

Solving system of linear equations in two, Three variables using inverse of a matrix.

SUBJECT : (Mathematics)
CHAPTER NUMBER: 04
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CHANGING YOUR TOMORROW

Solving system of linear equations in two, Three variables using inverse of a matrix.

Example:1

Solve the system of equations

$$2x + 5y = 1$$

$$3x + 2y = 7$$

Answer:

The system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution

Note that

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore

$$X = A^{-1}B = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence

$$x = 3, y = -1$$

Example:2

Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

Answer:

The system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3(2 - 3) + 2(4 + 4) + 3(-6 - 4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$$A_{11} = -1,$$

$$A_{21} = -5,$$

$$A_{31} = -1,$$

$$A_{12} = -8,$$

$$A_{22} = -6,$$

$$A_{32} = 9,$$

$$A_{13} = -10$$

$$A_{23} = 1$$

$$A_{33} = 7$$

Therefore

$$A^{-1} = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So

$$X = A^{-1}B = -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence

$$x = 1, y = 2 \text{ and } z = 3.$$

Example:3

Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations

$$x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$$

Answer:

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & 4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I$$

$$\Rightarrow \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

Given system of equation can be written in matrix form as

$$AX = B \Rightarrow X = A^{-1} B, \quad \dots(i)$$

$$\text{Where } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\text{We have } A^{-1} = \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

Now from (i)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -16 + 36 + 4 \\ -28 + 9 + 3 \\ 20 - 27 - 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 3, y = -2, z = -1$$

Example-4

Find A^{-1} where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$. Hence solve the system of equations

$$x + 2y - 3z = -4, 2x + 3y + 2z = 2, 3x - 3y - 4z = 11$$

SOLUTION We have,

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = -6 + 28 + 45 = 67 \neq 0$$

So, A is invertible.

Let C_{ij} be the co-factors of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} = -6, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} = 14,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 3 & -3 \end{vmatrix} = -15, \quad C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -3 \\ -3 & -4 \end{vmatrix} = 17,$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5, \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} = 9$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13, \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} = -8,$$

and, $C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$$\therefore \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{67} \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

The given system of equations is

$$x + 2y - 3z = -4$$

$$2x + 3y + 2z = -2$$

$$3x - 3y - 4z = 11$$

or, $AX = B$, where $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

As discussed above A is non-singular and so invertible. The inverse of A is given by (i).

The solution of the given system of equations is given by

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67} \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} = \frac{1}{67} \begin{bmatrix} 24 & +34 & +143 \\ -56 & +10 & -88 \\ 60 & +18 & -11 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 3, y = -2$ and $z = 1$ is the required solution.

Example-5

Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations

$$x + 3z = 9, \quad -x + 2y - 2z = 4, \quad 2x - 3y + 4z = -3.$$

Answer:

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} &= \begin{bmatrix} -2 - 9 + 12 & 0 - 2 + 2 & 1 + 3 - 4 \\ 0 + 18 - 18 & 0 + 4 - 3 & 0 - 6 + 6 \\ -6 - 18 + 24 & 0 - 4 + 4 & 3 + 6 - 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

$$[\because (A^T)^{-1} = (A^{-1})^T]$$

Now given system of equations can be written in matrix form as

$$AX = B, \text{ where, } A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 + 36 - 16 \\ 0 + 8 - 3 \\ 9 - 12 + 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 0, y = 5 \text{ and } z = 3$$

THANKING YOU
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