Chapter- 13

Probability

Conditional Probability

Let A and B be any two events, $B \neq \phi$ then P(A/B) denotes the conditional probability of occurrence of event A when B has already occurred.

Example:

Let a bag contain 2 red balls and 3 black balls. One ball is drawn from the bag and this ball is not replaced in the bag. The second ball is drawn from the bag.

Let B denotes the event of the occurrence of a red ball in the first draw and A denotes the event of the occurrence of a black ball in the second draw.

When a red ball has been drawn in the first draw, the number of balls left is 4 and out of these four balls one is a red ball and three are black balls.

:. P(A/B) = Probability of occurrence of a black ball in the second draw when a red ball has been drawn in the first draw = $\frac{3}{4}$

Definition:

Let E and F are two events associated with the sample space of a random experiment the conditional probability of the event E given that F has occurred i.e. P(E/F) is given by $P\left(E/F\right) = \frac{P(E \cap F)}{P(F)}$. provided $P(F) \neq 0$

$$= \frac{no\, of\ elementary\, events\ favorable\, to\ E \cap F}{No\ of\ elementary\, events\, which\, are\ favourable\, to\ F}$$

Properties of conditional probability

Let E and F be events of a sample space S of an experiment then.

1. $P\left(S/F\right) = P\left(F/F\right) = 1$
2. If A and B are any two events of a sample space S of F is an event of S such that $P(F) \neq 0$, then $P\left(A \cup B/F\right) = P\left(A/F\right) + P\left(B/F\right) - P\left(A \cap B/F\right)$.
3. $P\left(E'/F\right) = 1 - P\left(E/F\right)$

Problems

Problem-1

If P(A) = 08, P(B) = 0.5 and P(B/A) = 0.4 then find

(i) $P(A \cap B)$ (ii)P(A/B) (iii) $P(A \cup B)$

Answer:

(i) $P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)} \Rightarrow 0.4=\frac{P(A\cap B)}{0.8} \Rightarrow P(A\cap B)=0.8\times 0.4=0.32$

(ii) $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{0.32}{0.5}=\frac{32}{50}=\frac{16}{25}$

(iii) $P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.8+0.5-0.32=1.30-0.32=0.98$

Problem-2

Determine P(E/F) for following

A coin is tossed three times, where

E: head on third toss F: heads on first two toss.

Answer:

If a coin is tossed three times, then S={ (HHH),(HHT),(HTH),(HTT),(THH),(THT),(TTH),(TTT) }

Let E = Head occurs in third toss. So, E= {(HHH),(HTH),(THH),(TTH) }

And F = Heads in first two tosses. So, F ={(HHH), (HHT)}

Here, $E\cap F$ ={ (HHH) }

$$\therefore P\left(\frac{E}{F}\right)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{8}}{\frac{1}{4}}=\frac{4}{8}=\frac{1}{2}$$

Problem-3

Assume that each born child is equally likely to be a boy or a girl. If a family has two children. What is the conditional probability that both are girls given that?

(i) The youngest is a girl

(ii) At least one is a girl

Answer:

Let first and second girls are denoted by G1, G2, and boys by B1, B2.

Sample space = {(G1,G2), (G1,B2), (G2, B1), (B1, B2) }

Let A = both the children are girls = { (G1,G2) }

B = Youngest girl is a girl = { (G1,G2),(B1,G2) }

C = at least one girl = { (G1,B2),(G1,G2),(B1,G2) }

$A\cap B$ = { (G1,G2) } , $A\cap C$ = { (G1,G2) }

$P(A\cap B)=\frac{1}{4}, \quad P(C)=\frac{3}{4}$

(i) $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{4}}{\frac{2}{4}}=\frac{1}{2}$

(ii) $P\left(\frac{A}{C}\right)=\frac{P(A\cap C)}{P(C)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$

Problem-4

An instructor has a question bank consisting of 300 easy True/False questions. 200 difficult True/False questions. 500 easy multiple choice questions and 400 difficult multiple-choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple-choice question?

Answer:

	Easy	Difficult	Total
True/False	300	200	500
Multiple choice	500	400	900
Total	800	600	1400

Let E = easy questions , D = Difficult questions and T = true/false questions and M = Multiple choice question.

No of multiple-choice questions 500.

The total number of questions 1400.

$P(E \cap M)$ = probability of selecting an easy and multiple-choice question = $\frac{500}{1400}$

Total number of multiple-choice questions = 500 + 400 = 900

P(M) = Probability of selecting one multiple choice question = $\frac{900}{1400}$

$$\therefore\ P\left(\frac{E}{M}\right) = \frac{P(E \cap M)}{P(M)} = \frac{\frac{500}{1400}}{\frac{900}{1400}} = \frac{500}{900} = \frac{5}{9}$$

Multiplication theorem on probability

Let E and F are any two events, then, $P(E \cap F) = P(F).P\left(\frac{E}{F}\right)$ where and P(E/F) denotes the probability of occurrence of event E and F has already occurred.

Proof:

Let S be the sample space. In case of occurrence of event E when F has already occurred, F works as sample space and works as the event, therefore.

$$\therefore\ P\left(\frac{E}{F}\right) = \frac{|E \cap F|}{|F|} = \frac{\frac{|E \cap F|}{|S|}}{\frac{|F|}{|S|}} = \frac{P(E \cap F)}{P(F)}$$

$P(E \cap F) = P(F).P\left(\frac{E}{F}\right)$----------------(1)

Example – 1

An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Answer:

Let E and F denote respectively the event that the first and second ball drawn is black.

We have to find it $P(E \cap F)$ or $P(EF)$.

Now P (E) = P (black ball in first draw) = $\frac{10}{15}$

Also given that the first ball is black, i.e. event E has occurred, now there are 9 black balls and 5 white balls left in the urn. Therefore, the probability that the second ball is drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $P\left(\frac{E}{F}\right) = \frac{9}{14}$

By multiplication rule of probability, we have

$$P(E \cap F) = P(F).P\left(\frac{F}{E}\right) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$$

Example – 2

Three cards are drawn successively without replacement from the pack of 52 well-shuffled cards. What is the probability that the first two cards are kings and the third card drawn is an ace?

Answer:

Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. We have to find P(KKA)

Now, P(K) = $\frac{4}{52}$

Also, P (K/K) is the probability of the second king with the condition that one king has already been drawn. Now, there are three kings in (52 – 1) = 51 cards.

Therefore, P(K/K) = $\frac{3}{51}$

Lastly, P (A/KK) is the probability of a third drawn card to be an ace, with the condition that the two kings have already been drawn. Now there are four aces in the left 50 cards.

Therefore P(A/KK) = $\frac{4}{50}$

By multiplication law of probability, we have

P(KKA) = P(K) P(K/K) P(A/KK) = $\frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$

Independent events

If A and B are any two events, then $P(A\cap B)=P(B).P\left(A/B\right)$

If A and B are independent events, then the probability of occurrence of event A is not affected by the occurrence or non-occurrence of event B, therefore.

P (A/B) = P (A)

Hence $P(A\cap B)=P(A).P(B)$

Where A and B are independent events.

Remember:

1. $P\left(B/A\right)=\dfrac{P(A\cap B)}{P(A)}$

   Or $P(A\cap B)=P(A).P\left(B/A\right)$
2. If A and B are independent events P(B/A) = P(B).
3. Two events A and B are independent if and only if $P(A\cap B)=P(A).P(B)$
4. If A, B, and C are any three independent events then.

   $P(A\cap B\cap C)=P(A\cap(B\cap C))=P(A).P(B\cap C)=P(A).P(B).P(C)$

Note: General Case:

→ $P(A_1\cap A_2\cap A_3\cap....\cap A_n)=P(A_1).P(A_2).P(A_3)....P(A_n)$

Where $A_1,A_2,A_3.......A_n$ are independent events.

→ The events A and ϕ are independent then $P(A\cap\phi)=P(A).P(\phi)$

→ The events A and S are independent then $P(A\cap S)=P(A).P(S)$

→ Mutually exclusive events (none of which is an impossible event) are not the event and non-impossible independent events are not mutually exclusive.

Let A and B be non-impossible independent events, then

$P(A\cap B)=P(A).P(B)$ ——————(1)

Now

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=P(A)+P(B)-P(A)\times P(B)\neq P(A)+P(B)$$

$\because\ P(A)>0,\ P(\mathbf{B})>0,$

$\therefore\ P(A)\times P(B)>0$

Example: 1

Q- If A and B are independent events, then

(i) A and B’ are independent

(ii) A’ and B are independent.

(iii) A’ and B’ are independent

Answer:

Given, A and B are independent events, therefore

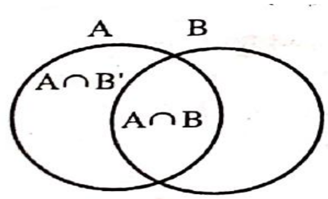
$P(A\cap B)=P(A).P(B)$ .... (1)

(i) Now $A=(A\cap B)\cup(A\cap B')$

$\therefore \quad P(A)=P(A\cap B)+P(A\cap B')$

[$\therefore A\cap B$ *and* $A\cap B'$ are mutually exclusive events]

$=P(A).P(B)+P(A\cap B')$



$\Rightarrow P(A\cap B')=P(A)-P(A).P(B)$

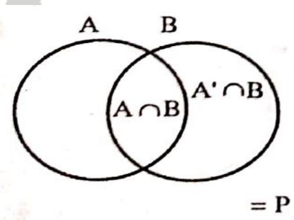
$=P(A)\left[1-P(B)\right]$

$=P(A).P(B') \qquad \left[\therefore 1-P(B)=P(B')\right]$

Hence A and B' are independent events.

(ii) $A'\cap B$ *and* $A\cap B$ are mutually exclusive events and $(A'\cap B)\cup(A\cap B)=B$

$\therefore P\left[(A'\cap B)\cup(A\cap B)\right]=P(B)$



$\Rightarrow P(A'\cap B)+P(A\cap B)=P(B)$

$\Rightarrow P(A'\cap B)=P(B)-P(A\cap B)$

$=P(B)-P(A).P(B)$ [From (1)]

$=P(B)\left[1-P(A)\right]$

$=P(B).P(A')$

$=P(A').P(B)$

Compound and conditional probability

Hence A' and B are independent events

(iii) $P(A'\cap B')=P(A\cup B)'=1-P(A\cup B)$

$=1-\left[P(A)+P(B)-P(A\cap B)\right]$

[By addition theorem of probability]

$=1-P(A)-P(B)+P(A).P(B)$ [From (1)]

$=1-P(A)-P(B),\left[1-P(A)\right]$

$=\left[1-P(A)\right]\left[1-P(B)\right]$

$=P(A')P(B')$

Hence A' and B' are independent event

Example-2

Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6 find.

(i) P (A and B) (ii) P (A or B) (iii) P(A and not B) (iv) P(neither A and B)

Answer:

Let A and B be an independent event

(i) Therefore P(A and B) = $P(A \cap B) = P(A).P(B) = 0.3 \times 0.6 = 0.18$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.18 = 0.72$

(iii) We have $P(A \cap B') = P(A) - P(A \cap B) = 0.3 - 0.18 = 0.12$

(iv) $P(A' \cap B') = P(A') \times P(B') = (1 - 0.3) \times (1 - 0.6) = 0.7 \times 0.4 = 0.28$

Example – 3

Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that.

(i) both balls are red
(ii) the First ball is black and the second is red.
(iii) One of them is black and the other is red.

Answer:

The total number of balls is 18.

The number of red balls is 8 and the number of blackball 10.

(i) Probability of getting the red ball in the first draw = $\frac{8}{18} = \frac{4}{9}$

The ball is replaced after the first draw.

Probability of getting the red ball in the second draw = $\frac{8}{18} = \frac{4}{9}$

Probability of getting both balls red = $\frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$

(ii) Probability of getting first ball black = $\frac{10}{18} = \frac{5}{9}$

The ball is replaced after the first draw probability of getting the second ball is red = $\frac{8}{18} = \frac{4}{9}$

Thus, the probability of getting the first ball as black and the second ball as red = $\frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$

(iii) Probability of getting the red ball in the first draw = $\frac{8}{18} = \frac{4}{9}$

The ball is replaced after the first draw probability of getting the second ball is black

= $\frac{10}{18} = \frac{5}{9}$

Thus, the probability of getting the first ball as black and the second ball as red =

$\frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$

Therefore the probability that one of them is black and the other is red

= probability of getting first ball black and second as red + probability of getting first ball red and second ball black = $\frac{20}{81} + \frac{20}{81} = \frac{40}{81}$

Example – 4

The probability of solving the specific problem independently by A and B is 1/2 and 1/3 respectively. If both try to solve the problem independently, find the probability that.

(i) The problem is solved

(ii) Exactly one of them solves the problem

Answer:

Probability of solving the problem by A = P(A)= $\frac{1}{2}$

Probability of solving the problem by B = P(B)= $\frac{1}{3}$

Since the problem is solved independently by A and B.

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

P(A') = 1 – P(A) = $1 - \frac{1}{2} = \frac{1}{2}$ and P(B') = 1 – P(B) = $1 - \frac{1}{3} = \frac{2}{3}$

(i) Probability that the problem is solved = $P(A \cup B)$

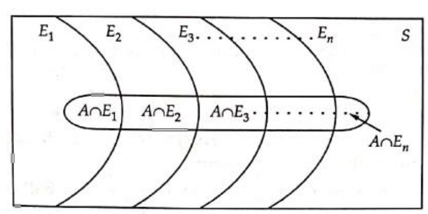
We have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

(ii) Probability of solving Exactly one of the problems is =

$$P(A) \times P(B') + P(B) \times P(A') = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Total Probability

Let S be the sample space and let $E_1, E_2, \ldots, E_n$ is n mutually exclusive and exhaustive events associated with a random experiment. If A is an event that occurs with E_1 or E_2 or------- En, then,



$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + \dots\dots\dots\dots\dots + P(E_n)P\left(\frac{A}{E_n}\right)$$

Proof:- Since $E_1, E_2, \dots, E_n$ is n mutually exclusive and exhaustive events. Therefore,

$S = E_1 \cup E_2 \cup E_3 \dots \cup E_n$, where $E_i \cap E_j = \phi$ *for* $i \neq j$

Clearly, $A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \dots \cup (A \cap E_n)$

$\therefore \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) \dots + P(A \cap E_n)$

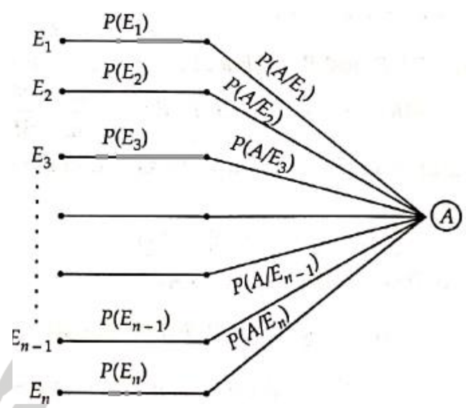
[By addition theorem]

But, $P(A \cap E_i) = P(E_i)P(A/E_i)$ for $i = 1, 2, \dots, n$

Hence,

$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \dots + P(E_n)P(A/E_n)$

The law of total probability as stated and proved above says that if an event A can occur in n mutually exclusive ways, then the probability of occurrence of A is the sum of the probabilities of all mutually exclusive ways as shown in the following tree diagram.



Example – 1

A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.

Solution:-

A red ball can be drawn in two mutually exclusive ways

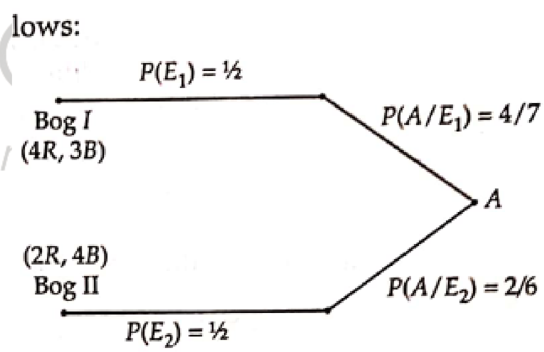
(i) Selecting bag I and then drawing a red ball from it.

(ii) Selecting bag I and then drawing a red ball from it.

Let E_1, E_2, and A denote the events defined as follows

E_1 = Selecting bag I, E_2 = selecting bag II and A = Drawing a red ball

Since one of the two bags is selected randomly.



$\therefore \quad P(E_1) = \frac{1}{2}$ *and* $P(E_2) = \frac{1}{2}$

Now $P(A/E_1)$ = Probability of drawing a red ball when the first bag has been chosen.

$= \frac{4}{7}$ [$\therefore$ The first bag contains 4 red and 3 black balls]

And, $P(A/E_2) =$ Probability of drawing a red ball when the second bag has been selected

$= \frac{2}{6}$ [$\therefore$ Second bag contains 2 red and 4 black balls)

Using the law of total probability, we have

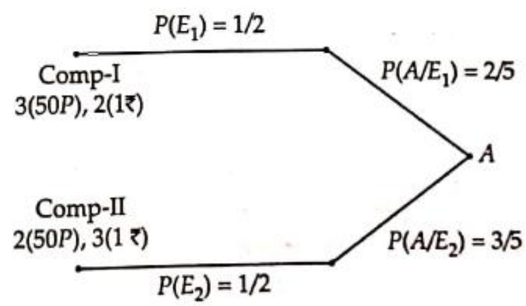
Required probability = $P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$

$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{2}{6} = \frac{19}{42}$

Example – 2

Find the probability of drawing a one-rupee coin from a pure with two compartments one of which contains 3 fifty-paisa coins and 2 one-rupee coins and the other contains 2 fifty paisa coins and 3 one rupee coins.

Solution:-



A one rupee coin can be drawn in two mutually exclusive ways

(i) Selecting compartment I and then drawing a rupee coin from it.

(ii) Selecting compartment II and then drawing a rupee coin from it.

Let E_1, E_2, and A be the events defined as follows:

E_1 = the first compartment of the purse is chosen

E_2 = the second compartment of the pure is chosen

A = a rupee coin is drawn from the pure

Since one of the two compartments is chosen randomly.

$\therefore \ P(E_1) = \frac{1}{2} = P(E_2)$

Also, $P(A/E_1)$ = Probability of drawing a rupee coin given that the first compartment of the purse is chosen

$= \frac{2}{5}$ The first compartment contains 3 fifth paisa coins and 2 one rupee coins and $P(A/E_2)$ = Probability of drawing a rupee coin given that the second compartment of the purse is chosen

$= \frac{3}{5}$ [$\therefore$ The second compartment contains 2 fifth paisa coins and 3 one rupee coins]

By the law of total probability

P (Drawing a one rupee coin) = $P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{1}{2}$

Example- 3

There are two bags, one of which contains 3 black and 4 white, balls, while the other contains 4 black and 3 white balls. A fair dice is cast, if face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turns up a ball is chosen from the second bag. Find the probability of choosing a black ball.

Solution:-

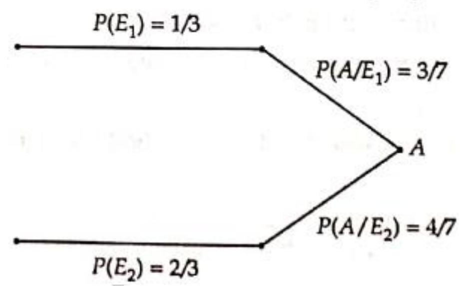
Let E_1, E_2 *and* A be the events defined as follows:

E_1 = The die shows 1 or 3, E_2 = The shows, 2, 4, 5 or 6 and A = the ball drawn is black.

We have, $P(E_1)=\frac{2}{6}=\frac{1}{3}$, $P(E_2)=\frac{4}{6},\frac{2}{3}$

If E_1 occurs, then the first bag is chosen and the probability of drawing a black ball from it is $\frac{3}{7}$

$\therefore \quad P(A/E_1)=\frac{3}{7}$



If E_2 occurs, then the second bag is chosen and the probability of drawing a black ball from it is $\frac{4}{7}$

$\therefore \quad P(A/E_2)=\frac{4}{7}$

Using the law of total probability, we obtain

$$P(A)=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)=\frac{1}{3}\times\frac{3}{7}+\frac{2}{3}\times\frac{4}{7}=\frac{11}{21}$$

Example -4

Two third of the students in a class are boys and the rest are girls. It is known that the probability of a girl getting the first class is 0.25 and that a boy getting a first class is 0.28. Find the probability that a student chosen at random will get a first-class mark in the subject.

Answer:

Let E_1, E_2, and A be the events defined as follows.

E_1 = a boy is chosen from the class,

E_2 = a girl is chosen from the class,

A = The students get the first-class mark.

Then $P(E_1)=\frac{2}{3}$ and $P(E_2)=\frac{1}{3}$, $P\left(\frac{A}{E_1}\right)=0.28$, $P\left(\frac{A}{E_2}\right)=0.25$

Using the law of probability, we obtain

$$P(A) = P(E_1)\times P\left(\frac{A}{E_1}\right)+P(E_2)\times P\left(\frac{A}{E_2}\right)=\frac{2}{3}\times 0.28+\frac{1}{3}\times 0.25=0.27$$

Bayes' Theorem

Partition of a set.

A family of sets $A_1, A_2, \ldots, A_n$ is said to form a partition of a set A if

(i) $A_1, A_2, \ldots, A_n$ are non-empty. (ii) $A_i \cap A_j = \phi$ for $i \neq j$ (iii) $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

Definition of Bayes' Theorem:

If $A_1, A_2, A_3..., A_n$ be n mutually exclusive and exhaustive events and A is an event that occurs together (in conjunction with either A_i i.e. (If $A_1, A_2..., A_n$ from the partition of the sample space S and A be an event then.

$$P\left(\frac{A_k}{A}\right)=\frac{P(A_k)\times P\left(\frac{A}{A_k}\right)}{P(A_1)\times P\left(\frac{A}{A_1}\right)+P(A_2)\times P\left(\frac{A}{A_2}\right)+..............+P(A_n)\times P\left(\frac{A}{A_n}\right)}$$

Proof: -

Since $A_1, A_2, ...A_n$ form, a partition of S. Therefore

(i) $A_1, A_2, ...A_n$ is non-empty sets.

(ii) $S = A_1 \cup A_2 \cup .... \cup A_n$

Now

$$A = A \cap S = A \cap (A_1 \cup A_2 \cup .... \cup A_n)$$
$$= (A \cap A_1) \cup (A \cap A_2) \cup .... \cup (A \cap A_n) \qquad \text{——— (1)}$$

Since $A_1, A_2 ... A_n$ are disjoint sets

\ $A \cap A_1, A \cap A_2, ... A \cap A_n$ are also disjoint.

From (1), by addition theorem

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + ... + P(A \cap A_n) \qquad \text{——— (2)}$$

Now

$$P\left(\frac{A_k}{A}\right)=\frac{P(A_k \cap A)}{P(A)}$$

$$=\frac{P(A_k \cap A)}{P(A_1 \cap A)+P(A_2 \cap A)+----------+P(A_n \cap A)} \qquad \text{From equation (2)}$$

$$\Rightarrow P\left(\frac{A_k}{A}\right)=\frac{P(A_k)\times P\left(\frac{A}{A_k}\right)}{P(A_1)\times P\left(\frac{A}{A_1}\right)+P(A_2)\times P\left(\frac{A}{A_2}\right)+..............+P(A_n)\times P\left(\frac{A}{A_n}\right)}$$

Because $P(A \cap B) = P(B).P\left(\frac{A}{B}\right)$, Hence, Proved.

Problems

Problem-1

The bag I contain 3 Red and 4 Black balls while Bag II contains 5 Red and 6 Black balls. One ball is drawn at random from one of the Bags and it is found to be Red. Find the probability that it was drawn from Bag II.

Answer:

Let A be the event of choosing Bag I, B be the event of choosing Bag II and R be the event of drawing a red ball.

P (A) = P (B) = $\frac{1}{2}$

Also $P\left(\frac{R}{A}\right) =$ P (drawing a red ball from the bag I) = $\frac{3}{7}$

And $P\left(\frac{R}{B}\right) =$ P (drawing a red ball from bag II) = $\frac{5}{11}$

Now, the probability of drawing a ball from bag II, being given that it is red;

By using Bayes' theorem, we have

$$\Rightarrow P\left(\frac{B}{C}\right) = \frac{P(B)\times P\left(\frac{C}{B}\right)}{P(A)\times P\left(\frac{C}{A}\right) + P(B)\times P\left(\frac{C}{B}\right)} = \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

Problem-2

A doctor is to visit a patient. From the experience, it is known that the probabilities that he will come by train, bus, scooter, or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probability that he will be late $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$ if he comes by trains, bus, and scooter respectively, but if he comes by other means of transport, then he will not be late. where he arrives, he is late. What is the probability that he comes by train?

Answer:

Let E be the event that the doctor visits the patient late and T_1, T_2, T_3, T_4 be the event that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then $P(T_1) = \frac{3}{10}$, $P(T_2) = \frac{1}{5}$, $P(T_3) = \frac{1}{10}$, $P(T_4) = \frac{2}{5}$, (given)

$P\left(\frac{E}{T_1}\right)$ = Probability that doctor arriving late comes by train = $\frac{1}{4}$

Similarly, $P\left(\frac{E}{T_2}\right)=\frac{1}{3}$, $P\left(\frac{E}{T_3}\right)=\frac{1}{12}$ and $P\left(\frac{E}{T_4}\right)=0$ since he is not late if he comes by other means of transport.

Therefore, by Bayes' theorem, we have

$P\left(\frac{T_1}{E}\right)=$ the probability that a doctor arriving late comes by train

$$\Rightarrow P\left(\frac{T_1}{E}\right)=\frac{P(T_1)P\left(\frac{E}{T_1}\right)}{P(T_1)P\left(\frac{E}{T_1}\right)+P(T_2)P\left(\frac{E}{T_2}\right)+P(T_3)P\left(\frac{E}{T_3}\right)+P(T_4)P\left(\frac{E}{T_4}\right)}$$

$$\Rightarrow P\left(\frac{T_1}{E}\right)=\frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0}=\frac{4}{40}\times\frac{120}{18}=\frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$

Problem-3

A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Answer:

Let E_1, E_2, and A be the events defined as follows

E_1 = six occurs, E_2 = six does not occur, and A = the man reports that it is a six.

Clearly, $P(E_1)=\frac{1}{6}$, $P(E_2)=\frac{5}{6}$

Now, $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die

And $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die.

= Probability that the man does not speak the truth = $1-\frac{3}{4}=\frac{1}{4}$

We have to find $P(E_1/A)$ i.e, the probability that there is six on the die given that the man has reported that there is six.

By Baye's theorem

$$P(E_1/A)=\frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}=\frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}}=\frac{3}{8}$$

Problem -4

In a test, an examinee either guesses or copies or knows the answer to a multiple-choice question with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$, the probability that his answer is correct, given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question, given that correctly answer it.

Solution: -

Let E_1, E_2, E_3 *and* A be the events defined as follows:

E_1 = Examinee guesses the answer, E_2 = Examine copies the answer, E_3 = Examinee knows the answer, and A = Examinee answers correctly.

Clearly, $P(E_1)=\frac{1}{3}, P(E_2)=\frac{1}{6}$

Since E_1, E_2, E_3 are mutually exclusive and exhaustive events

$\therefore \quad P(E_1)+P(E_2)+P(E_3)=1 \qquad \Rightarrow P(E_3)=1-\left(P(E_1)+P(E_2)\right)=1-\frac{1}{3}-\frac{1}{6}=\frac{1}{2}$

If E_1 has already occurred, the examinee guesses, then there are four choices out of which only one is correct. Therefore, the probability that he answers correctly given that he has guessed i.e $P(A/E_1)=\frac{1}{4}$. It is given that $P(A/E_2)=\frac{1}{8}$ and $P(A/E_3)$ = Probability that he answer correctly given that he knew the answer = 1

By Bayes' theorem

Required probability = $P(E_3/A)$

$$=\frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$=\frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}=\frac{24}{29}$$

Problem- 5

A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from

(i) CALCUTTA (ii) TATANAGAR?

Solution:-

Let E_1 be the event that the letter came from Calcutta and E_2 be the event that the letter came from TATANAGAR. Let A denote the event that two consecutive letters visible on the envelope are TA.

Since the letters have come either from Calcutta or TATANAGAR.

$$\therefore \quad P(E_1)=\frac{1}{2}=P(E_2)$$

If E_1 has occurred, then it means that the letter came from Calcutta. In the word CALCUTTA, there are 8 letters in which TA occurs in the end. Considering TA as one letter there are seven letters out of which one can be in 7 ways.

$$\therefore \quad P(A/E_1)=\frac{1}{7}$$

If E_1 has occurred, then the letter came from TATANAGAR. In the word TATANAGAR, there are 9 letters in which TA occurs twice. Considering one of the two TA's one letter there are 8 letters.

$$\therefore \quad P(A/E_2)=\frac{2}{8}$$

We have to find $P(E_1/A)$ *and* $P(E_2/A)$

(i) $P(E_1/A)=\dfrac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}=\dfrac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}}=\dfrac{4}{11}$

(ii) $P(E_2/A)=\dfrac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}=\dfrac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}}=\dfrac{7}{11}$

Problem- 6

Suppose that the reliability of the HIV test is specified as follows:

Of people having HIV, 90% of the test detects the disease but 10% go undetected. Of people free of HIV, 99% of the test is to judge HIV negative but 1% is diagnosed as HIV positive. From a large population of which only 0.1% has HIV, one person is selected randomly, given the HIV test, and the pathologist reports him/her as HIV positive. What is the probability that the person actually has HIV?

Answer:

Consider the following events

E_1 = The person selected is actually having HIV

E_2 = The person selected is not having HIV

A = The person's HIV test is diagnosed as positive.

We have

$P(E_1)$ = 0.1% = 0.001 , $P(E_2) = 1 - P(E_1)$ =1 – 0.001 = 0.009

$P\left(\frac{A}{E_1}\right)$= probability that person tested as HIV positive given that he/she is actually having positive = $\frac{90}{100} = 0.9$

$P\left(\frac{A}{E_2}\right)$= probability that person tested as HIV positive given that he/she is actually not having positive = $\frac{1}{100} = 0.001$

Required probability

$$= P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} = \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

Random variable and its probability distribution: -

Binomial Probability: -

If an experiment is repeated n times. Under similar conditions, we say that n trials of the experiment have been made.

Binomial Theorem on Probability: -

Statement: -

Let E be an event.

Let P = The probability of occurrence of event E in one trial.

$q = 1 - p$ = The probability of non-occurrence of event E is one trial. $(\because p + q = 1)$

Let X = The Number of successes i.e. number of times event E occurs in n trials.

Then the probability of occurrence of event E exactly r times, in n trials is denoted by $p(x = r)$ or $P(r) = {}^n n_r p^r q^{n-r}$ = (r + 1)th term in the expansion of $(q + p)^n$.

Example: -

When a die is thrown, sample space

$S = \{1,2,3,4,5,6\}$ Let $E = \{5,6\}$

$\therefore$ P = Probability of occurrence of event E when a die is thrown once,

$= \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3} \quad \therefore q = 1 - p = \frac{2}{3}$

Now the probability of occurrence of event E (i.e. probability of occurrence of a number greater than 4) three times when a die is thrown 10 times are given by

$P(X=3) = {}^{10}C_3 p^3 q^{10-3} = {}^{10}C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7.$

Probability Distribution: -

Random Variable: -

A random variable is a real-valued function defined over the sample space of an experiment.

In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment.

A random variable is usually denoted by capital X, Y, Z…….. etc.

Discrete random variables:

A random variable that can take only a finite or an accountably infinite number of values is called a discrete random variable.

The probability distribution of a random variable: -

If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

Example: -

Probability distribution when two coins are tossed;

Let X denote the number of heads that occurred then

P (X = 0) = probability of occurrence of zero head $= P(TT) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

P (X = 1) = probability of occurrence of one head $= P(HT) + P(TH) = \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2} = \frac{1}{2}$

$P(x=2)$ = Probability of occurrence of two heads $= P(HH) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

Thus the probability distribution when two coins are tossed is as given below.

X	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Problems

Problem-1

A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(x)$	O	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Determine

(i) K (ii) $P(x<3)$ (iii) $P(x>6)$ (iv) $P(O<x<3)$

Answer:

(i) It is known that the sum of probabilities of the probability distribution of the random variable is 1.

$\therefore\ 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow 10k^2+9k-1=0 \Rightarrow (10k-1)(k+1)=0 \Rightarrow k=-1,\frac{1}{10}$

But -1 cannot be possible. So $k=\frac{1}{10}$

(ii) $P(X<3)=0+k+2k=0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$

(iii) $P(X>6)=7k^2+k=\frac{7}{100}+\frac{1}{10}=\frac{17}{10}$

(iv) $P(0<X<3)=k+2k=\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$

Problem-2

From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Answer:

It is given that out of 30 bulbs 6 are defective.

Therefore, the number of non-defective bulbs 30 – 6 = 24

Four bulbs are drawn from the lot with replacement.

Let X be the random variable that denotes the number of defective bulbs.

P (X = 0)=P (4 – non defective and 0 defective)= ${}^4C_0\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}=\frac{256}{625}$

P (X = 1)=P (3 – non defective and 1 defective)= ${}^4C_1\times\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}=\frac{256}{625}$

P (X = 2)=P (2 – non defective and 2 defective)= ${}^4C_2\times\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}=\frac{96}{625}$

P (X = 3)=P (1 – non defective and 3 defective)= ${}^4C_3\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}=\frac{16}{625}$

P (X = 0)=P (0 – non defective and 4 defective)= ${}^4C_4\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}=\frac{1}{625}$

Therefore the probability distribution is

X	0	1	2	3	4
P(X)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Problem- 3

The random variable X has a probability distribution P(X) of the following form, where k is some number:

$$P(X=x)=\begin{cases} k & ,if\ x=0 \\ 2k & ,if\ x=1 \\ 3k & ,if\ x=2 \\ 0 & ,otherwise \end{cases}$$

(i) Determine the value of k

(ii) Find $P(X<2), P(X\le 2)\ and\ P(X\ge 2)$

Answer: -

(i) The probability distribution of X is

$$\begin{array}{cccc} X: & 0 & 1 & 2 \\ P(X): & k & 2k & 3k \end{array}$$

The given distribution of probabilities will be a probability distribution, if

$P(X=0)+P(X=1)+P(X=2)=1$

$\Rightarrow k+2k+3k=1$

$\Rightarrow 6k=1 \quad \Rightarrow k=\frac{1}{6}$

(ii) $P(X<2)=P(X=0)+P(X=1)=k+2k=3k=\frac{3}{6}=\frac{1}{2}$

(iii) $P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=k+2k+3k=6k=1$

(iv) $P(X\ge 2)=1-P(X<2)=1-\frac{1}{2}=\frac{1}{2}$

Problem-4

Find the probability distribution of X, the number of heads in two tosses of a coin (or a simultaneous toss of two coins)

Answer: -

When two coins are tossed, there may be 1 head, 2 heads or no head at all. Thus, the possible values of X are 0, 1, 2

Now,

Now,

$P(X=0)=P(Getting\ no\ head)=P(TT)=\frac{1}{4}$

$P(X=1)=P(Getting\ no\ head)=P(HT\ or\ TH)=\frac{2}{4}=\frac{1}{2}$

$P(X=2)=P(Getting\ two\ heads)=P(HH)=\frac{1}{4}$

Thus, the required probability distribution of X is given by

$$\begin{array}{cccc} X: & 0 & 1 & 2 \\ P(X): & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

Problem- 5

Three cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.

Answer: -

Let X denote the number of aces in a sample of 3 cards drawn from a well-shuffled pack of 52 playing cards. Since there are four aces in the pack, therefore in the sample of 3 cards drawn either there can be no ace or there can be one ace or two aces or three aces. Thus, X can take values 0, 1, 2, and 3.

Now, $P(X=0)$ = Probability of getting on ace = Probability of getting 3 other cards =

$$\frac{^{48}C_3}{^{52}C_3}=\frac{4324}{5525}$$

$P(X=1)=$ Probability of getting one ace and two other cards = $\frac{^{4}C_1 \times ^{48}C_2}{^{52}C_3}=\frac{1128}{5525}$

$P(X=2)=$ Probability of getting two aces and one other card = $\frac{^{4}C_2 \times ^{48}C_1}{^{52}C_3}=\frac{72}{5525}$

And, $P(X=3)$ = Probability of getting 3 aces = $\frac{^{4}C_3}{^{52}C_3}=\frac{1}{5525}$ the probability distribution of random variable X is given by

X:	0	1	2	3
P (X) :	$\frac{4324}{5525}$	$\frac{1128}{5525}$	$\frac{72}{5525}$	$\frac{1}{5525}$

Problem-6

An urn contains 4 white and 6 red balls. Four balls are drawn at random from the urn. Find the probability distribution of the number of white balls

Answer:

Let X denote the number of white balls drawn from the urn. Since there are 4 white balls, therefore X can take values 0, 1, 2, 3, 4.

P (X = 0) = Probability of getting no white ball = Probability that 4 balls drawn are red =

$$\frac{^{6}C_4}{^{10}C_4}=\frac{1}{14}$$

P (X = 1) = Probability of getting one white ball = $\frac{^{4}C_1 \times ^{6}C_3}{^{10}C_4}=\frac{8}{21}$

$P(X=2) =$ Probability of getting two white balls = $\frac{{}^4C_2 \times {}^6C_2}{{}^{10}C_4} = \frac{6}{14}$

$P(X=3)$ = Probability of getting three white balls = $\frac{{}^4C_3 \times {}^6C_1}{{}^{10}C_4} = \frac{4}{35}$

$P(X=4) =$ Probability of getting 4 white balls = $\frac{{}^4C_4}{{}^{10}C_4} = \frac{1}{210}$

Thus, the probability distribution of X is given by

X :	0	1	2	3	4
$P(X)$:	$\frac{1}{14}$	$\frac{8}{21}$	$\frac{6}{14}$	$\frac{4}{35}$	$\frac{1}{210}$

Mean of a random variable: -

Let x be a random variable that takes values $x_1, x_2, ..., x_n$ with corresponding probabilities $p_1, p_2 .... p_n$.

Then

Mean = $\frac{\sum_{i=1}^{n} p_i x_i}{\sum P_i} = \sum p_i x_i$ because $\sum P_i = 1$

Note: -

(i) Mean of the Binominal Distribution = $\mu = np$

(ii) Mean of random variable otherwise known as expectation of x. it is denoted by $\sum(X) = \sum p_i x_i$.

Problem-1

Let a pair of dice be thrown and the random variable x be the sum of the numbers that appear on the two dice. Find the mean or expectation of x.

Answer:

The sample space of the experiment consists of 36 elementary events in the form of order pairs (x_i, y_i) where x_i = 1, 2, 3, 4, 5, 6 and y_i = 1, 2, 3, 4, 5, 6

The random variable X i.e. the sum of the numbers on the two dies takes values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12.

Now, $P(X=2) = P(\{(1,1)\}) = \frac{1}{36}$

$P(X=3) = P(\{(1,2),(2,1)\}) = \frac{2}{36}$

$P(X=4) = P(\{(1,3),(2,2),(3,1)\}) = \frac{3}{36}$

$P(X=5)=P(\{(1,4),(2,3),(3,2),(4,1)\})=\frac{4}{36}$

$P(X=6)=P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\})=\frac{5}{36}$

$P(X=7)=P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\})=\frac{6}{36}$

$P(X=8)=P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\})=\frac{5}{36}$

$P(X=9)=P(\{(3,6),(4,5),(5,4),(6,3)\})=\frac{4}{36}$

$P(X=10)=P(\{(4,6),(5,5),(6,4)\})=\frac{3}{36}$

$P(X=11)=P(\{(5,6),(6,5)\})=\frac{2}{36}$

$P(X=12)=P(\{(6,6)\})=\frac{1}{36}$

The probability distribution of X is

X or x_i	2	3	4	5	6	7	8	9	10	11	12
$P(X)$ or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Therefore,

$$\mu=E(X)=\sum_{i=1}^{n}x_ip_i=2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}$$
$$+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}$$
$$=\frac{2+6+12+20+30+42+40+36+30+22+12}{36}=7$$

Problem-2

In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice, but to quit as and when he gets a six. Find the expected value of the amount he wins/loses.

Answer:

Probability of getting six in a throw of die = $\frac{1}{6}$

Probability of not getting six in the throw of die = $1-\frac{1}{6}=\frac{5}{6}$

Now if the man gets a six in the first throw then the probability of getting six = $\frac{1}{6}$

If he does not get a six in the first throw and he gets a six in the second throw.

$\therefore$ Probability of getting six = $\frac{5}{6}\times\frac{1}{6}=\frac{5}{36}$

If he does not get a six in the first two throws and he gets a six in the third throw

$\therefore$ Probability of getting six = $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

If he does not get a six in three throws

$\therefore$ Probability of not getting 6 in any throw = $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$

When he gets a six in the first throw, he will receive Rs. 1.

When he gets a six in the second throw, he will receive $(1-1)=Rs.0$

When he gets a six in the third throw, he will receive, $(-1-1+1)=Rs.(-1)$, i.e. lose Rs. 1

When he does not get a six in any throw he will receive $(-1-1-1)=Rs.(-3)$, i.e., lose Rs. 3

$\therefore$ Expected value $\frac{1}{6}\times1+\frac{5}{36}\times0+\frac{25}{216}\times-1+\frac{125}{216}\times(-3)$

$=\frac{1}{6}-\frac{25}{216}-\frac{375}{216}=\frac{36-25-375}{216}=\frac{-364}{216}=-1.68$

So he will lose Rs. 1.68

Problem-3

Two numbers are selected randomly (without replacement) from the first six positive integers. Let X denotes the larger of two numbers obtained. Find the expectation of X.

Answer:

First six positive integers are 1, 2, 3, 4, 5, 6

If two numbers are selected at random from above six numbers then sample space S is given by S = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}

Hence, n(S) = 30 and X is random variable, which may have values 2, 3, 4, 5 or 6.

Therefore, required probability distribution is given as

$P(X=2)=$ probability of event getting $(1,2),(2,1)=\frac{2}{30}$

$P(X=3)=$probability of event getting $(1,3),(2,3),(3,1),(3,2)=\frac{4}{30}$

$P(X=4)=$ probability of event getting $(1,4),(2,4),(3,4),(4,1),(4,2),(4,3)=\frac{6}{30}$

$P(X=5)=$ probability of event getting $(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4)=\frac{8}{30}$

$P(X=6)=$ probability of event getting

$(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5)=\frac{10}{30}$

If is represented in tabular form as

X	2	3	4	5	6
$P(X)$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{6}{30}$	$\frac{8}{30}$	$\frac{10}{30}$

$\therefore$ Required mean = $E(X)=\sum p_i x_i$

$$=2\times\frac{2}{30}+3\times\frac{4}{30}+4\times\frac{6}{30}+5\times\frac{8}{30}+6\times\frac{10}{30}=\frac{4+12+24+40+60}{30}=\frac{140}{30}=\frac{14}{3}=4\frac{2}{3}$$

Variance and Standard Deviation of random variables: -

Let x be a random variable whose possible value $x_1, x_2, \ldots x_n$ occurs with probabilities respectively.

Then the variance of x

i.e. Var (X) = $\sum_{i=1}^{n}(x_i)^2 p(x_i)-\left(\sum_{i=1}^{n}(x_i)p(x_i)\right)^2$

i.e. Var (X) = $\sum(X)^2-\left(\sum(X)\right)^2$

Note: -

Standard Deviation = $\sqrt{Variance}$

Problem-1

A class has 15 students whose ages 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19, 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. what is the probability distribution of the random variable X? find the mean, variance, and standard derivation of X.

Answer:

Here, X = 14, 15, 16, 17, 18, 19, 20, 21 and

$P(X=14)=\frac{2}{15}, P(X=15)=\frac{1}{15}, P(X=16)=\frac{2}{15}, P(X=17)=\frac{3}{15},$

$P(X=18)=\frac{1}{15}, P(X=19)=\frac{2}{15}, P(X=20)=\frac{3}{15}, P(X=21)=\frac{1}{15}$

Probability distribution

X:	14	15	16	17	18	19	20	21
$P(X)$:	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Now $\sum X\ P(X) = \frac{28+15+32+51+18+38+60+21}{15}$

$\Rightarrow E(X) = \frac{263}{15}$

$\Rightarrow \sum X^2 P(X) = \frac{392+225+512+867+324+722+1200+441}{15}$

$\Rightarrow E(X^2) = \frac{4683}{15}$

Mean $(\mu) = E(X) = \sum X\ P(X) = \frac{263}{15} = 17.53$

$$Var(X) = E(X^2) - [(E(X))]^2 = \frac{4683}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{4683\times 15 - 263\times 263}{225} = \frac{1076}{225} = 4.78$$

$S.D(X) = \sqrt{\text{var}(X)} = \sqrt{4.78} = 2.19$

Problem-2

In a meeting 70% of the members favor and 30% oppose a certain proposal. A member is selected at random and we take x = 0 if he is opposed and x=1 if he is in favor. Find $E(x)$ and $\text{var}(x)$.

Answer:

Here, X = 0, 1

$P(X,0) = \frac{30}{100}, P(X=1) = \frac{70}{100}$

$E(X) = \sum_{i=1}^{n} p_i x_i = 0\times\frac{30}{100} + 1\times\frac{70}{100} = 0.7$

$E(X^2) = \sum X^2 P(X) = 0\times\frac{30}{100} + 1\times\frac{70}{100} = 0.7$

$Var(X) = E(X^2) - [E(X)]^2 = 0.7 - (0.7)^2 = 0.7 - 0.49 = 0.21$

Problem- 3

Three numbers are selected at random (without replacement) from the first six positive integers. If X denotes the smallest of the three numbers obtained, find the probability distribution of X. Also, find the mean and variance of the distribution.

Answer:

The first six positive integers are 1, 2, 3, 4, 5 and 6

If three numbers are selected at random from the above six numbers then the number of elements in sample space S is given by.

i.e. $n(s) = {}^6C_3 = \frac{6!}{3!3!} = \frac{6\times5\times4}{3\times2} = 20$

Here X, the smallest of the three numbers obtained, is a random variable X may have values 1, 2, 3, and 4. Therefore, the required probability distribution is given as

$P(X=1)$ = Probability of event getting 1 as smallest number

$= \frac{{}^5C_2}{20} = \frac{5!}{2!3!\times20} = \frac{5\times4}{2\times20} = \frac{10}{20} = \frac{1}{2}$ [${}^5C_2 \equiv$ selection of two numbers out of 2, 3, 4, 5, 6]

$P(X=2) =$ The probability of events gets 2 as the smallest number.

$= \frac{{}^4C_2}{20} = \frac{4!}{2!2!\times20} = \frac{6}{20} = \frac{3}{10}$ [${}^4C_2 \equiv$ selection of two numbers out of 3, 4, 5, 6]

$P(X=3) =$ Probability of events getting 3 as the smallest number

$= \frac{{}^3C_2}{20} = \frac{3!}{2!1!\times20} = \frac{3}{20}$ [${}^3C_2 \equiv$ selection of two numbers out 4, 5, 6]

$P(X=4) =$ The probability of events gets 4 as the smallest number.

$= \frac{{}^2C_2}{20} = \frac{1}{20}$ [${}^2C_2 \equiv$ selection of two numbers out of 5, 6]

The required probability distribution table is

X or x_i	1	2	3	4
P(X) or p_i	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$

Mean = $E(X)=\sum p_i x_i$

$= \quad 1\times\frac{1}{2}+2\times\frac{3}{10}+3\times\frac{3}{20}+4\times\frac{1}{20}$

$=\frac{1}{2}+\frac{6}{10}+\frac{9}{20}+\frac{4}{20}=\frac{10+12+9+4}{20}=\frac{35}{20}=\frac{7}{4}$

Variance = $\sum x_i^2 p_i-(EX)^2$

$=\left\{1^2\times\frac{1}{2}+2^2\times\frac{3}{10}+3^2\times\frac{3}{20}+4^2\times\frac{1}{20}\right\}-\left(\frac{7}{4}\right)^2$

$=\frac{1}{2}+\frac{12}{10}+\frac{27}{20}+\frac{16}{20}-\frac{49}{16}=\frac{10+24+27+16}{20}-\frac{49}{16}$

$=\frac{27}{20}-\frac{49}{16}=\frac{308-245}{80}=\frac{63}{80}$

Problem – 4

There are 4 cards numbered 1 to 4, with one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

Answer

If two cards, from four cards having numbers 1, 2, 3, 4 each are drawn at random then sample space S is given by

S = (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (4, 1), (4, 2), (4, 3) (3, 1), (3, 2), (3, 4)}

Let X, the sum of the numbers, be a random variable. X may have values 3, 4, 5, 6, 7

Now, $P(X=3)=$ Probability of event getting (1, 2), (2, 1) = $\frac{2}{12}=\frac{1}{6}$

$P(X=4)=$ Probability of event getting (1, 3), (3, 1) = $\frac{2}{12}=\frac{1}{6}$

$P(X=5)=$ Probability of event getting (1, 4), (4, 1), (2, 3), (3, 2) = $\frac{4}{12}=\frac{1}{3}$

$P(X=6)=$ Probability of event getting (4, 2), (2, 4) = $\frac{2}{12}=\frac{1}{6}$

$P(X=7)=$ Probability of event getting (4, 3), (3, 4) = $\frac{2}{12}=\frac{1}{6}$

Thus, the probability distribution is represented in tabular form as

X	3	4	5	6	7
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
X, P(X)	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{3}$	$\frac{6}{6}$	$\frac{7}{6}$

$X^2P(X)$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{3}$	$\frac{36}{6}$	$\frac{49}{6}$

$\therefore$ Mean = $\sum X.P(X) = \frac{3}{6}+\frac{4}{6}+\frac{5}{3}+\frac{6}{6}+\frac{7}{6} = \frac{3+4+10+6+7}{6} = \frac{30}{6} = 5$

Variance $\sum X^2P(X) - \left(\sum X.P(X)\right)^2$

$$=\left(\frac{9}{6}+\frac{16}{6}+\frac{25}{3}+\frac{36}{6}+\frac{49}{6}\right)=(5)^2=\frac{9+16+50+36+49}{6}-25$$

$$=\frac{160}{6}-25=\frac{160-150}{6}=\frac{10}{6}=\frac{5}{3}$$

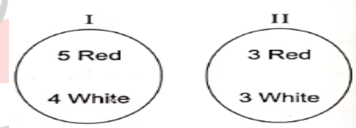
Additional problems

Problem-1

Bag I contains 5 red and 4 white balls and Bag II contains 3 red and 3 white balls. Two balls are transferred from the bag I to bag II and then one ball is drawn from bag II. If the ball drawn from bag II is red, then find the probability that one red and one white ball are transferred from the bag I to bag II.

Answer:

Let E_1, E_2, E_3, and A be an event such that



E_1 = Both transferred balls from the bag I to bag II are red.

E_2 = Both transferred balls from a bag I to bag II are white.

E_3 = Out of two transferred balls one is red and the other is white.

A = Drawing a red ball from bag II

$$P(E_1)=\frac{{}^5C_2}{{}^9C_2}=\frac{5\times4}{9\times8}=\frac{20}{72}=\frac{5}{18}$$

$$P(E_2)=\frac{{}^4C_2}{{}^9C_2}=\frac{4\times3}{9\times8}=\frac{12}{72}=\frac{3}{18}$$

$$P(E_3)=\frac{{}^5C_1\times{}^4C_1}{{}^9C_2}=\frac{5\times4\times2}{9\times8}=\frac{40}{72}=\frac{10}{18}$$

$$P\left(\frac{A}{E_1}\right)=\frac{5}{8},\ P\left(\frac{A}{E_2}\right)=\frac{3}{8};\ P\left(\frac{A}{E_3}\right)=\frac{4}{8}$$

We require $P\left(\frac{E_3}{A}\right)$

Now, by Bayes' theorem

$$P\left(\frac{E_3}{A}\right)=\frac{P(E_3).P\left(\frac{A}{E_3}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)}$$

$$\frac{\frac{10}{18}\times\frac{4}{8}}{\frac{5}{18}\times\frac{5}{8}+\frac{3}{18}\times\frac{3}{8}+\frac{10}{18}\times\frac{4}{8}}=\frac{\frac{40}{144}}{\frac{25}{144}+\frac{9}{144}+\frac{40}{144}}=\frac{40}{144}\times\frac{144}{74}=\frac{20}{37}$$

Problem-2

A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

Answer:

Let E_1, E_2, E_3, E_4 and A be events defined as

E_1 = the lost card is a spade card, E_2 = the lost card is a non spade card, and A = drawing three spade cards from the remaining cards.

Now, $P(E_1)=\frac{13}{52}=\frac{1}{4},\ P(E_2)=\frac{39}{52}=\frac{3}{4}$

$$P\left(\frac{A}{E_1}\right)=\frac{{}^{12}C_3}{{}^{51}C_3}=\frac{220}{20825};\quad P\left(\frac{A}{E_2}\right)=\frac{{}^{13}C_3}{{}^{51}C_3}=\frac{286}{20825}$$

Here, required probability = $P\left(\frac{E_1}{A}\right)$

$$=\frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)}=\frac{\frac{1}{4}\times\frac{220}{20825}}{\frac{1}{4}\times\frac{220}{20825}+\frac{3}{4}\times\frac{286}{20825}}$$

$$=\frac{220}{220+3\times 286}=\frac{220}{1078}=\frac{10}{49}$$

Problem-3

In a hockey match, both teams A and B scored the same no of goals up to the end of the game, so to decide the winner, the referee asked both the captains to through the die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find the respective probabilities of winning the match.

Answer:

Let E_1, E_2 be two events such that

E_1 = The captain of team 'A' gets a six

E_2 = The captain of team 'B' gets a six

Here, $P(E_1)=\frac{1}{6}$, $P(E_2)=\frac{1}{6}$ $P(E_1)'=1-\frac{1}{6}=\frac{5}{6}$, $P(E_2)'=1-\frac{1}{6}=\frac{5}{6}$

Now, P (winning the match by team A) = $\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+...$

$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2.\frac{1}{6}+\left(\frac{5}{6}\right)^4.\frac{1}{6}+...=\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}=\frac{1}{6}\times\frac{36}{11}=\frac{6}{11}$$

P (winning the match by team B) = $1-\frac{6}{11}=\frac{5}{11}$

(Note: If a be the first term and r the common ratio then the sum of infinite terms $S_\infty=\frac{a}{1-r}$]

Problem-4

The random variable X can take only the values 0, 1, 2, 3. Given that $P(X=0)=P(X=1)=p \text{ and } P(X=2)=P(X=3)$ such that $\sum p_i x_i^2=2\sum p_i x_i$, find the value of p

Answer:

Given X is a random variable with values 0, 1, 2, 3. Given probability distributions areas

$X(x_i)$	0	1	2	3
$P(x)(p_i)$	p	p	a	a
$x_i p_i$	0	p	2a	3a
$x_i^2 p_i$	0	p	4a	9a

$\therefore \quad \sum x_i p_i=0+p+2a+3a=P+5a$

$\sum x_i^2 p_i p_i=0+p+4a+9a=p+13a$

According to question

$\sum p_i x_i^2=2\sum p_i x_i$

$p+13a=2p+10a \quad \Rightarrow \quad p=3a$

Also $p+p+a+a=1 \quad \Rightarrow 2p+2a=1$

$2a=1-2p \quad \Rightarrow a=\frac{1-2p}{2}$

$\therefore \quad p=3\times\frac{(1-2p)}{2} \quad \Rightarrow \quad 2p=3-6p$

$\Rightarrow \; 8p=3 \quad \Rightarrow \; p=\frac{3}{8}$

Problem-5

The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits it is $\frac{2}{5}$. If each one of A and B shoots at the target, what is the probability that

(i) the target is hit? (ii) exactly one of them hits the target?

Answer:

Let P(A) = Probability that A hits the target = $\frac{1}{3}$

P(B) = Probability that B hits the target = 2/5

(i) P(target is hit) = P (at least one of a, B hits)

= 1 – P (none hits)

$= 1-\frac{2}{3}\times\frac{3}{5}=\frac{9}{15}=\frac{3}{5}$

(ii) P(exactly one of them hits) = P (A and $\bar{B}$ *or* $\bar{A}$ and B) = $P\left(A\cap\bar{B}\cup\bar{A}\cap B\right)$

$$=P\left(A\right)\times P\left(\bar{B}\right)+P\left(\bar{A}\right)\times P\left(B\right)=\frac{1}{3}\times\frac{3}{5}+\frac{2}{3}\times\frac{2}{5}=\frac{7}{15}$$