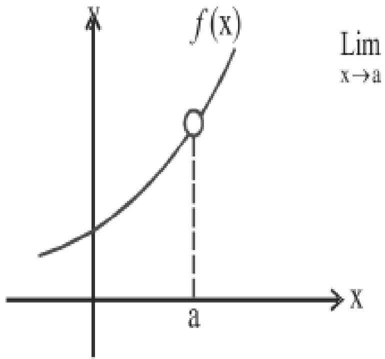
Figure 10.3

From the above idea the function shown in figure 10.1 is continuous.

The function shown in fig 10.2 has a hole at a point and hence not continuous.

The function shown in fig. 10.3 has a jump at a point and hence is not continuous.

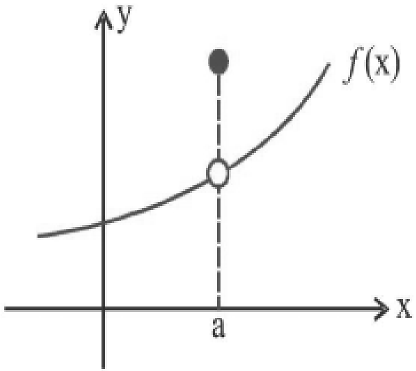
Different types of Discontinuity



$\lim_{x \to a} f(x) \to$ exist finitely.

$f(a) \to$ does not exist.

missing point discontinuity at x = a

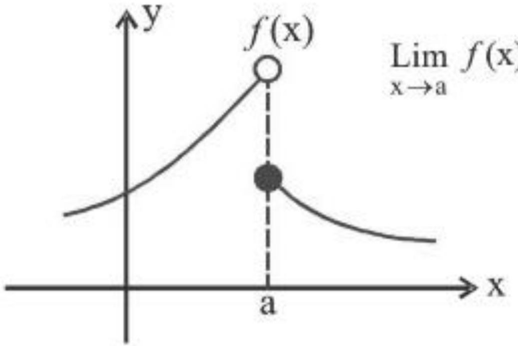


$\lim_{x \to a} f(x) \to$ exists finitely

$f(a) \to$ exists.

But, $\lim_{x \to a} f(x) \neq f(a)$

Isolated point discontinuity at x = a



$\lim_{x \to a} f(x) \to$ does not exists

non-removable discontinuity at x=a

Mathematical Definition of Continuity

A function $f : D \to R$ is said to be continuous at $x = c$

$f : D \to R$

i.e. if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(x)$

i.e. LHL = RHL = Functional value

Otherwise the function will be discontinuous at $x = c$

Conclusion

As the function $f(x)$ is continuous at $x = a$ if $LHL = RHL = f(a)$

But we know that when $LHL = RHL = l$ (say)

Then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = l$

Thus the function f(x) will be continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

i.e. Limiting value = Functional value.

Example

Examine the continuity of the function $f(x) = 2x^2 - 1 \ at \ x = 3$

Example

Check the continuity of the function $f(x) = \begin{cases} 3x + 5, & if x \geq 2 \\ x^2, & if x < 2 \end{cases}$ $at\ x = 2$

Assignment

1. Find the value of k so that the function continuous at $x = 2$

$$f(x) = \begin{cases} 2x + 1, x < 2 \\ k, \quad\quad x = 2 \\ 3x - 1, x > 2 \end{cases}.$$

2. Examine the continuity of the function $f(x) = 5x - 3 \ at \ x = -3$

CONTINUOUS FUNCTION AND ITS PROPERTIES

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERENTIABILITY

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

Example:

Show that the function $f(x) = \begin{cases} \frac{2x^2-3x-2}{x-2}, if x \neq 2 \\ 5 \quad if x = 2 \end{cases}$ $is\ continuous\ at\ x = 2$

Example:

Discuss the continuity of $f(x)$ when $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, & if\ x \neq 0 \\ 5, & if\ x = 0 \end{cases}$ at $x = 0$.

Example:

Find the value of k so that the function $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & if \;\; x \neq 2 \\ k, & if \;\; x = 2 \end{cases}$ is continuous at $x = 2$.

Definition

A real function 'f' is said to be continuous if it is continuous at every point in the domain of 'f'.

Suppose 'f' is a function defined on a closed interval $[\mathrm{a}\,, b]$,then for 'f' to be continuous, it needs to be continuous at every point in $[\mathrm{a}\,, b]$, including the end points a and b.

Example:- Prove that the constant function $f(x) = k$ is continuous.

List of Some Continuous Functions

Sl. NO.	Function f(x)	Interval in which f(x) is continuous
1.	Constant function $f(x) = C$	$(-\infty, \infty)$
2.	x^n, n is an integer ≥0	$(-\infty, \infty)$
3.	x^{-n},n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\|x - a\|$	$(-\infty, \infty)$
5.	$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomial in x	$(-\infty, \infty) - \{x; q(x) = 0\}$
7.	$\sin x$	$(-\infty, \infty)$
8.	$\cos x$	$(-\infty, \infty)$
9.	$\tan x$	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	$\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
11.	$\sec x$	$(-\infty, \infty) - \{(2n+1)\}$
12.	$cosec\, x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

Algebra of Continuous Functions

Suppose f and g be two real functions which are continuous at $x = c$ then

- $f + g$ is continuous at $x = c$
- $f - g$ is continuous at x=c
- $f . g$ is continuous at $x = c$
- $\left(\frac{f}{g}\right)$ is continuous at $x = c$, provided $g(c) \neq 0$
- If f and g be two functions such that fog is defined at c and if f is continuous at $g(c)$. Then fog is continuous at $x = c$

Example:

Show that the function defined by $f(x) = |cosx|$ is a continuous function.

Example:

Show that the function f given by $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & if\ x \neq 0 \\ -1, & if\ x = 0 \end{cases}$ is discontinuous at $x = 0$

Example:

If the function $f(x) = \begin{cases} 3ax + b, if\ x > 1 \\ 11, \qquad\quad if\ x = 1 \\ 5ax - 2b, if\ x < 1 \end{cases}$ is continuous at $x = 1$, find the values of a and b.

Assignment

1. Find k, if $f(x) = \begin{cases} k \sin\frac{\pi}{2}(x+1), & if\ x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & if\ x > 0 \end{cases}$ is continuous at $x = 0$.

2. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points.

Here $[x]$ denotes the greatest integer less than or equal to x.

3. If $f(x) = \begin{cases} 1, if\ x \le 3 \\ ax + b, if\ 3 < x < 5 \\ 7, if\ 5 \le x \end{cases}$, Determine the values of a and b so that the function $f(x)$ is continuous.

4. Determine the value k so that the function $f(x)$ is continuous at $x = 0$, Where

If $f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & if\ -1 \le x < 0 \\ \frac{2x+1}{x-1}, & if\ 0 \le x \le 1 \end{cases}$.

DIFFERENTIABILITY

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERTIABILITY

DIFFERENTIABILITY OF A FUNCTION AT A POINT

A function f is said to have a derivative at $x = a$ or differentiable at $x = a$ iff $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists finitely.

The value of this limit is called the derivative of f at 'a' and is denoted by $f'(a)$.

A function $y = f(x)$ is differentiable at $x = a$ if L.H.D. = R.H.D. at $x = a$.

i.e. $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Differentiable Function

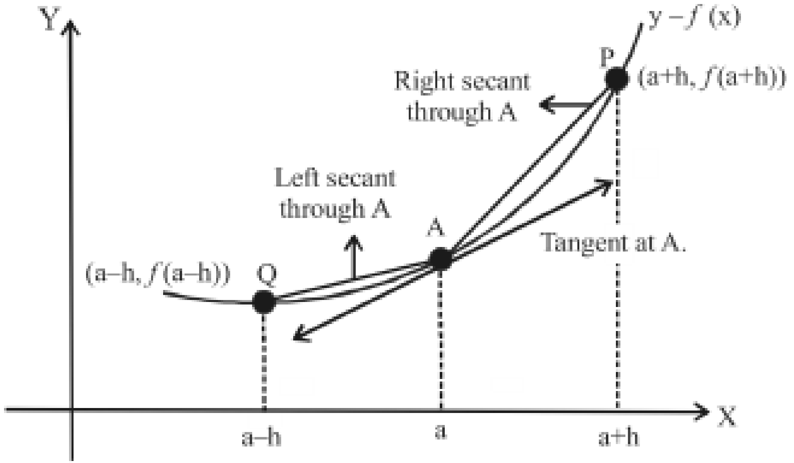
We define the derivative of the function $f(x)$ at $x = a$ as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Or $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists finitely.

Now the question in our mind when does not exists ?

For this let us consider:



Slope of the right hand secant $= \frac{f(a+h)-f(a)}{h}$ as $h \to 0, P \to A$ and

$the\ secant\ (AP) \to tangent\ at\ A$

$\Rightarrow$ The right hand derivative $= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} =$

$Slope\ of\ tangent\ at\ A\ (when\ approached\ from\ right) = f'\ (a^+)$

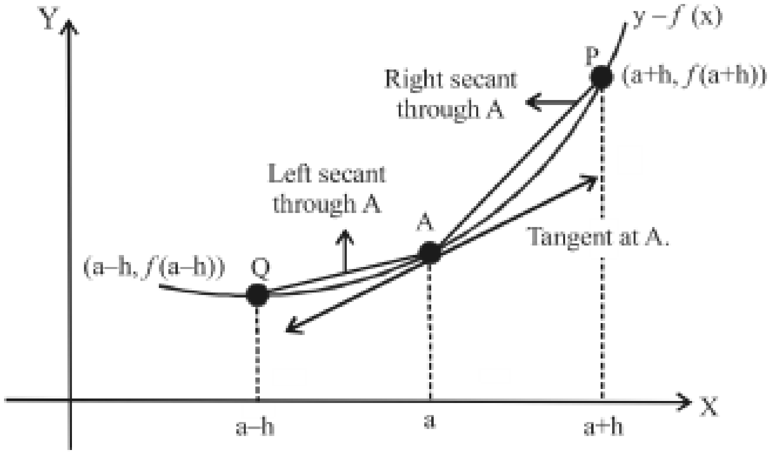
Differentiable Function

Slope of the left hand secant $= \frac{f(a-h)-f(a)}{-h}$ as $h \to 0, Q \to A$ and

$the\ secant\ (AQ) \to tangent\ at\ A$

$\Rightarrow$ The left hand derivative $= \lim\limits_{h\to 0} \frac{f(a-h)-f(a)}{-h} =$

$Slope\ of\ tangent\ at\ A\ (when\ approached\ from\ left) = f'\ (a^-)$



If the $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist, then f is not differentiable at $x = a$

In other words, we say that the function f is differentiable at the point ‘a’ if both Left-hand derivative(LHD) and Right-hand derivative (RHD) are finite and equal

I.e. When LHD = RHD

$\Leftrightarrow \lim\limits_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

EXAMPLE

Prove that the function $f(x) = \begin{cases} 1 + x, if\ x \leq 2 \\ 5 - x, if\ x > 2 \end{cases}$ is not differenttiable at x = 2.

EXAMPLE

Show that the function $f(x) = \begin{cases} 12x - 13, & if\ x \leq 3 \\ 2x^2 + 5, & if\ x > 3 \end{cases}$ is differenttiable at x = 3.

NOTE: Differentiable functions are continuous but every continuous functions may or may not be differentiable.

EXAMPLE

For what choice of a and b is the function $f(x) = \begin{cases} x^2, & if\ x \leq 2 \\ ax + b, & if\ x > 2 \end{cases}$ is differenttiable at x = 2.

Rules of Differentiation

For any two differentiable functions u and v, the following rules are as a part of algebra of derivatives

- $(ku)' = ku'$, where k is a constant
- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + v'u$ (Product rule)
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Some formulas on Derivatives

- $\frac{d}{dx}(k) = 0\ , where\ k\ is\ a\ constant$
- $\frac{d}{dx}\ (x^n) = n.\, x^{n-1}$
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

SOME FORMULAES OF DERIVATIVE

1. $\frac{d(k)}{dx} = 0$, where k is constant.
2. $\frac{d(x^n)}{dx} = nx^{n-1}$
3. $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$
4. $\frac{d(e^x)}{dx} = e^x$
5. $\frac{d(a^x)}{dx} = a^x \log a$
6. $\frac{d(\log x)}{dx} = \frac{1}{x}, x > 0$

SOME FORMULAES OF DERIVATIVE

- $\frac{d(\sin x)}{dx} = \cos x$
- $\frac{d(\cos x)}{dx} = -\sin x$
- $\frac{d(\tan x)}{dx} = \sec^2 x$
- $\frac{d(\cot x)}{dx} = -\cos e\, c^2 x$
- $\frac{d(\sec x)}{dx} = \sec x \,.\tan x$
- $\frac{d(\sec x)}{dx} = \sec x \,.\tan x$
- $\frac{d(\cos ecx)}{dx} = -\cos e\, cx.\cot x$

Assignment

1. Find the value of p and q so that the function $f(x) = \begin{cases} x^2 + 3x + p, if\ x \le 1 \\ qx + 2, \quad if\ x > 1 \end{cases}$

   is differentiable at $x = 1$

2. Let $f(x) = x|x|$, for all $x \in R$. Discuss the derivability of $f(x)$ at $x = 0$.

DERIVATIVE OF COMPOSITE FUNCTIONS(CHAIN RULE)

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERTIABILITY

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

DERIVATIVE OF COMPOSITION FUNCTIONS

CHAIN RULE

Let f be the real-valued function which is the composition of two functions u & v. i.e. $f = uov$.

Where u & v are differentiable functions and uov is also a differentiable function?

$\Rightarrow \frac{df}{dx} = \frac{duov}{dx} = \frac{duov}{dv} \times \frac{dv}{dx}$, Provided all the derivatives in the statement exists.

Soppose we want to find the derivative of f where $f(x) = (2x+1)^3$.

Now $\frac{df(x)}{dx} = \frac{d(2x+1)^3}{dx} = \frac{d(8x^3+12x^2+6x+1)}{dx} = 24x^2 + 24x + 6 = 6(2x+1)^2$

We observe that, if we take $g(x) = 2x + 1$ and $h(x) = x^3$

Then $f(x) = hog(x) = (2x+1)^3 \Rightarrow \frac{df(x)}{dx} = f^{/}(x) = \frac{dhog(x)}{dg(x)} \times \frac{dg(x)}{dx}$

I.e. $\frac{d(2x+1)^2}{d(2x+1)} \times \frac{d(2x+1)}{dx} \Rightarrow f^{/}(x) = 3 \times (2x+1)^2 \times 2 = 6(2x+1)^3$

Example

Find $\frac{dy}{dx}$, if $y = sin(x^2 + 1)$

Example

Find $\frac{dy}{dx}$, if $y = log(\,tan\,x)$

Example

Find $\frac{dy}{dx}$,if $y = e^{\sin(x^2)}$

Example

Find $\frac{dy}{dx}$, if $y = (x^2 + x + 1)^4$

Example

Find $\frac{dy}{dx}$, if $y = \frac{1}{\sqrt{a^2 - x^2}}$

Example

Find $\frac{dy}{dx}$, if $y = sin^3 x$

Example

Find $\frac{dy}{dx}$, if $y = log(\sec x + \tan x)$

Assignment

1. Find $\frac{dy}{dx}$, if $y = e^{x \sin x}$

2. Find $\frac{dy}{dx}$, if $y = \frac{sin(ax+b)}{cos(cx+d)}$.

3. Find $\frac{dy}{dx}$, if $y = cos(x^3).sin(x^3)$

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERTIABILITY

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

FORMULAE

1. $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

2. $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

3. $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

4. $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$

5. $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

6. $\frac{d(\cos ec^{-1} x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

SOME IMPORTANT SUBSTITUTIONS

- $\sqrt{a^2 - x^2}$------------- > put $x = a\sin\theta$ or $x = a\cos\theta$
- $\sqrt{x^2 + a^2}$------------- > put $x = a\tan\theta$ or $x = a\cot\theta$
- $\sqrt{x^2 - a^2}$------------ > put $x = a\sec\theta$ or $x = a\,cos\,e\,c\theta$
- $\frac{2x}{1+x^2}$, $\frac{2x}{1-x^2}$, $\frac{1-x^2}{1+x^2}$,$\frac{3x-x^3}{1-3x^2}$ ------------ > put $x = \tan\theta$
- $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$ ------------ > put $x = a\cos 2\theta$
- $2x^2 - 1$ ------------ > put $x = \cos\theta$
- $1 - 2x^2$ ------------ > put $x = \cos\theta$
- $3x - 4x^3$ ------------ > put $x = \sin\theta$
- $4x^3 - 3x$ ------------ > put $x = \cos\theta$

Example:

Find $\frac{dy}{dx}$, if $y = sin^{-1} 2\,x\sqrt{1 - x^2}$

Example:

Find $\frac{dy}{dx}$, if $y = sin^{-1}\frac{2x}{1+x^2}$

Example:

Find $\frac{dy}{dx}$, if $y = sec^{-1}\frac{1}{2x^2-1}$

Example:

Find $\frac{dy}{dx}$, if $y = tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Assignment

1. Find $\frac{dy}{dx}$, if $y = cot^{-1}\left(\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right)$.

2. Find $\frac{dy}{dx}$,if $y = tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$.

Derivative of Exponential and Logarithmic Functions

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Derivative Of Implicit Functions

Definition:

Considering $x - y - 2 = 0$ and $x + sin(xy) - y = 0$

In the first case, we can solve for y and rewrite the relationship $y = x - 2$.

But in the second case, it does not seem that there is an easy way to solve for y. When a relationship between x and y is expressed in a way that is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x. In the 2nd case, it is implicit that y is a function of x and we say that the relationship of the second type mentioned above gives function implicitly.

Derivative Of Implicit Functions

Definition:

Considering $x - y - 2 = 0$ and $x + sin(xy) - y = 0$

In the first case, we can solve for y and rewrite the relationship $y = x - 2$.

But in the second case, it does not seem that there is an easy way to solve for y. When a relationship between x and y is expressed in a way that is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x. In the 2nd case, it is implicit that y is a function of x and we say that the relationship of the second type mentioned above gives function implicitly.

Working Rule

Given that $f(x, y) = 0$ to find $\frac{dy}{dx}$

Step I Write the given expression.

Step II Differentiate both sides of the expression with respect to x.

Step III Bring all the terms containing $\frac{dy}{dx}$ to L.H.S. and the remaining terms on R.H.S.

Step IV Simplify to get $\frac{dy}{dx}$

Example

Find $\frac{dy}{dx}$, if $x + y = 10$

Example

Find $\frac{dy}{dx}$, if $2x^2 + y^2 + xy = a$

Example

Find $\frac{dy}{dx}$, if $\frac{y}{x} + sin^2 y = k$

Example

Find $\frac{dy}{dx}$, if $x^3 + x^2y + cos(xy) + y^3 = 81$

Assignment

1. Find $\frac{dy}{dx}$ when $(x^2 + y^2)^2 = xy$.

2. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = -\frac{-1}{(1+x)^2}$.

3. If $siny = x\ sin(a+y)$, prove that $\frac{dy}{dx} = \frac{sin^2(a+y)}{sina}$.

Derivative of Exponential and Logarithmic Functions

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

Derivative of Exponential and Logarithmic Functions

We have learned about the derivatives of the functions of the form $\{f(x)\}^n$, $n^{f(x)}$, where $f(x)$ is a function of x and n is the constant.

In this section, we will mainly discuss derivatives of the functions of the form $\{f(x)\}^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x.

Derivative of Exponential and Logarithmic Functions

To find out the derivative

Let $y = f(x)^{g(x)}$

Taking logarithm on both sides we get

$\Rightarrow log\, y = g(x).log\{f(x)\}$

Differentiating w. r. t. x we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x) \times \frac{1}{f(x)} \times f^{/}(x) + log\{f(x)\} \times g^{/}(x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{g(x) \times \frac{1}{f(x)} \times f^{/}(x) + log\{f(x)\} \times g^{/}(x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (f(x))^{g(x)}\left\{g(x) \times \frac{1}{f(x)} \times f^{/}(x) + log\{f(x)\} \times g^{/}(x)\right\}$$

Example

Find the derivative of $y = x^x$

Example

Find $\frac{dy}{dx}$, if $y = (sin\, x)^{log\, x}$

Assignment

1. Differentiate the following function with respect to x .

$i)\ x^{\cos^{-1}x}$ $ii)\ logx^{logx}$ $iii)\ x^{\sin x-\cos x}$

Derivative of Exponential and Logarithmic Functions

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Example

Find the derivative of $y = (log\,x)^x + x^{log\,x}$

Example

Find $\frac{dy}{dx}$, if $x^y + y^x = 2$.

Example

Find $\frac{dy}{dx}$ if $(cos\, x)^y = (sin\, y)^x$.

Example

Find $\frac{dy}{dx}$, if $y = \frac{\left(\sqrt{1-x^2}\right)(2x-3)^{1/3}}{(x^2+2)^{2/3}}$.

Assignment

Find $\frac{dy}{dx}$ when .

1. $y = x^{x} + x^{\frac{1}{x}}$

2. $x^{y} + y^{x} = (x + y)^{x+y}$

3. $y = e^{\sin x} + (\tan x)^{x}$

4. If $e^{x} + e^{y} = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$.

Derivatives of Functions in Parametric Forms

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Derivatives of Functions in Parametric Forms:

Definition

If x and y are two functions of a third variable t , say $x = f(t)\ and\ y = g(t)$ the functions x and y are called parametric functions and t is called the parameter.

Here $x = f(t)\ and\ \ y = g(t)$ is called parametric form.

To find $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy/_{dt}}{dx/_{dt}}$

Working Rule for finding $\frac{dy}{dx}$ when the function in parametric form

To find the derivative of a function in parametric form, we have the working rule

Step I Write the given parametric form of the equation say $x = f(t)$ and $y = g(t)$

Step II Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

Step III Find $\frac{dy}{dx}$ using the formulae $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, provided $\frac{dx}{dt} \neq 0$

Example:

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$,find $\frac{dy}{dx}$

Example:

If $x = log\,t + sin\,t\,, y = e^{t} + cos\,t$ find $\frac{dy}{dx}$

Problems based on derivative of a function w.r.t other function

Working Rule:

1. If the derivative of $f(x)$ w.r.t $g(x)$ is to be determined, then let $u = f(x)\ and\ v = g(x)$

2. Find $\frac{du}{dx}$ and $\frac{dv}{dx}$

3. Now $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Example:

Find the derivative $tan\, x$ with respect to $sin\, x$

Example:

If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then prove that $\frac{dy}{dx} = -\frac{y}{x}$

Example:

If $x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta)$ find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

Assignment

1. If $sinx = \frac{2t}{1+t^2}$ and $tany = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$

2. If $x = a(\theta - sin\theta)$ and $y = a(1 + cos\theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

3. If $x = \frac{1+logt}{t^2}$, $y = \frac{3+2\,logt}{t}$, find $\frac{dy}{dx}$.

Second Order Derivatives

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Second-Order Derivative

Definition

If a function $f(x)$ is differentiable, then it's derivative $f'(x)$ is called the first-order derivative of f.

If $f'(x)$ is again differentiable, then its derivative is called the second-order derivative of f.

Notations:

(i) First-order derivative of $y = f(x)$ can be denoted by $f'(x)$ or $\frac{dy}{dx}$ or y_1 or y'

(ii) Second-order derivative of $y = f(x)$ can be denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$ or y_2 or y''

Working Rule for Computation of Second order derivative

To compute the second-order derivative of any function(except the function in the parametric form) we first compute the first-order derivative and then differentiate again to get the second-order derivative.

For the function in parametric form:

If x and y are functions of a third variable t (called parameter), then first find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ then, find $\frac{dy}{dx}$ using the formula. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

To find $\frac{d^2y}{dx^2}$, use the following

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{1}{\frac{dy}{dx}}$$

Example

Find the second-order derivative of $tan^{-1} x$.

Example

If $x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta)$ find $\frac{dy}{dx}$. Also, find $\frac{d^2y}{dx^2}$.

Example

If $y = tan^{-1} x$, then prove that $(1 + x^2)y_2 + 2xy_1 = 0$

Example

If $x\cos(a+y) = cos\, y$, then prove that $\frac{dy}{dx} = \frac{cos^2(a+y)}{sina}$.

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$.

Assignment

1. If $e^x(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

2. If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2y_2+2x(x^2+1)y_1=2$.

3. If $y=e^{\tan x}$, prove that $cos^2x\frac{d^2y}{dx^2}-(1+\sin 2x)\frac{dy}{dx}=0$

Rolle’s Theorem

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

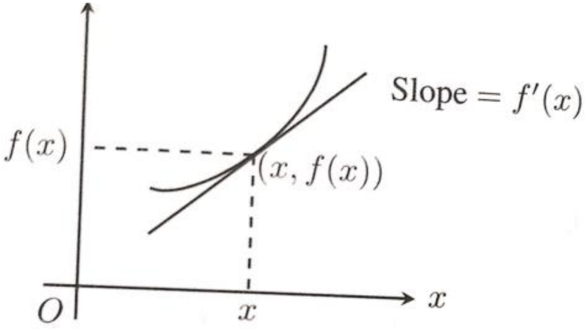
Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

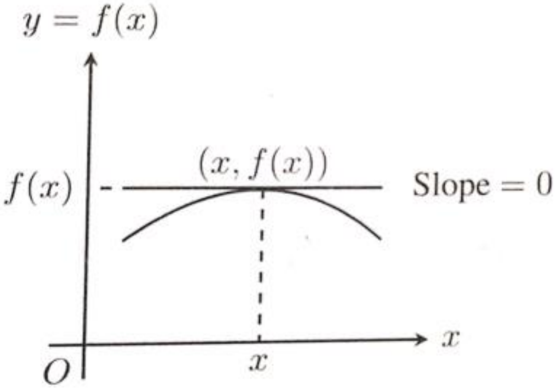
Introduction

In this section, we will discuss two important theorems of Calculus, which are Rolle’s and Lagrange’s Mean Value Theorem. These theorems have some important applications relating to the behaviour of f and f'.

Geometrically Representation of Derivatives

We have already discussed that geometrically $\frac{dy}{dx}$ or $f'(x)$ represents the slope of the tangent to the curve $y = f(x)$ at the point (x, y) or $(x, f(x))$on the curve as shown in the figure.





Rolle's Theorem

Statement: Rolle's Theorem state that

Let $f: [a, b] \to R$ be a function such that

(i) f is continuous on the closed interval $[a, b]$.

(ii) f is differentiable in open interval (a, b).

(iii) $f(a) = f(b)$.

Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical Interpretation of Rolle's Theorem

Let $f: [a, b] \to R$ all the three conditions of Rolle's theorem

Geometrically $f(x)$ is a real-valued function defined on $[a, b]$ such that

(i) The curve $y = f(x)$ is continuous between the points $A(a, f(a))$ and $B(b, f(b))$

(ii) The curve $y = f(x)$ has a unique tangent (with finite slope) at every point between A and B. And

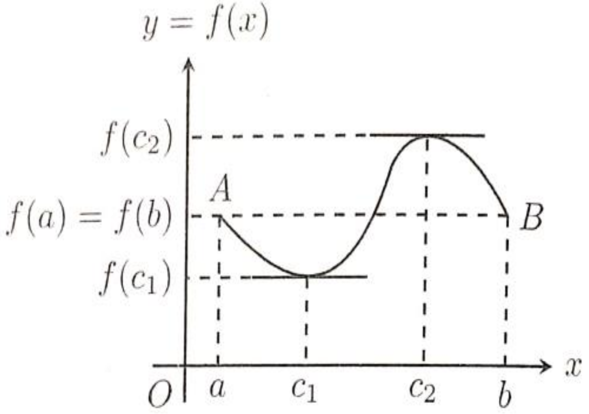
(iii) The ordinates of the curve $y = f(x)$ at the end points of the interval $[a, b]$ are equal.

Then there exists at least one point $c \in (a, b)$ on the curve between A and B where the tangent is parallel to the x-axis.

i.e. The slope of the tangent is 0.

In other words, there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

In the figure, there are two points $(c_1, f(c_1))$ and $(c_2, f(c_2))$ on the curve between A and B where the tangent is parallel to the x –axis.



Example

Verify Rolle's theorem (If applicable) for the function $f(x) = x^3 + 3x^2 - 24x - 80$ on $[-4,5]$.

Assignment

1. Verify Rolle's theorem for $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$.

2. Verify Rolle's theorem for $f(x) = x^2 + 5x + 6$ on $[-3, -2]$.

3. Verify Rolle's theorem for $f(x) = \sin x + \cos x$ on $\left[0, \ \frac{\pi}{2}\right]$.

Lagrange’s Mean Value Theorem

SUBJECT : MATHEMATICS
CHAPTER NUMBER:5
CHAPTER NAME : CONTINUITY AND DIFFERETIABILITY

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: **1800 120 2316**
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

Lagrange's Mean Value Theorem(LMVT)

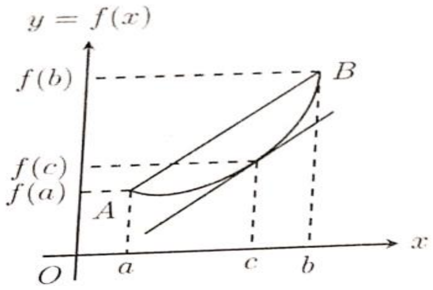
Statement: Let $f:[a,b]\rightarrow R$ be a function such that

(i) f is continuous on the closed interval $[a,b]$ and

(ii) f is differentiable in open interval (a,b).

Then, there exists a real number $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation of Lagrange's Mean Value Theorem



Let $f: [a, b] \rightarrow R$ satisfy both the conditions of Lagrange's MVT.

Geometrically, $f(x)$ is a real-valued function defined on $[a, b]$ such that

(i) The curve $y = f(x)$ is continuous between the points $A(a, f(a))$ and $B(b, f(b))$

And (ii) the curve $y = f(x)$ has a unique tangent (with finite slope) at every point between A and B.

Then there exists at least one point $(c, f(c))$ on the curve between A and B where the tangent is parallel to the secant AB. i. e. slope of the tangent is equal to the slope of the secant AB given by $\frac{f(b)-f(a)}{b-a}$.

In other words, there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

In this figure, we see that there is one point $(c, f(c))$ on the curve between A and B where the tangent is parallel to the secant joining the endpoints A and B of the curve.

Example

Verify Lagrange's mean value theorem for $f(x) = 3x^2 - 5x + 1$ defined in the interval [2,5].

Assignment

1. Verify Lagrange's mean value for the function $f(x) = (x-3)(x-6)(x-9)$ on the interval $[3, 5]$.

2. Verify Lagrange's mean value for the function $f(x) = x^2 - 3x + 2$ on the interval $[-1, 2]$.

3. Verify Lagrange's mean value for the function $f(x) = x + \frac{1}{x}$ on the interval $[1, 3]$.

THANKING YOU

ODM EDUCATIONAL GROUP