Adjoint and inverse of a square matrix

SUBJECT : (Mathematics)
CHAPTER NUMBER: 04
CHAPTER NAME : Determinant

ODM EDUCATIONAL GROUP
Changing your Tomorrow

CHANGING YOUR TOMORROW

Website: www.odmegroup.org
Email: info@odmps.org
Toll Free: 1800 120 2316
Sishu Vihar, Infocity Road, Patia, Bhubaneswar- 751024

Adjoint and Inverse of a Matrix:-

- Adjoint of a matrix is the transpose of the matrix of cofactors of the given matrix
- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- Matrix formed by the cofactors are $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

adj A = Transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

ODM EDUCATIONAL GROUP
Changing your Tomorrow

Note:-

For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The adj A can also obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21} i.e

$$\text{adj}\,A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & \text{all1} \end{bmatrix}$$

Theorem – 1

If A be any given square matrix of order n, then A (adj A) = (adj A) A = |A| I, where I is the identity matrix of order n.

Verification:-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to |A| and otherwise zero, we have

$$A(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I$$

Similarly, we can show $(\text{adj}A)A = |A| I$

Hence A (adj A) = (adj A) A = AI

Singular and No singular matrix

A square matrix a is said to be singular if $|A| = 0$

A square matrix A is said to be non-singular if $|A| \neq 0$

Theorem – 2

If A and B are non-singular matrices of the same order, then AB and BA are also non singular matrices of the same order.

Theorem – 3

The determinant of the product of matrices is equal to product of their respective determinants, that is $|AB| = |A||B|$, where A and B are square matrices of the same order.

ODM EDUCATIONAL GROUP
Changing your Tomorrow

Remark

We know that, $(\text{adj}\,A)A = |A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

Writing determinants of matrices on both sides, we have

$$|(\text{adj}\,A)A| = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$|(\text{adj}\,A)||A| = |A^3| \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|(\text{adj}\,A)||A| = |A^3|(1)$$

$$|(\text{adj}\,A)| = |A|^2$$

In general, if A is a square matrix of order n, then $|\text{adj}(A)| = |A|^{n-1}$

Theorem – 4

A square matrix a is invertible if and only if A is non-singular matrix

Verification:-

Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix b of order n such that $AB = BA = I$

Now AB = I, So |AB| = I or |A| |B| = 1 (since |I|, |AB| = |A| |B|). This gives $|A| \neq 0$. Hence A is non-singular.

Conversely, let a be non-singular. Then $|A| \neq 0$

Now A(adj A) = (adj A) A = |A| I (Theorem 1)

$$A\left(\frac{1}{|A|}\operatorname{adj} A\right) = \left(\frac{1}{|A|}\operatorname{adj} A\right)A = I$$

$$AB = BA = I, \text{ where } B = \frac{1}{|A|}\operatorname{adj} A$$

A is invertible and $A^{-1} = \frac{1}{|A|}\operatorname{adj} A$

Properties:-

1. $A.(adj(A)) = adj(A).A = |A|.I_n$

2. $|adj(A)| = |A|^{n-1}$

3. $adj(adj(A)) = |A|^{(n-1)^2}$

4. $adj(A^T) = (adj(A))^T$

5. $adj(AB) = adj(B).adj(A)$

6. $adj(A^m) = (adj(A))^m$

7. $adj(KA) = K^{n-1}adj(A)$

8. If A is singular then $|adj(A)| = 0$

9. $(A^T)^{-1} = (A^{-1})^T$

10. $(AB)^{-1} = B^{-1}.A^{-1}$

11. $(A^K)^{-1} = (A^{-1})^K, K \in N$

12. $adj(A^{-1}) = (adj(A))^{-1}$

13. $(A^{-1})^{-1} = A$

14. $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

15. $AB = AC \Rightarrow B = C \text{ iff } |A| \neq 0$

Example:

If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A *adj* A = |A| I. Also find A^{-1}.

Answer:

We have $|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1 \neq 0$

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $$adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $$A\,(adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| . I$$

Also $$|A|^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example:

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Answer:

We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since, $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore $B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence $(AB)^{-1} = B^{-1} A^{-1}$

Example:

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

Answer:

We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now $A^2 - 4A + I = O$

Therefore $AA - 4A = -I$

or $A\,A(A^{-1}) - 4\,AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\,(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

THANKING YOU

ODM EDUCATIONAL GROUP