Chapter- 5

Continuity and Differentiability

INTRODUCTION

In this chapter, we will discuss two very important concepts of mathematics continuity and the differentiability of real functions. Also, discuss the relationship between them. To understand these concepts well one should know the concept of limits which was in Class – XI

Limits:

Let $a \in R$and 'f' be a real-valued function in real variable x defined at the points in an open interval containing 'a' except possible at 'a'. Then we say that limit of the function $f(x)$ is a real number l as x tends to 'a'. If the value of $f(x)$ approaches l as x approaches 'a'. Which is denoted by $\lim_{x\to a} f(x) = l$

Here x can approach 'a' on a real number line in two ways, either from the left or from the right of a. This leads to two limits as the left-hand limit (LHL) and the Right-hand limit (RHL).

The left-hand limit is the value of f(x) approaches l as x approaches 'a' from the left of a. It is denoted by $\lim_{x\to a^-} f(x)$

The right-hand limit is the value of $f(x)$ approaches as x approaches 'a' from the right of 'a'. It is denoted by $\lim_{x\to a^+} f(x)$

Existence of Limit

Whenever $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$

Then $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = l$

$LHL = \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

$RHL = \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

Some Important results on the limit

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x\to 0} \frac{\tan x}{x} = 1$ (c) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (d) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

(e) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ (f) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n.a^{n-1}$ (g) $\lim_{x\to a} \frac{\sin^{-1} x}{x} = 1$ (h) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

If $a \in R$ and but, f, g be real-valued functions then

(a) $\lim_{x\to a} f(x) = k\lim_{x\to a} f(x)$ (K is constant) (b) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(c) $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$ (d) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}; \lim_{x\to a} g(x) \neq 0$

Intuitive Idea of continuity

Let 'f' be a real-valued function in any interval and let $y = f(x)$. Then we can represent the function by a graph in XY –plane. The function 'f' is continuous when we try to draw the graph in one stoke, i.e without lifting the pen from the plane of the paper. Roughly, a function is continuous if its graph is a single unbroken curve with no holes or jumps.

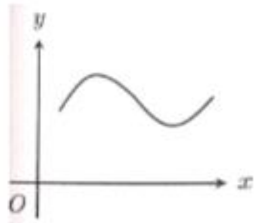


Figure 10.1

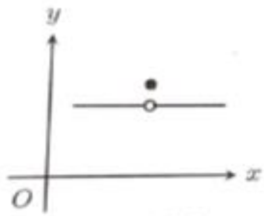


Figure 10.2

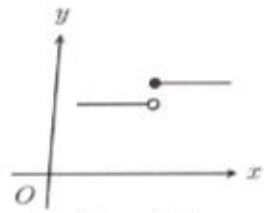


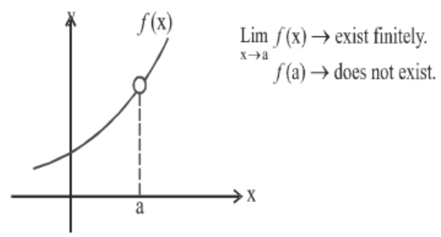
Figure 10.3

From the above idea, the function shown in figure 10.1 is continuous.

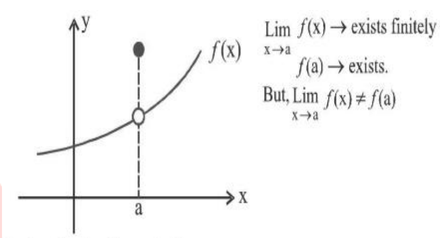
The function shown in fig 10.2 has a hole at a point and hence is not continuous.

The function is shown in fig. 10.3 has a jump at a point and hence is not continuous.

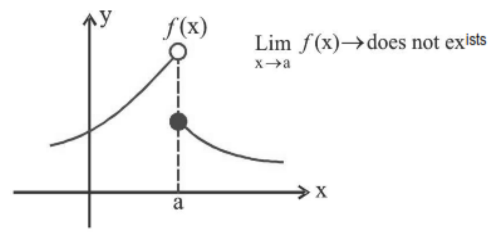
Different types of Discontinuity



missing point discontinuity at x = a



Isolated point discontinuity at x = a



non-removable discontinuity at x=a

Mathematical definition of Continuity

A function $f: D \to R$ is said to be continuous at $x = c$

i.e. if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(x)$

i.e. LHL = RHL = Functional value

Otherwise the function will be discontinuous at $x = c$

Conclusion

As the function $f(x)$ is continuous at x=a if LHL = RHL = f(a)

But we know that when LHL=RHL= ℓ (say)

Then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = \ell$

Thus the function f(x) will be continuous at x=a if $\lim_{x \to a} f(x) = f(a)$

i.e. Limiting value = Functional value.

Example

Examine the continuity of the function $f(x) = 2x^2 - 1 at x = 3$

Solution:-

Given $f(x) = 2x^2 - 1$ LHL = $\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h)$

$= \lim_{h \to 0} [2(3 - h)^2 - 1]$ $= 2[(3 - 0)^2 - 1] = 17$

RHL = $\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 + h)$ $= \lim_{h \to 0} [2(3 + h)^2 - 1] = 2(3)^2 - 1 = 17$

As LHL=RHL=f(3)

So f is continuous at x =3

Example:(Exemplar)

Check the continuity of the function $f(x) = \begin{cases} 3x + 5 & if x \geq 2 \\ x^2 & if x < 2 \end{cases}$ at x = 2

Solution:

Given that $f(2) = 3.2 + 5 = 11$

$$LHL = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}(2-h^2) = (2-0)^2 = 4$$

$$RHL \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h)+5 = 3(2+0)+5 = 11$$

As, so f is not continuous at x=2.

$LHL \neq RHL$

Example:

Show that the function $f(x) = \begin{cases} \frac{2x^2-3x-2}{x-2} & if x \neq 2 \\ 5 & if x = 2 \end{cases}$ is continuous at x = 2

Solution:

Given that f(2)=5

Now $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{2x^2-3x-2}{x-2} = \lim_{x\to 2} \frac{2x^2-4x+x-2}{x-2}$

$$= \lim_{x\to 2} \frac{(x-2)(2x+1)}{x-2} = \lim_{x\to 2} \frac{2x+1}{1} = 2.2+1 = 5$$

As $\lim_{x\to 2} f(x) = f(2)$ so f is continuous at x = 2

Example:

Discuss the continuity of $f(x)$when $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, if x \neq 0 \\ 5, if x = 0 \end{cases}$ $at x = 0$

Solution:

Given that $f(o) = 5$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$

$$= 2\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 2(\ell)^2 = 2$$

Since $\lim_{x\to 0} f(x) \neq f(0)$ Hence f is not continuous at x=0

Example:

Find the value of k so that the function $f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & x \neq 2 \\ k, & if x = 2 \end{cases}$ is continuous at x=2.

Solution:

Given that $f(2) = k$

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{2^{x+2}-16}{4^x-16} = \lim_{x\to 2} \frac{2^x.2^2-16}{4^x-16} = \lim_{x\to 2} \frac{4(2^x-4)}{(2^x)^2-(4)^2}$

$$= \lim_{x\to 2} = \frac{4}{2^x+4} = \frac{4}{2^2+4} = \frac{1}{2}$$

As f(x) is continuous at x = 2

$\lim_{x\to 2} f(x) = f(2)$

$\Rightarrow \frac{1}{2} = k$ so, $k = \frac{1}{2}$

Definition:

A real function 'f' is said to be continuous if it is continuous at every point in the domain of 'f'. Suppose 'f' is a function defined on a closed interval [a, b], then for 'f' to be continuous, it needs to be continuous at every point of [a, b] including the endpoints a and b.

Example:- Prove that the constant function f(x) = k is continuous.

Solution:-Let 'c' be any real number

Here $f(c) = k$ for every $c \in R$

$\lim_{x\to c} f(x) = \lim_{x\to c} k = k and \lim_{x\to c} f(x) = f(x)$

Since for any real number 'c', the function 'f' is continuous.

The function f is continuous at every real number.

List of some continuous functions

Sl. NO.	Function f(x)	Interval in which f(x) is continuous
1.	Constant C	$(-\infty, \infty)$
2.	$x^n, n$is an integer ≥0	$(-\infty, \infty)$
3.	x^{-n},n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\lvert x - a \rvert$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomial in x	$(-\infty, \infty) - \{x; q(x) = 0\}$
7.	$\sin x$	$(-\infty, \infty)$
8.	$\cos x$	$(-\infty, \infty)$
9.	$\tan x$	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2}: n \in I\right\}$
10.	$\cot x$	$(-\infty, \infty) - \{n\pi: n \in I\}$
11.	$\sec x$	$(-\infty, \infty) - \{(2n+1)\}$
12.	$cosec\, x$	$(-\infty, \infty) - \{n\pi: n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

Algebra of Continuous Functions:

Suppose 'f' and 'g' be two real functions at a real number 'c' then

(a) $f + g$ is continuous at $x = c$

(b) $f - g$ is continuous at $x = c$

(c) $f.g$ is continuous at $x = c$

(d) $\left(\frac{f}{g}\right)$ is continuous at $x = c$ (provided $g(c) \neq 0$

(e) If 'f' and 'g' be two functions such that fog is defined at c and if 'f' is continuous at $g(c)$. Then fog is continuous at c.

Example:-

Show that the function defined by $f(x) = |\cos x|$ is continuous.

Solution:- The function ‘f’ may be thought of as a composition gof of the two functions ‘g’ and ‘h,
Where $g(x) = |x| and h(x) = \cos x$

$$goh(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)$$

Since both ‘g’ and ‘h’ are continuous functions so ‘f’ is continuous.

Example:

If the function $f(x) = \begin{cases} 3ax + b & if x > 1 \\ 11 & if x = 1 \\ 5ax - 2b & if x = 1 \end{cases}$ is continuous at x = 1, find the values of a and b.

Continuity at x = 1 we have, $f(1) = 11$.

$$\lim_{x\to 1+} [f(x)] = \lim_{x\to 1+} [3ax + b]$$

$= \lim_{h\to 0}[3a(1 + h) + b]$ [By putting $x = 1 + h$]

$$\lim_{x\to 1-} [f(x)] = \lim_{x\to 1-} [5ax - 2b]$$

$= \lim_{h\to 0}[5a(1 - h) - 2b]$ [By putting $x = 1 - h$]

$= 5a - 2b.$

So, f is continuous at x = 1 if

i.e, $\lim_{x\to 1+} [f(x)] = \lim_{x\to 1-} [f(x)] = f(1)$

i.e $3a + b = 5a - 2b = 11$

i.e $3a + b = 11 and 5a - 2b = 11$

i.e $a = 3, b = 2$

Example:

Find k, if $f(x) = \begin{cases} k \sin\frac{\pi}{2}(x + 1) & if x \le 0 \\ \frac{\tan x - \sin x}{x^3} & if x > 0 \end{cases}$ is continuous at x = 0.

Solution:

Given $f(x) = \begin{cases} k \sin\frac{\pi}{2}(x + 1) & if x \le 0 \\ \frac{\tan x - \sin x}{x^3} & if x > 0 \end{cases}$

Continuity at x = 0 we have $f(0) = k$

$$\lim_{x\to 0+} [f(x)]$$

$$= \lim_{x\to 0+} \left[\frac{\tan x - \sin x}{x^3}\right]$$

$= \lim_{h\to 0} \left[\frac{\tan \quad h - \sin \quad h}{h^3}\right]$ [By Putting $x = 0 + h$]

$$= \lim_{h\to 0} \left[\frac{\tan h\,(1 - \cos \quad h)}{h^3}\right]$$

$$= \lim_{h\to 0} \left[\frac{\tan h\left(2\sin^2\frac{h}{2}\right)}{h^3}\right]$$

$$= \lim_{h\to 0}\left[\frac{\tan h}{h} \times \frac{2\sin^2\frac{h}{2}}{4\left(\frac{h}{2}\right)}\right]$$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{\tan h}{h}\right]\lim_{h\to 0}\left[\frac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}\right] = \frac{1}{2}$$

$$\lim_{x\to 0-}[f(x)]$$

$$= \lim_{x\to 0-}\left[k\sin\frac{\pi}{2}(x+1)\right]$$

$= \lim_{h\to 0}\left[k\sin\frac{\pi}{2}\{(0-h)+1\}\right]$ [By putting $x = 0 - h$]

$$= k\sin\frac{\pi}{2} = k$$

So, f is continuous at $x = 0$ if $\lim_{x\to 0+}[f(x)] = \lim_{x\to 0-}[f(x)] = f(0)$

i.e, $\frac{1}{2} = k = k$

i.e, $k = \frac{1}{2}$

Example:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here [x] denotes the greatest integer less than or equal to x.

Solution: Let $n \in I$

Then, $\lim_{x\to n^-}[x] = n - 1$

$\therefore$ $[x] = n - 1 \forall x \in [n-1, n]$and $g(n) = n - n = 0$ [$\therefore$[n] = n because $n \in I$]

Now, $\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x - [x]) = \lim_{x\to n^-} x - \lim_{x\to n^-}[x]$

$$= n - (n-1) = 1$$

Also, $\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x - [x]) = \lim_{x\to n^+} x - \lim_{x\to n^+}[x] = n - n = 0$

Thus, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$. Hence, g(x) is discontinuous at all integral points.

Example:

Determine the values of **a** and **b** so that the function f (x) is continuous. $If f(x) = \begin{cases} 1, if x \leq 3 \\ ax + b, if 3 < x < 5. \\ 7, if 5 \leq x \end{cases}$

Sol:

The given function is continuous at each x in R so at x=3 and x=5.

At $x - 3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$$\Rightarrow \lim_{h\to 0} f(3-h) = \lim_{h\to 0} f(3+h) = 1$$

$$\Rightarrow \lim_{h\to 0} 1 = \lim_{h\to 0} a(3+h) + b = 1$$

$$\Rightarrow 1 = 3a + b = 1$$

$\Rightarrow 3a + b = 1$ .... (1)

At x=5

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$
$$\Rightarrow \lim_{h\to 0} f(5-h) = \lim_{h\to 0} f(5+h) = 7$$
$$\Rightarrow \lim_{h\to 0} a(5-h) + b = \lim_{h\to 0} 7 = 7$$
$$\Rightarrow 5a + b = 7 = 7$$

$\Rightarrow 5a + b = 7$ .... (2)

Solving equation (1) and (2) we have a = 3 and b = -8

Example:

Determine the value k so that the function f(x) is continuous at x=0, Where

$f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if -1 \le x < 0 \\ \frac{2x+1}{x-1}, & if 0 \le x \le 1 \end{cases}$ at $x = 0$

Solution:

∴ LHL = $\lim_{x\to 0^-} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$

$$= \lim_{x\to 0^-} \left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right).\left(\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)$$
$$= \lim_{x\to 0^-} \frac{1+kx-1+kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$
$$= \lim_{x\to 0^-} \frac{2kx}{x\sqrt{1+kx}+\sqrt{1-kx}}$$
$$= \lim_{h\to 0} \frac{2k}{x\sqrt{1+k}+\sqrt{1+kh}} = \frac{2k}{2} = k$$

And $f(0) = \frac{2\times 0+1}{0-1} = -1$

$\Rightarrow k = -1$ LHL = RHL = f(0)

DIFFERENTIABILITY AT A POINT

A function f is said to have a derivative at x=a or differentiable at x=a iff $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists finitely.

The value of this limit is called the derivative of f at 'a' and is denoted by f'(a).

NOTE: A function y = f(x) is differentiable at x = a if L.H.D. = R.H.D. at x = a.

i.e. $\lim_{h\to 0} \frac{f(a-b)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Differentiable Function

We define the derivative of the function f(x) at x=a as:

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} or f'(a) = \lim_{x\to a} \frac{f(a)-f(a)}{x-a}$, exists finitely.

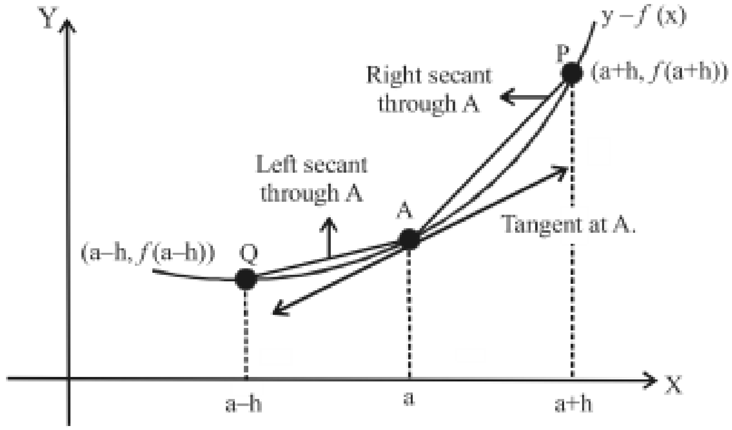
Now the question in our mind when does not exists?

For this let us consider

The slope of the Right-hand secant = $\frac{f(a+h)-f(a)}{h}$ as $h \to 0, P \to A$ and secant $(AP) \to$ tangent at A

⇒ Right-hand derivative = $\lim\limits_{h\to 0}\left(\frac{f(a+b)-f(a)}{h}\right)$

= Slop of the tangent at A (when approached from the right $f'(a^+)$.



The slope of the left-hand secant = $\frac{f(a-b)-f(a)}{-h}$ as $h \to 0, Q \to A$ and secant $AQ \to$ tangent at A

⇒ Left-hand derivative = $\lim\limits_{h\to 0}\left(\frac{f(a-h)-f(a)}{-h}\right)$

= slope of the tangent at A (where approached from left) $f'(a^-)$

If $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$ does not exist, then f is not differentiable at x = a

In other words, we say that the function f is differentiable at the point 'a' if both the Left-hand derivative(LHD) and Right-hand derivative (RHD) are finite and equal

I.e. $\Leftrightarrow \lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0}\frac{f(a-b)-f(a)}{-h}$ When LHD = RHD

EXAMPLE – 1

Prove that the function $f(x) = \begin{cases} 1+x, & if x \le 2 \\ 5-x, & if x > 2 \end{cases}$ is not differentiable at x = 2.

Proof : Here f(2) = 1 + 2 = 3 , f(2 – h) = 1 + 2 – h = 3 – h and f(2+ h) = 5 – (2 +h) = 3 – h

L.H.D = $\lim\limits_{h\to 0}\frac{f(2-h)-f(2)}{-h} \qquad = \lim\limits_{h\to 0}\frac{3-h-3}{-h} = 1$

RHD = $\lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}\frac{3-h-3}{h}$ as L.H.D is not equal to R.H.D. hence not differentiable

EXAMPLE – 2

Show that $f(x) = \begin{cases} 12x-13, & x \le 3 \\ 2x^2+5, & x > 3 \end{cases}$ is a differentiable function at x = 3.

Solution:

Here, $f(3) = 12 \times 3 - 3 = 23, f(3-h) = 12(3-h) - 13 = 23 - 12h$

$$f(3+h) = 2(3+h)^2 + 5 = 2(9+h^2+6h) + 5 = 23 + 2h^2 + 12h$$

LHD = $\lim\limits_{h\to 0}\frac{f(3-h)-f(3)}{-h} = \lim\limits_{h\to 0}\frac{23-12h-23}{-h} = \lim\limits_{h\to 0}\frac{-12h}{-h} = 12$

RHD = $\lim\limits_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0}\frac{23+12h^2+12h-23}{h} = \lim\limits_{h\to 0}\frac{12h+12}{1} = 12$

As, LHD = RHD hence it is a differentiable function at x = 3.

NOTE: Every Differentiable function is continuous but every continuous function may or may not be differentiable.

EXAMPLE -3

For what choice of a and b is the function $f(x) = \begin{cases} x^2 & ,x \leq 2 \\ ax+b & ,x > 2 \end{cases}$ is differentiable at x = 2

Solution:

We know that every differentiable function is continuous.

So, $f(2) = 4, LHL = \lim_{h\to 2^-} x^2 = \lim_{h\to 2}(2-h)^2 = 4, RHL = \lim_{h\to 2^-} ax+b = \lim_{h\to 0} a(2+h)+b = 2a+b$

Therefore, $2a+b = 4$.....................(1)

Here $f(2) = 2^2 = 4, f(2-h) = (2-h)^2 = 4+h^2-4h, f(c+h) = a(2+h)+b$

LHD = $\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{4+h^2-4h-4}{-h} = \lim_{h\to 0}\frac{h^2-4h}{-h} = \lim_{h\to 0}\frac{h-4}{-1} = 4$

RHD = $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{2a+ah+-4}{h} = \lim_{h\to 0}\frac{4+ah-4}{h} = \lim_{h\to 0}\frac{ah}{h} = a$

Therefore, A = 4 and from equation (1) b = -4

Rules of Differentiation:

For any two differentiable functions $u$and v, The following rules are established as a part of the algebra of derivatives.

1. $(ku)^/ = ku^/, wherekisany$constant
2. 2. $(u \pm v)^/ = u^/ \pm v^/$
3. $(uv)^/ = u^/v + v^/u$ (product rule)
4. $\left(\frac{u}{v}\right)^/ = \frac{vu^/-uv^/}{v^2}$ (quotient rule)

SOME FORMULAE OF DERIVATIVE

1. $\frac{d(k)}{dx} = 0$, where k is constant.
2. $\frac{d(x^n)}{dx} = nx^{n-1}$
3. $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$
4. $\frac{d(e^x)}{dx} = e^x$
5. $\frac{d(a^x)}{dx} = a^x \log a$
6. $\frac{d(\log x)}{dx} = \frac{1}{x}, x > 0$
7. $\frac{d(\sin x)}{dx} = \cos x$
8. $\frac{d(\cos x)}{dx} = -\sin x$
9. $\frac{d(\tan x)}{dx} = \sec^2 x$
10. $\frac{d(\cot x)}{dx} = -\cos e c^2 x$
11. $\frac{d(\sec x)}{dx} = \sec x.\tan x$
12. $\frac{d(\sec x)}{dx} = \sec x.\tan x$
13. $\frac{d(\cos ecx)}{dx} = -\cos e cx.\cot x$

DERIVATIVE OF COMPOSITION FUNCTIONS (CHAIN RULE)

To study the derivative of composition functions, we start with an illustrative example, say we want to find the derivative of f where $f(x) = (2x+1)^3$.

Now $\frac{df(x)}{dx} = \frac{d(2x+1)^3}{dx} = \frac{d(8x^3+12x^2+6x+1)}{dx} = 24x^2 + 24x + 6 = 6(2x+1)^2$

We observe that, if we take $g(x) = 2x + 1$ and $h(x) = x^3$

Then $f(x) = hog(x) = (2x+1)^3 \Rightarrow \frac{df(x)}{dx} = f^{/}(x) = \frac{dhog(x)}{dg(x)} \times \frac{dg(x)}{dx}$

I.e. $\frac{d(2x+1)^2}{d(2x+1)} \times \frac{d(2x+1)}{dx} \Rightarrow f^{/}(x) = 3 \times (2x+1)^2 \times 2 = 6(2x+1)^3$

The advantage of such observation is that it simplifies the calculation of finding the derivative.

CHAIN RULE

Let f be the real-valued function which is the composition of two functions u & v. i.e. $f = uov$.

Where u & v are differentiable functions and uov is also a differentiable function?

$\Rightarrow \frac{df}{dx} = \frac{duov}{dx} = \frac{duov}{dv} \times \frac{dv}{dx}$, Provided all the derivatives in the statement exists.

PROBLEMS

Problem-1: Find $\frac{dy}{dx}$,if $y = sin(x^2 + 1)$

Answer: Given $y = sin(x^2+1) \Rightarrow \frac{dy}{dx} = \frac{d(sin(x^2+1))}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(sin(x^2+1))}{d(x^2+1)} \times \frac{d(x^2+1)}{dx} \Rightarrow \frac{dy}{dx} = cos(x^2+1) \times 2x$$

Problem- 2: Find $\frac{dy}{dx}$ if $y = log(\tan x)$

Answer: Given$y = log(\tan x) \Rightarrow \frac{dy}{dx} = \frac{d(log(\tan x))}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(log(\tan x))}{d\tan x} \times \frac{d\tan x}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\tan x} \times sec^2 x$$

Problem- 3: Find $\frac{dy}{dx}$ if $y = e^{\sin(x^2)}$

Answer: Given $y = e^{\sin(x^2)} \Rightarrow \frac{dy}{dx} = \frac{d(e^{\sin(x^2)})}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(e^{\sin(x^2)})}{d\sin(x^2)} \times \frac{d\sin(x^2)}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{dy}{dx} = e^{\sin(x^2)} \times cos(x^2) \times 2x$$

Problem-4: Find $\frac{dy}{dx}$ if $y = (x^2+x+1)^4$

Answer: Given $y = (x^2+x+1)^4 \Rightarrow \frac{dy}{dx} = \frac{d(x^2+x+1)^4}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(x^2+x+1)^4}{d(x^2+x+1)} \times \frac{d(x^2+x+1)}{dx} \Rightarrow \frac{dy}{dx} = 4(x^2+x+1)^3 \times (2x+1)$$

Problem-5: Find $\frac{dy}{dx}$, if $y = \frac{1}{\sqrt{a^2-x^2}}$

Answer: Given $y = \frac{1}{\sqrt{a^2-x^2}} \Rightarrow \frac{dy}{dx} = \frac{d(a^2-x^2)^{-1/2}}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{d(a^2-x^2)^{-1/2}}{d(a^2-x^2)} \times \frac{d(a^2-x^2)}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{2}(a^2-x^2)^{-3/2} \times (-2x) = \frac{x}{(a^2-x^2)^{\frac{3}{2}}}$

Problem- 6: Find $\frac{dy}{dx}$, if $y = sin^3 x$

Answer: Given $y = sin^3 x \Rightarrow \frac{dy}{dx} = \frac{d(sin^3 x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{d(sin^3 x)}{d\sin x} \times \frac{d\sin x}{dx} = 3\,sin^2 x \times cos\,x$

Problem- 7: Find $\frac{dy}{dx}$, if $y = log(sec\,x + tan\,x)$

Answer: Given that $y = log(sec\,x + tan\,x) \Rightarrow \frac{dy}{dx} = \frac{d(log(sec\,x+tan\,x))}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(log(sec\,x + tan\,x))}{d(sec\,x + tan\,x)} \times \frac{d(sec\,x + tan\,x)}{dx}$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{sec\,x+tan\,x} \times (sec\,x\,.\,tan\,x + sec^2\,x) = sec\,x$

Problem-8: Find $\frac{dy}{dx}$, if $y = e^{x\sin x}$

Answer: Given that $y = e^{x\sin x} \Rightarrow \frac{dy}{dx} = \frac{d(e^{x\sin x})}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(e^{x\sin x})}{d(x\sin x)} \times \frac{d(x\sin x)}{dx} \Rightarrow \frac{dy}{dx} = e^{x\sin x} \times \{x\cos x + \sin x\}$$

Problem- 9: Find $\frac{dy}{dx}$, if $y = \frac{sin(ax+b)}{cos(cx+d)}$

Answer: Given that $y = \frac{sin(ax+b)}{cos(cx+d)} \Rightarrow \frac{dy}{dx} = \frac{d\left\{\frac{sin(ax+b)}{cos(cx+d)}\right\}}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{cos(cx+d) \times \frac{d(sin(ax+b))}{d(ax+b)} \times \frac{d(ax+b)}{dx} - sin(ax+b) \times \frac{d(cos(cx+d))}{d(cx+d)} \times \frac{d(cx+d)}{dx}}{cos^2(cx+d)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos(cx+d) \times cos(ax+b) \times a + sin(ax+b) \times sin(cx+d) \times c}{cos^2(cx+d)}$$

Problem- 10: Find $\frac{dy}{dx}$, if $y = cos(x^3).\,sin(x^3)$

Answer: Given $y = cos(x^3).\,sin(x^3) \Rightarrow \frac{dy}{dx} = \frac{d(cos\,x^3.sin\,x^3)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d(cos\,x^3\,.\,sin\,x^3)}{dx} \Rightarrow \frac{dy}{dx} = cos\,x^3 \frac{d(sin\,x^3)}{dx^3} \times \frac{dx^3}{dx} + sin\,x^3 \frac{d(cos\,x^3)}{dx^3} \times \frac{dx^3}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (cos\,x^3)(cos\,x^3)3x^2 - (sin\,x^3)(sin\,x^3)3x^2 \Rightarrow \frac{dy}{dx} = (cos\,x^3)^2 3x^2 - (sin\,x^3)^2 3x^2$$

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

FORMULAE

1) $\frac{d(sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d(cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

3) $\frac{d(tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

4) $\frac{d(cot^{-1}x)}{dx} = \frac{-1}{1+x^2}$

5) $\frac{d(sec^{-1}x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

6) $\frac{d(cosec^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

SOME IMPORTANT SUBSTITUTIONS

1) $\sqrt{a^2 - x^2}$------------- > put $x = a\sin\theta$ or $x = a\cos\theta$

2) $\sqrt{x^2+a^2}$-------------> put $x = a\tan\theta$ or $x = a\cot\theta$

3) $\sqrt{x^2-a^2}$------------> put $x = a\sec\theta$ or $x = a\,cos\,e\,c\theta$

4) $\frac{2x}{1+x^2}, \frac{2x}{1-x^2}, \frac{1-x^2}{1+x^2}, \frac{3x-x^3}{1-3x^2}$------------> put $x = \tan\theta$

5) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$------------> put $x = a\cos 2\theta$

6) $2x^2-1$ ------------> put $x = \cos\theta$

7) $1-2x^2$ ------------> put $x = \cos\theta$

8) $3x-4x^3$ ------------> put $x = \sin\theta$

9) $4x^3-3x$ ------------> put $x = \cos\theta$

PROBLEMS

Problem -1: Find $\frac{dy}{dx}$, if $y = \sin^{-1} 2x\sqrt{1-x^2}$

Answer: Given $y = \sin^{-1} 2x\sqrt{1-x^2}$, let $x = \sin\theta \Rightarrow \theta \Rightarrow \sin^{-1} x$

$$\Rightarrow y = \sin^{-1} 2\sin\theta\sqrt{1-\sin^2\theta}$$
$$\Rightarrow y = \sin^{-1} 2\sin\theta\cos\theta$$
$$\Rightarrow y = \sin^{-1}\sin 2\theta$$

$\Rightarrow y = 2\theta \Rightarrow y = 2\sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Problems-2: Find $\frac{dy}{dx}$, if $y = \sin^{-1}\frac{2x}{1+x^2}$

Answer:

Method -1: (Using the substitution method)

Given$y = \sin^{-1}\frac{2x}{1+x^2}$, let $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\Rightarrow y = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} \Rightarrow y = \sin^{-1}\sin 2\theta \Rightarrow y = 2\theta \Rightarrow y = 2\tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Method -2: (Direct method)

Given $y = \sin^{-1}\frac{2x}{1+x^2} \Rightarrow y = 2\tan^{-1} x$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Problems-3: Find $\frac{dy}{dx}$, if $y = \sec^{-1}\frac{1}{2x^2-1}$

Answer: Given $y = \sec^{-1}\frac{1}{2x^2-1}$, let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$\Rightarrow y = \sec^{-1}\frac{1}{(2\cos^2 x)-1} \Rightarrow y = \sec^{-1}\frac{1}{\cos 2\theta}$$
$$\Rightarrow y = \sec^{-1}\sec 2\theta \Rightarrow y \Rightarrow 2\theta$$
$$\Rightarrow y = 2\cos^{-1}\theta \Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

Problems-4: Find $\frac{dy}{dx}$, if $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Answer: Given that $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, let $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\Rightarrow y = tan^{-1}\left(\frac{\sqrt{1+tan^2\theta}-1}{tan\,\theta}\right)$$

$$\Rightarrow y = tan^{-1}\left(\frac{sec\,\theta - 1}{tan\,\theta}\right)$$

$$\Rightarrow y = tan^{-1}\left(\frac{1-cos\,\theta}{sin\,\theta}\right) \Rightarrow y = tan^{-1}\left(\frac{2\,sin^2\left(\frac{\theta}{2}\right)}{2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)}\right)$$

$$\Rightarrow y = tan^{-1}\,tan\left(\frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$$

$$\Rightarrow y = \frac{tan^{-1}x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\times\frac{1}{1+x^2}$$

Problems-5: Find $\frac{dy}{dx}$, if $y = cot^{-1}\left(\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right)$

Answer: Given that $y = cot^{-1}\left(\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right)$

$$\Rightarrow y = cot^{-1}\left(\frac{\sqrt{sin^2\left(\frac{x}{2}\right)+cos^2\left(\frac{x}{2}\right)+2\,sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}+\sqrt{sin^2\left(\frac{x}{2}\right)+cos^2\left(\frac{x}{2}\right)-2\,sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}}{\sqrt{sin^2\left(\frac{x}{2}\right)+cos^2\left(\frac{x}{2}\right)+2\,sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}-\sqrt{sin^2\left(\frac{x}{2}\right)+cos^2\left(\frac{x}{2}\right)-2\,sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}}\right)$$

$$\Rightarrow y = cot^{-1}\left(\frac{\sqrt{\left\{cos\left(\frac{x}{2}\right)+sin\left(\frac{x}{2}\right)\right\}^2}+\sqrt{\left\{cos\left(\frac{x}{2}\right)-sin\left(\frac{x}{2}\right)\right\}^2}}{\sqrt{\left\{cos\left(\frac{x}{2}\right)+sin\left(\frac{x}{2}\right)\right\}^2}-\sqrt{\left\{cos\left(\frac{x}{2}\right)-sin\left(\frac{x}{2}\right)\right\}^2}}\right)$$

$$\Rightarrow y = cot^{-1}\left(\frac{cos\left(\frac{x}{2}\right)+sin\left(\frac{x}{2}\right)+cos\left(\frac{x}{2}\right)-sin\left(\frac{x}{2}\right)}{cos\left(\frac{x}{2}\right)+sin\left(\frac{x}{2}\right)-cos\left(\frac{x}{2}\right)+sin\left(\frac{x}{2}\right)}\right)$$

$$\Rightarrow y = cot^{-1}\left(\frac{2\,cos\left(\frac{x}{2}\right)}{2\,sin\left(\frac{x}{2}\right)}\right) \Rightarrow y = cot^{-1}\,cot\left(\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Problems-6: Find $\frac{dy}{dx}$,if $y = tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Answer: Given that $y = tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$, let $x = cos\,2\,\theta \Rightarrow \theta = \frac{cos^{-1}x}{2}$

$$\Rightarrow y = tan^{-1}\left(\frac{\sqrt{1+cos\,2\,\theta}-\sqrt{1-cos\,2\,\theta}}{\sqrt{1+cos\,2\,\theta}+\sqrt{1-cos\,2\,\theta}}\right) \Rightarrow y = tan^{-1}\left(\frac{\sqrt{2}\,cos\,\theta-\sqrt{2}\,sin\,\theta}{\sqrt{2}\,cos\,\theta+\sqrt{2}\,sin\,\theta}\right)$$

By dividing by $\sqrt{2}\,cos\,\theta$ both numerator and denominator we get

$$\Rightarrow y = tan^{-1}\left(\frac{1-tan\,\theta}{1+tan\,\theta}\right) \Rightarrow y = tan^{-1}\left(\frac{tan\left(\frac{\pi}{4}\right)-tan\,\theta}{1+tan\left(\frac{\pi}{4}\right)tan\,\theta}\right)$$

$\Rightarrow y = tan^{-1}\,tan\left(\frac{\pi}{4}-\theta\right) \Rightarrow y = \frac{\pi}{4}-\theta$ $\qquad \Rightarrow y = \frac{\pi}{4}-\frac{cos^{-1}x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

DERIVATIVE OF IMPLICIT FUNCTIONS

Until now we have been differentiating various functions given in the form of $y = f(x)$. But the functions don't need to be always expressed in this form.

For example:

Considering $x - y - 2 = 0$ and $x + sin(xy) - y = 0$

In the first case, we can solve for y and rewrite the relationship $y = x - 2$.

But in the second case, it does not seem that there is an easy way to solve for y. When a relationship between x and y is expressed in a way that is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x. In the 2nd case, it is implicit that y is a function of x and we say that the relationship of the second type mentioned above gives function implicitly.

PROBLEMS

Problem-1: Find $\frac{dy}{dx}$, if $x + y = 10$

Answer: Given that $x + y = 10$

Differentiating w. r. t. x we get.

$$\Rightarrow \frac{d(x+y)}{x} = \frac{d(10)}{dx} \Rightarrow \frac{dx}{dx} + \frac{dy}{dx} = 0$$

$$\Rightarrow 1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$$

Problem-2: Find $\frac{dy}{dx}$, if $2x^2 + y^2 + xy = a$

Answer: Given that $2x^2 + y^2 + xy = a$

Differentiating both sides w. r. t. x we get.

$$\Rightarrow \frac{d(2x^2 + y^2 + xy)}{dx} = \frac{da}{dx} \Rightarrow \frac{2dx^2}{dx} + \frac{dy^2}{dx} + \frac{dxy}{dx} = 0$$

$$\Rightarrow 4x + \frac{dy^2}{dy} \times \frac{dy}{dx} + x\frac{dy}{dx} + y\frac{dx}{dx} = 0 \Rightarrow 4x + 2y \times \frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx}(2y + x) = -(4x + y) \Rightarrow \frac{dy}{dx} = \frac{-(4x+y)}{(2y+x)}$$

Problem- 3: Find $\frac{dy}{dx}$, if $\frac{y}{x} + sin^2 y = k$

Answer: Given that $\frac{y}{x} + sin^2 y = k$

Differentiating both sides w. r. t. x we get.

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{dx} + \frac{d(sin^2 y)}{dx} = \frac{dk}{dx} \Rightarrow \frac{x\frac{dy}{dx} - y}{x^2} + \frac{d(sin^2 y)}{d(sin\, y)} \times \frac{d(sin\, y)}{dy} \times \frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} - y + x^2 2\, sin\, y\, .\, cos\, y\frac{dy}{dx} = 0 \Rightarrow (x + x^2\, sin\, 2\, y)\frac{dy}{dx} = y \qquad \Rightarrow \frac{dy}{dx} = \frac{y}{(x + x^2\, sin\, 2y)}$$

Problem-4: Find $\frac{dy}{dx}$, if $x^3 + x^2 y + cos(xy) + y^3 = 81$

Answer: Given that $x^3 + x^2 y + cos(xy) + y^3 = 81$

$$\Rightarrow \frac{d(x^3 + x^2 y + cos(xy) + y^3)}{dx} = 0$$

$$\Rightarrow \frac{dx^3}{dx} + \frac{d(x^2 y)}{dx} + \frac{d\{cos(xy)\}}{d(xy)} \times \frac{d(xy)}{dx} + \frac{d(y^3)}{dy} \times \frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left(x^2\frac{dy}{dx} + y(2x)\right) - sin(xy) \times \left(x\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y(2x) - x\,sin(xy)\frac{dy}{dx} - y\,sin(xy) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (3y^2 + x^2 - x\,sin(xy))\frac{dy}{dx} = y\,sin(xy) - 3x^2 - 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\,sin(xy) - 3x^2 - 2xy}{(3y^2 + x^2 - x\,sin(xy))}$$

The derivative of logarithm and exponential functions

We have learned about the derivatives of the functions of the form$\{f(x)\}^n$, $n^{f(x)}$, where $f(x)$ is a function of x and n is the constant. In this section, we will mainly discuss derivatives of the functions of the form $\{f(x)\}^{g(x)}$where $f(x)$ and $g(x)$ are functions of x. To find out the derivative

Let $y = f(x)^{g(x)}$

Taking logarithm both sides we get

$$\Rightarrow log\,y = g(x).log\{f(x)\}$$

Differentiating w. r. t. x we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x) \times \frac{1}{f(x)} \times f^/(x) + log\{f(x)\} \times g^/(x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{g(x) \times \frac{1}{f(x)} \times f^/(x) + log\{f(x)\} \times g^/(x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (f(x))^{g(x)}\left\{g(x) \times \frac{1}{f(x)} \times f^/(x) + log\{f(x)\} \times g^/(x)\right\}$$

Alternately, we can write

$y = f(x)^{g(x)} = e^{g(x).log\{f(x)\}}$

$$\Rightarrow \frac{dy}{dx} = e^{g(x).log\{f(x)\}}\left\{g(x) \times \frac{1}{f(x)} \times f^/(x) + log\{f(x)\} \times g^/(x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = f(x)^{g(x)}\left\{g(x) \times \frac{1}{f(x)} \times f^/(x) + log\{f(x)\} \times g^/(x)\right\}$$

Problem-1: Find the derivative of $y = x^x$

Method-1

Answer: Given $y = x^x$ --------(i)

Taking log both sides we get

$$\Rightarrow log\,y = x\,log\,x$$

Differentiating both sides with respect to x we get

$$\Rightarrow \frac{d(log\,y)}{dx} = \frac{d(x\,log\,x)}{dx}$$

$$\Rightarrow \frac{d(log\,y)}{dy} \times \frac{dy}{dx} = x\frac{d(log\,x)}{dx} + (log\,x).1$$

$$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = x.\frac{1}{x} + log\,x$$

$\Rightarrow \frac{dy}{dx} = y(1 + log\,x) \Rightarrow \frac{dy}{dx} = x^x(1 + log\,x)$ from equation (i)

Method- 2

Given $y = x^x$

$$\Rightarrow y = e^{\log x^x} = e^{x \log x}$$

Differentiating with respect to x we get

$$\Rightarrow \frac{dy}{dx} = \frac{de^{x \log x}}{d(x \log x)} \times \frac{d(x \log x)}{dx}$$

$\Rightarrow \frac{dy}{dx} = e^{x \log x} \times \left\{x \frac{d(\log x)}{dx} + \log x \frac{dx}{dx}\right\}$ $\Rightarrow \frac{dy}{dx} = e^{x \log x}\left(\frac{x}{x} + \log x\right) \Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$

Problem-2: Find the derivative of $y = (\sin x)^{\log x}$

Answer: Given that $y = (\sin x)^{\log x}$

$$\Rightarrow y = e^{(\log x)(\log\sin x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d\left(e^{(\log x)(\log\sin x)}\right)}{d\left((\log x)(\log\sin x)\right)} \times \frac{d\left((\log x)(\log\sin x)\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{(\log x)(\log\sin x)}\left\{\log x . \frac{d(\log\sin x)}{d(\sin x)} \times \frac{d(\sin x)}{dx} + \log\sin x \frac{d \log x}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log x}{\sin x} \times \cos x + \frac{\log\sin x}{x}\right\}$$

Problem-3: Find the derivative of $y = (\log x)^x + x^{\log x}$

Answer: Given that $y = (\log x)^x + x^{\log x}$

$$\Rightarrow y = e^{x(\log\log x)} + e^{(\log x)(\log x)}$$

Differentiating with respect to x we get

$$\Rightarrow \frac{dy}{dx} = \frac{d\left(e^{x(\log\log x)}\right)}{d\left(x(\log\log x)\right)} \times \frac{d\left(x(\log\log x)\right)}{dx} + \frac{d\left(e^{(\log x)^2}\right)}{d(\log x)^2} \times \frac{d(\log x)^2}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{x(\log\log x)}\left\{x \frac{d(\log\log x)}{d(\log x)} \times \frac{d(\log x)}{dx} + (\log\log x)\frac{dx}{dx}\right\} + e^{(\log x)^2}\left\{\frac{d(\log x)^2}{d(\log x)} \times \frac{d(\log x)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x\left\{x \frac{1}{\log x} \times \frac{1}{x} + \log\log x\right\} + x^{\log x}\left\{\frac{2 \log x}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log\log x\right\} + x^{\log x}\left\{\frac{2 \log x}{x}\right\}$$

Alternate method :

Given that $y = (\log x)^x + x^{\log x}$

Let $u = (\log x)^x$--------(1) and $v = x^{\log x}$------------(2)

So the above equation becomes $y = u + v$

Differentiating w. r. t. x we get $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ------------(*)

Taking log both sides in equation (1) we get

$$\Rightarrow \log u = x(\log\log x)$$

Now, differentiating u with respect to x, we get

$$\Rightarrow \frac{d(\log u)}{du} \times \frac{du}{dx} = x \frac{d(\log\log x)}{d(\log x)} \times \frac{d(\log x)}{dx} + (\log\log x)\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \frac{x}{log\, x} \times \frac{1}{x} + (loglog\, x)$$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{1}{log\, x} + (loglog\, x)\right)$$

$\Rightarrow \frac{du}{dx} = (log\, x)^x \left(\frac{1}{log\, x} + (loglog\, x)\right)$--------------(3)

Again taking log both sides in equation (2) we get

$$\Rightarrow log\, v = (log\, x)(log\, x)$$

Now, differentiating $u$with respect to x, we get

$$\Rightarrow \frac{d(log\, v)}{dv} \times \frac{dv}{dx} = x\frac{d(log\, x)^2}{d(log\, x)} \times \frac{d(log\, x)}{dx}$$

$$\Rightarrow \frac{1}{v} \times \frac{dv}{dx} = 2(log\, x) \times \frac{1}{x}$$

$\Rightarrow \frac{dv}{dx} = v\left(\frac{2\,log\, x}{x}\right) \Rightarrow \frac{dv}{dx} = x^{log\, x}\left(\frac{2\,log\, x}{x}\right)$---------(4)

Hence equation (*) becomes

$\frac{dy}{dx} = (log\, x)^x \left(\frac{1}{log\, x} + (loglog\, x)\right)$+$x^{log\, x}\left(\frac{2\,log\, x}{x}\right)$

Problem-3: Find $\frac{dy}{dx}$ if $x^y + y^x = 2$.

Answer: Given that $x^y + y^x = 2$

Let $u = x^y$-----------(1) and $v = y^x$------------(2)

So, the above equation becomes $u + v = 2$

Differentiating w. r. t. x we get $\frac{du}{dx} + \frac{dv}{dx} = 0$-----------(3)

Now, in equation (1) taking the log on both sides we get

$$log\, u = y\, log\, x$$

Differentiating w. r. t. x we get

$$\Rightarrow \frac{d(log\, u)}{du} \times \frac{du}{dx} = y\frac{d(log\, x)}{dx} + log\, x\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = y\frac{1}{x} + log\, x\frac{dy}{dx} \Rightarrow \frac{du}{dx} = u\left(y\frac{1}{x} + log\, x\frac{dy}{dx}\right)$$

$\Rightarrow \frac{du}{dx} = x^y\left(y\frac{1}{x} + log\, x\frac{dy}{dx}\right)$-----------------(4)

Again, in equation (2) taking the log on both sides, we get

$$log\, v = x\, log\, y$$

Differentiating w. r. t. x we get

$$\Rightarrow \frac{d(log\, v)}{du} \times \frac{dv}{dx} = x\frac{d(log\, y)}{dy}\frac{dy}{dx} + log\, y\frac{dx}{dx}$$

$$\Rightarrow \frac{1}{v} \times \frac{dv}{dx} = x\frac{1}{y}\frac{dy}{dx} + log\, y \Rightarrow \frac{dv}{dx} = v\left(x\frac{1}{y}\frac{dy}{dx} + log\, y\right)$$

$\Rightarrow \frac{dv}{dx} = y^x\left(x\frac{1}{y}\frac{dy}{dx} + log\, y\right)$-----------------(5)

So equation (3) becomes

$$x^y\left(y\frac{1}{x}+\log x\frac{dy}{dx}\right)+y^x\left(x\frac{1}{y}\frac{dy}{dx}+log\,y\right)=0$$

$$\Rightarrow\left(y.x^{y-1}+x^y\log x\frac{dy}{dx}\right)+\left(x.y^{x-1}\frac{dy}{dx}+y^x\,log\,y\right)=0$$

$$\Rightarrow\frac{dy}{dx}(x^y\,log\,x+x.y^{x-1})=-(y.x^{y-1}+y^x\,log\,y)$$

$$\Rightarrow\frac{dy}{dx}=\frac{-(yx^{y-1}+y^x\,log\,y)}{(x^y\,log\,x+x.y^{x-1})}$$

Problem-4: Find $\frac{dy}{dx}$ if $(cos\,x)^y=(sin\,y)^x$.

Answer: Given that $(cos\,x)^y=(sin\,y)^x$

Taking log both sides we get

$$\Rightarrow y\,log(cos\,x)=x\,log(sin\,y)$$

Differentiating w. r. t. x we get

$$\Rightarrow y\frac{d\{log(cos\,x)\}}{d(cos\,x)}\times\frac{d(cos\,x)}{dx}+log(cos\,x)\frac{dy}{dx}=x\frac{d\{log(sin\,y)\}}{d(sin\,y)}\times\frac{d(sin\,y)}{dy}\times\frac{dy}{dx}+log(sin\,y)$$

$$\Rightarrow\frac{-y\,sin\,x}{cos\,x}+log(cos\,x)\frac{dy}{dx}=\frac{x\,cos\,y}{sin\,y}\times\frac{dy}{dx}+log(sin\,y)$$

$$\Rightarrow(log(cos\,x)-x\,cot\,y)\frac{dy}{dx}=y\,tan\,x+log(sin\,y)$$

$$\Rightarrow\frac{dy}{dx}=\frac{y\,tan\,x+log(sin\,y)}{(log(cos\,x)-x\,cot\,y)}$$

Problem-5: Find $\frac{dy}{dx}$ if $y=\frac{(\sqrt{1-x^2})(2x-3)^{1/3}}{(x^2+2)^{2/3}}$.

Answer: Given that $y=\frac{(\sqrt{1-x^2})(2x-3)^{1/3}}{(x^2+2)^{2/3}}$

Taking log both sides we get

$$\Rightarrow log\,y=log\left\{\left(\sqrt{1-x^2}\right)(2x-3)^{1/3}\right\}-log(x^2+2)^{2/3}$$

$$\Rightarrow log\,y=log(1-x^2)^{1/2}+log(2x-3)^{1/3}-log(x^2+2)^{2/3}$$

$$\Rightarrow log\,y=\frac{1}{2}log(1-x^2)+\frac{1}{3}log(2x-3)-\frac{2}{3}log(x^2+2)$$

Differentiating w. r. t. x we get

$$\Rightarrow\frac{d(log\,y)}{dy}\times\frac{dy}{dx}$$

$$=\frac{1}{2}\frac{d\{log(1-x^2)\}}{d(1-x^2)}\times\frac{d(1-x^2)}{dx}+\frac{1}{3}\frac{d\{log(2x-3)\}}{d(2x-3)}\times\frac{d(2x-3)}{dx}$$

$$-\frac{2}{3}\frac{d\{log(x^2+2)\}}{d(x^2+2)}\times\frac{d(x^2+2)}{dx}\Rightarrow\frac{1}{y}\times\frac{dy}{dx}=\frac{-2x}{2(1-x^2)}+\frac{2}{3(2x-3)}-\frac{2\times 2x}{3(x^2+2)}$$

$$\Rightarrow\frac{dy}{dx}=y\left(\frac{-x}{(1-x^2)}+\frac{2}{3(2x-3)}-\frac{4x}{3(x^2+2)}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\left(\sqrt{1-x^2}\right)(2x-3)^{1/3}}{(x^2+2)^{2/3}}\right)\left(\frac{-x}{(1-x^2)} + \frac{2}{3(2x-3)} - \frac{4x}{3(x^2+2)}\right)$$

Derivatives of Functions in Parametric Forms:

If x and y are two functions of a third variable t, say $x = f(t)\ and\ y = g(t)$ the functions x and y are called parametric functions, and t is called the parameter.

Here $x = f(t)\ and\ y = g(t)$ is called parametric form. To find $\frac{dy}{dx}$

Here $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Working Rule for finding $\frac{dy}{dx}$ when the Parametric Equations are Given

To find the derivative of a function in parametric form, we have the working rule

Step I Write the given parametric form of the equation say $x = f(t)\ and\ y = g(t)$

Step II Find $\frac{dy}{dt}\ and\ \frac{dx}{dt}$

Step III Find $\frac{dy}{dx}$ using the formulae $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, provided $\frac{dx}{dt} \neq 0$

Example:

If $x = log\, t + sin\, t, y = e^t + cos\, t . find \frac{dy}{dx}$

Solution

$x = log\, t + sin\, t \therefore \frac{dx}{dt} = \frac{1}{t} + cos\, t$ and $y = e^t + cos\, t \therefore \frac{dy}{dt} = e^t - sin\, t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t - sin\, t}{\frac{1}{t} + cos\, t} = \frac{t(e^t - sin\, t)}{1 + t\, cos\, t}$$

Problems based on derivative of a function w.r.t other function.

Working Rule:

1. If the derivative of $f(x)$ w.r.t $g(x)$ is to be determined, then let $u = f(x)\ and\ v = g(x)$
2. Find $\frac{du}{dx}\ and\ \frac{dv}{dx}$
3. Now $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Example: Find the derivative $tan\, x$ with respect to $sin\, x$.

Ans:

Now $\frac{du}{dx} = sin^2\, x\ and\ \frac{dv}{dx} = cos\, x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{sec^2\, x}{cos\, x} = sec^2\, x$$

Let $u = tan\, x\ and\ v = sin\, x$ we have to find $\frac{du}{dv}$

Example:

If $x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta)$ find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$

Solution:

Given $x = a(\theta - sin\,\theta) \Rightarrow \frac{dx}{d\theta} a(1 - cos\,\theta)$ and $y = a(1 - cos\,\theta) \Rightarrow \frac{dy}{dx} = a\,sin\,\theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\,sin\,\theta}{a(1-cos\,\theta)} = \frac{sin\,\theta}{1-cos\,\theta}$ At $\theta = \frac{\pi}{2}, \frac{dy}{dx} = \frac{sin\frac{\pi}{2}}{1-cos\frac{\pi}{2}} = 1$

Second-Order Derivatives:

Definition

If a function $f(x)$ is differentiable, then it's derivative $f'(x)$is called the first-order derivative of f. If $f'(x)$ is again differentiable, then its derivative is called the second-order derivative of f.

Notations:

(i) First-order derivative of $y = f(x)$ can be denoted by $f'(x) or\ \frac{dy}{dx}$ or y_1 or y'

(ii) Second-order derivative of $y = f(x)$ can be denoted by $f''(x)\ or\ \frac{d^2y}{dx^2}$ or $y_2\ or\ y''$

Working Rule for Computation of Second-order derivative:

To compute the second-order derivative of any function(except the function in the parametric form) we first compute the first-order derivative and then differentiate again to get the second-order derivative.

For the function in parametric form:

If x and y are functions of a third variable t (called parameter), then first find $\frac{dx}{dt}\,and\,\frac{dy}{dt}$ and then, find $\frac{dy}{dx}$ using the formula.

To find $\frac{d^2y}{dx^2}$, use the following

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{1}{\frac{dy}{dx}}$$

Example: Find the second-order derivative of $tan^{-1} x$.

Solution:

Let $y = tan^{-1} x$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$$= \frac{d(1+x^2)^{-1}}{d(1+x^2)}\frac{d}{dx}(1+x^2)$$

$$= (-1)(1+x^2)^{-2}.2x = -\frac{2x}{(1+x^2)^2}$$

Example:

If $x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta)$ find $\frac{dy}{dx}$. Also, find $\frac{d^2y}{dx^2}$.

Sol:

Now, $\frac{dx}{d\theta} = a(1 - cos\,\theta) and \frac{dy}{dx} = a\,sin\,\theta$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = cot\frac{\theta}{2}$

Again, differentiating both sides w.r.t to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(cot\frac{\theta}{2}\right) = \frac{d}{d\left(\frac{\theta}{2}\right)}\left(cot\frac{\theta}{2}\right).\frac{d\left(\frac{\theta}{2}\right)}{dx}$$

$$= -cos\,e\,c^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx}$$

$$= -\frac{1}{2}cos\,e\,c^2\frac{\theta}{2}.\frac{1}{a(1-cos\,\theta)}$$

$$= -\frac{1}{2}cos\,e\,c^2\frac{\theta}{2}.\frac{1}{2a\,sin^2\frac{\theta}{2}} = -\frac{1}{4a}cos\,e\,c^2\frac{\theta}{2}$$

EXAMPLE:

If $y = tan^{-1}x$, then prove that $(1+x^2)y_2 + 2xy_1 = 0$

Solution.

Given $y = tan^{-1}x$

Differentiating both sides w.r.t x, we get $y_1 = \frac{1}{1+x^2}\,(1+x^2)y_1 = 1$

Differentiating w.r.t x we have

$$(1+x^2)y_2 + y_1(2x) = 0$$

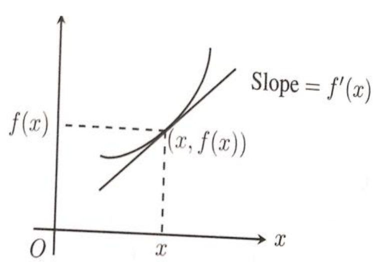
$$(1+x^2)y_2 + 2xy_1 = 0$$

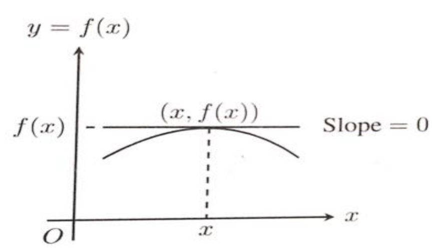
Rolle's And Lagrange's Mean Value Theorem

Introduction:

In this section, we will discuss two important theorems of Calculus, which are Rolle's and Lagrange's Mean Value Theorem. These theorems have some important applications relating to the behaviour of f and f'.

We have already discussed that geometrically $\frac{dy}{dx}$ or $f'(x)$ represents the slope of the tangent to the curve $y = f(x)$ at the point (x, y) or $\left(x, f(x)\right)$ on the curve as shown in the figure.





Rolle's Theorem

Statement: Rolle's Theorem state that

Let $f: [a, b] \to R$ be a function such that

(i) f is continuous on the closed interval $[a, b]$.

(ii) f is differentiable in open interval (a, b).

(iii) $f(a) = f(b)$.

Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical Interpretation of Rolle's Theorem

Let $f: [a, b] \rightarrow R$ all the three conditions of Rolle's theorem

Geometrically $f(x)$ is a real-valued function defined on $[a, b]$ such that

(i) The curve y= f(x) is continuous between the points $A(a, f(a))$ and $B(b, f(b))$

(ii) The curve $y = f(x)$ has a unique tangent (with finite slope) at every point between A and B. And

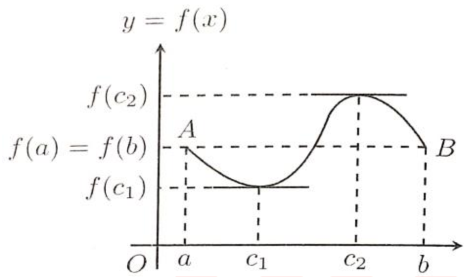
(iii) The ordinates of the curve $y = f(x)$ at the endpoints of the interval $[a, b]$ are equal.

Then there exists at least one point $c \in (a, b)$ on the curve between A and B where the tangent is parallel to the x-axis.

i.e. The slope of the tangent is 0.

In other words, there exists at least one point $c \in (a, b)$such that $f'(c) = 0$.

In the figure, there are two points $(c_1, f(c_1)) and (c_2, f(c_2))$on the curve between A and B where the tangent is parallel to the x-axis.



Lagrange's Mean Value Theorem(LMVT)

Statement: Let$f: [a, b] \rightarrow R$ be a function such that

(i) f is continuous on the closed interval $[a, b]$. and

(ii) f is differentiable in open interval (a, b).

Then, there exists a real number $c \in (a, b)$such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation of Lagrange's Mean Value Theorem

Let $f: [a, b] \rightarrow R$satisfy both the conditions of Lagrange's MVT.

Geometrically, $f(x)$ is a real-valued function defined on $[a, b]$ such that

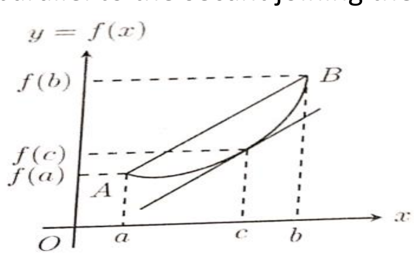
(i) The curve $y = f(x)$ is continuous between the points $A(a, f(a)) and B(b, f(b))$

And (ii) the curve $y = f(x)$ has a unique tangent (with finite slope) at every point between A and B.

Then there exists at least one point $(c, f(c))$ on the curve between A and B where the tangent is parallel to the secant AB. i. e slope of the tangent is equal to the slope of the secant AB given by $\frac{f(b)-f(a)}{b-a}$.

In other words, there exists at least one point $c \in (a, b)$such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

In this figure, we see that there is one point $(c, f(c))$on the curve between A and B where the tangent is parallel to the secant joining the endpoints A and B of the curve.



Problem:

Verify Rolle's theorem (If applicable) for the function $f(x) = x^3 + 3x^2 - 24x - 80$ on $[-4,5]$.

Solution:

Given $f(x) = x^3 + 3x^2 - 24x - 80$ on $[-4,5]$

Clearly $f(x)$ is defined for all $x \in [-4,5]$

Since a polynomial function is continuous and differentiable everywhere.

Therefore, (i) $f(x)$ is continuous on $[-4,5]$

(ii) $f(x)$ is differentiable on (-4,5) and $f'(x) = 3x^2 + 6x - 24$

(iii) $f(-4) = 0$, $f(5) = 0$ so $f(-4) = f(5)$,

Since, the conditions of the Rolle's theorem are satisfied

To prove there exist at least one value of $c \in (-4,5)$such that $f'(c) = 0$

For $f'(c) = 0$

$\Rightarrow 3c^2 + 6c - 24 = 0$ $\Rightarrow c^2 + 2c - 8 = 0$

$\Rightarrow c = -4,2$ $\Rightarrow c = 2 \in (-4,5)$

Hence, Rolle's theorem is verified for $f(x) = x^3 + 3x^2 - 24x - 80\ on[-4,5]$

Problem:

Verify Lagrange's mean value theorem for $f(x) = 3x^2 - 5x + 1$ defined in the interval [2,5].

Solution: Given that $f(x) = 3x^2 - 5x + 1$on [2,5]

$f(x)$ is defined for all $x \in$ [2,5]

Since the polynomial function is continuous and differentiable everywhere.

Therefore, (i) $f(x)$ is continuous on [2, 5]

(ii) $f(x)$ is differentiable on (2, 5) and $f'(x) = 6x - 5$

Since, the conditions of Lagrange's mean value theorem are satisfied.

Then, there must exist at least one value $c \in (2,5)$ such that $f'(c) = \frac{f(5)-f(2)}{5-2}$

$\Rightarrow$ $6c - 5 = \frac{51-3}{3}$ $\Rightarrow 6c - 5 = 16$ $\Rightarrow c = \frac{7}{2} \in (2,5)$

Hence, Lagrange's mean value theorem is verified for $f(x) = 3x^2 - 5x + 1 on[2,5]$.