

Chapter- 04

Determinants

It is the value (or number) associated with a square matrix. Only square Matrices have their determinants. The matrices which are not square do not have their determinants.

If $A = [a_{ij}]$ is a square matrix of order 'n' then the determinant of A is denoted by $\det A$ or $|A|$ and

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

The determinant of a square matrix of order 1:-

If $A = [a_{11}]$ is a square matrix of order 1, then $|A| = |a_{11}| = a_{11}$

Example:- If $A = [7]$ and $B = [-4]$ find $|A|$ and $|B|$

Here, $|A| = |7| = 7$, $|B| = |-4| = -4$

The determinant of a square matrix of order – 2:-

If $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2 \times 2}$ is a square matrix of order 2, the determinant of a is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example:- Evaluate (i) $\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix}$ (ii) $\begin{vmatrix} -a & b \\ -b & -a \end{vmatrix}$

Solution:- (i) $\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix} = 2(-5) - (-3)(4) = -10 + 12 = 12$

$$(ii) \begin{vmatrix} -a & b \\ -b & -a \end{vmatrix} = a^2 + b^2$$

The determinant of a square matrix of order – 3:-

The determinant of a square matrix of order -3 is the sum of the product of each element a_{ij} in the i th row (or j th column) with $(-1)^{i+j}$ times the determinant of the matrix obtained by leaving the i th row and j th column.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(i) Using elements of the first row

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 6 & -2 & 3 \end{vmatrix} \\ &= 2 \begin{vmatrix} 0 & 5 \\ -2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix} - 1 \begin{vmatrix} 4 & 0 \\ 6 & -2 \end{vmatrix} \\ &= 2(0+10) - 3(12-30) - 1(-8-0) \\ &= 20 + 54 + 8 = 82 \end{aligned}$$

(ii) Using elements of the second column

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 6 & -2 & 3 \end{vmatrix} = -3 \begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} \\ &= -3(12-30) + 0 + 2(10+4) \\ &= 54 + 28 = 82 \end{aligned}$$

Example:- Evaluate $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Solution:- $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

Example:- Solve for x: $\begin{vmatrix} 2x & 5 \\ 1 & 3 \end{vmatrix} = 7$

Solution:- $\begin{vmatrix} 2x & 5 \\ 1 & 3 \end{vmatrix} = 7$

$$\Rightarrow 6x - 5 = 7$$

$$\Rightarrow 6x = 12$$

$$\Rightarrow x = 2$$

Example – 3 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, find x

Solution:- $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$

$$\Rightarrow 2x^2 - 40 = 18 - 40$$

$$\Rightarrow 2x^2 = 18$$

$$\Rightarrow x^2 = 9$$

$$\Rightarrow x = \pm 3$$

Example – 4 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then show that $|3A| = 9|A|$

Solution:- Now $3A = 3 \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 3 & 12 \end{bmatrix}$

$$\Rightarrow |3A| = \begin{vmatrix} 6 & 9 \\ 3 & 12 \end{vmatrix} = 6 \times 12 - 9 \times 3 = 72 - 27 = 45 \dots\dots\dots (1)$$

And $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 2 \times 4 - 1 \times 3 = 8 - 3 = 5$

$$\Rightarrow 9|A| = 9 \times 5 = 45 \dots\dots\dots (2)$$

From equation (1) and (2) we get $|3A| = 9|A|$

Example:- If $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix}$, prove that $|2A| = 8|A|$

Solution:- $2A = 2 \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 2 \\ -2 & 4 & 6 \\ 6 & 2 & -8 \end{bmatrix}$

$$\Rightarrow |2A| = \begin{vmatrix} 4 & -6 & 2 \\ -2 & 4 & 6 \\ 6 & 2 & -8 \end{vmatrix} \quad \text{Expanding by } R_1, \text{ we get}$$

$$= 4 \begin{vmatrix} 4 & 6 \\ 2 & -8 \end{vmatrix} - (-6) \begin{vmatrix} -2 & 6 \\ 6 & -8 \end{vmatrix} + 2 \begin{vmatrix} -2 & 4 \\ 6 & 2 \end{vmatrix}$$

$$= 4(-32 - 12) + 6(16 - 36) + 2(-4 - 24)$$

$$= -176 - 120 - 56 = -352 \dots\dots\dots \text{Changing your Tomorrow} \blacksquare (1)$$

And $|A| = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix}$ Expanding by R_1 we get.

$$= 2 \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 3 & -4 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 2(-8 - 3) + 3(4 - 9) + 1(-1 - 6)$$

$$= -22 - 15 - 7 = -44$$

$$\Rightarrow 8|A| = 8(-44) = -352 \dots\dots\dots (2)$$

From (1) and (2) we get $|2A| = 8|A|$

Home Work:-

NCERT Book – Exercise – 4.1

Properties of Determinants:-

Property – 1

The value of the determinant remains unchanged if its rows and columns are interchanged.

Example:- Verify property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & 7 \end{vmatrix}$

Solution:- Expanding the determinant along with the first row we have

$$\begin{aligned} \Delta &= \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & 7 \end{vmatrix} = 2 \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3) \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5 \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} \\ &= 2(0 - 20) + 3(-42 - 4) + 5(30 - 0) \\ &= -40 - 138 + 150 = -28 \end{aligned}$$

By interchanging rows and columns, we get

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} \quad \text{Expanding by the first column} \\ &= 2 \begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3) \begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5 \begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix} \\ &= 2(0 - 20) + 3(-42 - 4) + 5(30 - 0) \\ &= -40 - 138 + 150 = -28 \end{aligned}$$

Clearly $\Delta = \Delta_1$

Hence, property 1 proved

Changing your Tomorrow

Property – 2

If any two rows (or columns) of a determinant are interchanged, then sign of determinants changes.

Example:- Verify property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Solution:- $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = -28$ (As shown in the previous example)

Interchanging rows R_2 and R_3 e.g $R_2 \leftrightarrow R_3$ we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

Expanding the determinant Δ_1 along the first row, we have.

$$\Delta_1 = 2 \begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3) \begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5 \begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}$$

$$= 2(20 - 0) + 3(4 + 42) + 5(0 - 30)$$

$$= 40 + 138 - 150 = 28$$

Hence property 2 is verified.

Property – 3

If any two rows (or columns) of a determinant are identical (all corresponding elements are the same), then the value of the determinant is zero.

Proof:- If we interchange identical rows (or column) of a determinant Δ , then Δ does not change. However, by property – 2, it follows that Δ has changed its sign.

$$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} \quad R_1 \leftrightarrow R_3$$

$$\therefore \Delta = -\Delta$$

$$\Rightarrow 2\Delta = 0$$

$$\Rightarrow \Delta = 0$$

Example:- Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Solution:- $\Delta = 3(6 - 6) - 2(6 - 9) + 3(4 - 6)$
 $= 0 - 2(-3) + 3(-2)$
 $= 6 - 6 = 0$

Here R_1 and R_3 are identical

Property – 4

If each element of a row (or a column) of a determinant is multiplied by a constant K , then its value gets multiplied by K .

Verification:- Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding by R_1

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \dots \dots \dots (1)$$

Now Δ_1 be the determinant obtained by multiplying the elements of the first row by K .

$$\Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding by the first row

$$\Rightarrow \Delta_1 = K_1 a_1 (b_2 c_3 - b_3 c_2) - Kb_1 (a_2 c_3 - a_3 c_2) + Kc_1 (a_2 b_3 - a_3 b_2)$$

$$\Rightarrow \Delta_1 = K [a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)]$$

$$\Rightarrow \Delta_1 = K\Delta \quad \text{From equation (1)}$$

Hence, $\begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Remember:-

By this property, we can take any common factor from any one row or any one column of a given determinant

If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero.

Example:- $\Delta = \begin{vmatrix} b_1 & b_2 & b_3 \\ Kb_1 & Kb_2 & Kb_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$ ($\because R_1$ and R_2 are proportional)

Example:- Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Solution:- $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

$$\begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

Homework:-

Prove that $\begin{vmatrix} 2 & 7 & 16 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$. Using properties of determinant.

Property – 5

If some or all elements of a row or column of a determinant are expressed as the sum of two (or more) terms, then determinant can be expressed as a sum of two (or more) determinants.

Example:- (1)

$$(i) \begin{vmatrix} a_1 + a_2 & b_1 & c_1 \\ a_3 + a_4 & b_2 & c_2 \\ a_5 + a_6 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_2 & c_2 \\ a_5 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_1 & c_1 \\ a_4 & b_2 & c_2 \\ a_6 & b_3 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_1 + k_1 & a_2 + k_2 & a_3 + k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(iii) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + \lambda_1 & b_2 & b_3 + \lambda_2 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ \lambda_1 & 0 & \lambda_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example – 2 Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Solution:- $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

$= 0 + 0 = 0$

(by property 5 and property 3 and 4)

Example:- Using properties of determinants without expanding prove that $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:- $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$

$= 0 + 0 = 0$ RHS

Property – 6

Changing your Tomorrow

If to each element of any row or column of a determinant, the equimultiples of corresponding elements of other rows (or column) are added, then the value of the determinant remains the same, i.e the value of the determinant remains same if we apply the operation

$R_i \rightarrow R_i + KR_j$ or $C_i \rightarrow C_i + KC_j$

Verification:- Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$

Now Δ_1 can be written as

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} = \Delta + K \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$\Rightarrow \Delta_1 = \Delta + K.0$

$\Rightarrow \Delta_1 = \Delta$

Note:-

(a) We can add or subtract any two-row or any two-column of a determinant then the value of the determinant remains unchanged.

(b) We also can add the entire row or all columns at a time of a determinant remain unchanged.

Example:-
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+a_2 & a_2 & a_3 \\ b_1+b_2 & b_2 & b_3 \\ c_1+c_2 & c_2 & c_3 \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} a_1+a_2+a_3 & a_2 & a_3 \\ b_1+b_2+b_3 & b_2 & b_3 \\ c_1+c_2+c_3 & c_2 & c_3 \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1+b_1+c_1 & a_1+b_1+c_1 & a_1+b_1+c_1 \end{vmatrix} \quad R_3 \rightarrow R_1 + R_2 + R_3$$

Example – 1 Without expanding, prove that $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution:-
$$= \begin{vmatrix} x+y+z & x+y+z & z+x+y \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 \rightarrow R_1 + R_2$$

Taking $x+y+z$ common from R_1

$$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \quad (\because R_1 \text{ and } R_3 \text{ are identical})$$

$$= (x+y+z) \times 0 = 0$$

Problem NCERT Exercise – 4.2:-

Problem – 1, Using properties of the determinant show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:- LHS
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} 0 & a-b & a^2 - b^2 \\ 0 & b-c & b^2 - c^2 \\ 1 & c & c^2 \end{vmatrix} \quad \text{Taking common } (a-b), (b-c) \text{ from } R_1 \text{ and } R_2 \text{ respect}$$

$$= [(a-b)(b-c)] \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(0-0+1(b+c)-1(a+b)) \quad \text{By expanding by } C_1$$

$$= (a-b)(b-c)(b+c-a-b)$$

$$= (a-b)(b-c)(c-a) \quad \text{RHS}$$

Home Work:-

NCERT Exercise 4.2 question number 8 (i)

Problem – 2

Using properties of determinants, show that $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Solution:-

LHS $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} \quad R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$

$$= \begin{vmatrix} x-y & x^2-y^2 & yz-zx \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix} \quad \text{Taking } (x-y)(y-z) \text{ common from } R_1 \text{ and } R_2 \text{ respectively}$$

$$= (x-y)(y-z) \begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix} \quad R_2 \rightarrow R_2 - R_1$$

$$= (x-y)(y-z) \begin{vmatrix} 1 & x+y & -z \\ 0 & z-x & z-x \\ z & z^2 & xy \end{vmatrix} \quad \text{Taking } (z-x) \text{ common from } R_2$$

$$= (x-y)(y-z)(z-x) \begin{vmatrix} 1 & x+y & -z \\ 0 & 1 & 1 \\ z & z^2 & xy \end{vmatrix} \quad \text{Expanding by } C_1$$

$$= (x-y)(y-z)(z-x) \{1(xy-z^2) - 0 + z(x+y+z)\}$$

$$= (x-y)(y-z)(z-x) \{xy-z^2+zx+zy+z^2\}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx) = \text{RHS}$$

Problem – 3 Using properties of Determinants show that $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

Solution:- LHS $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2 + R_3$

$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Taking common $(5x+4)$ from R_1

$= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$

$= (5x+4) \begin{vmatrix} 0 & 0 & 1 \\ x-4 & 4-x & 2x \\ 0 & x-4 & x+4 \end{vmatrix}$

Taking $(4-x)$ common from C_2

$= (5x+4)(4-x) \begin{vmatrix} 0 & 0 & 1 \\ x-4 & 1 & 2x \\ 0 & -1 & x+4 \end{vmatrix}$

Expanding by R_1 we get

$= (5x+4)(4-x) \{0-0+(-1)\}(x-4)-0 \times 1\}$

$= (5x+4)(4-x)(4-x)$

$= (5x+4)(4-x)^2 = \text{RHS}$

Home Work:-

NCERT Exercise – 4.2 Question number 10 (ii) and question number (11)

Problem – 4 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution:- LHS $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$R_1 \rightarrow R_1 - R_2 - R_3$

$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Expanding along R_1 we obtain

$= \Delta = 0 - (-2c) \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b) \begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= 2c(ab + b^2 - bc) - 2b(bc - c^2 - ac)$

$= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc$

$= 4abc = \text{RHS}$

Problem – 5 If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1 + xyz = 0$

Solution:- $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{By property – 5})$$

$$\Rightarrow (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

For 1st determinant using ($c_2 \leftrightarrow c_2$ and then $c_1 \leftrightarrow c_2$)

For 2nd determinant taking x, y, z common from R_1, R_2 and R_3 respectively

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz) = 0 \quad R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$$

Taking out a common factor $(y-x), (z-x)$ common from R_2 and R_3 respectively.

$$\Rightarrow (1 + xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} = 0$$

$$\Rightarrow (1 + xyz)(y-x)(z-x)(z-y) = 0, \text{ expanding by } C_1$$

Since, $\Delta = 0$ and x, y, z are all different i.e $x - y \neq 0, y - z \neq 0, z - x \neq 0$

$$\therefore \Delta = (1 + xyz) = 0 \quad (\text{Proved})$$

Problem – 6 Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + bc + ca + ab$

Solution:- LHS = $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking a, b, c common from $R_1, R_2,$ and R_3 respectively

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad \text{Applying } R_1 \rightarrow R_1 + R_2 + R_3$$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad \text{Taking } \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{ common from}$$

$$= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix} \quad \text{Expanding by } R_1 \text{ we get}$$

$$= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab = \text{RHS}$$

Homework:-

NCERT Exercise 4.2 question number (12)

Problem No. – 7:-

NCERT Exercise – 4.2 question (13)

Prove by using properties of determinants

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Solution:- From LHS, taking b and a common from C₁ and C₂ respectively we get.

$$= b.a \begin{vmatrix} \frac{1+a^2-b^2}{b} & 2b & -2b \\ 2a & \frac{1-a^2+b^2}{a} & 2a \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix} \quad \text{Now multiplying b with } R_1 \text{ and } R_2 \text{ respectively we get}$$

$$= \begin{vmatrix} 1+a^2-b^2 & 2b^2 & -2b^2 \\ 2a^2 & 1-a^2+b^2 & 2a^2 \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix} \quad \text{Applying } C_1 \rightarrow C_1 + C_2, C_2 \rightarrow C_2 + C_3$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b^2 \\ 1+a^2+b^2 & 1+a^2+b^2 & 2a^2 \\ 0 & -(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} \quad \text{Taking common } (1+a^2+b^2) \text{ from both } C_1 \text{ and } C_2 \text{ we get}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b^2 \\ 1 & 1 & 2a^2 \\ 0 & -1 & 1-a^2-b^2 \end{vmatrix} \quad \text{Now expanding by } C_1 \text{ we get}$$

$$= (1+a^2+b^2)^2 \{ 1\{(1-a^2-b^2)+2a^2\} - 1(0-2b^2) + 0 \}$$

$$= (1+a^2+b^2)(1+a^2+b^2) = (1+a^2+b^2+c^2)^3$$

Problem No. – 8:- NCERT Exercise 4.2 question number 14

Prove by using properties of determinant that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Solution:- From LHS taking a, b, c common from R₁, R₂, and R₃ respectively.

$$= abc \begin{vmatrix} \frac{a^2+1}{a} & b & c \\ a & \frac{b^2+1}{b} & c \\ a & b & \frac{c^2+1}{c} \end{vmatrix} \quad \text{Taking a, b, c multiplying in } C_1, C_2 \text{ and } C_3 \text{ respectively}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad \text{Taking } (1+a^2+b^2+c^2) \text{ common from } C_1$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} \quad R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & b^2 & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2) \times 1 = 1+a^2+b^2+c^2 \quad \text{(By expanding by } R_1)$$

Problem – 9:- If a, b, c are positive and unequal, show that the value of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is -ve}$$

Solution:- Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \quad \text{Taking } (a+b+c) \text{ common from } C_1$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \quad R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad \text{Expanding by } C_1 \text{ we get}$$

$$= (a+b+c) [(c-b)(b-c) - (a-c)(a-b)]$$

$$= (a+b+c) (-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= -\frac{1}{2} (a+b+c) (2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= -\frac{1}{2} (a+b+c) [(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Which is -ve $a+b+c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

Problem – 10:- If a, b, c are in AP find value of $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+b & 7y+9 & 10y+c \end{vmatrix}$

Solution:- Give a, b, c are in AP so, $2b = a + c$ (1)

From LHS Applying $R_1 \rightarrow R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 6y+10-6y-10 & 12y+16-12y-16 & 18y+a+c-18y-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{(From equation 1)}$$

$$= 0$$

Problem -11:- Show that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

Solution:- From LHS $R_1 \rightarrow xR_1, R_2 \rightarrow yR_2, R_3 \rightarrow zR_3$ and dividing by xyz we get

$$= \frac{1}{xyz} \begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common x, y, z from C_1, C_2 and C_3 respectively, we get

$$= \frac{xyz}{xyz} \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} \quad C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$$

$$= \begin{vmatrix} (y+z)^2 & x^2 - (y+z)^2 & x^2 - (y+z)^2 \\ y^2 & (x+z)^2 - y^2 & 0 \\ z^2 & 0 & (x+y)^2 - z^2 \end{vmatrix} \quad \text{Taking common factor } (x+y+z) \text{ from } C_2 \text{ and } C_3$$

$$= (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x - (y+z) & x - (y+z) \\ y^2 & (x+z) - y & 0 \\ z^2 & 0 & (x+y) - z \end{vmatrix} \quad \text{Applying } R_1 - (R_2 + R_3)$$

$$= (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x - y + z & 0 \\ z^2 & 0 & x + y - z \end{vmatrix} \quad \text{Applying } C_2 \rightarrow \left(C_2 + \frac{1}{y}C_1\right) \text{ and } C_3 \rightarrow \left(C_3 + \frac{1}{z}C_1\right)$$

$$= (x+y+z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+y & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x+y \end{vmatrix} \quad \text{Expanding by } R_1, \text{ we get}$$

$$= (x+y+z)^2 2yz((x+z)(x+y) - yz) = (x+y+z)^2 2yz(x^2 + xy + xz)$$

$$= (x+y+z)^3 2xyz$$

Aria of a Triangle:- In co-ordinate Geometry, we have studied that area of a Δ whose vertices are

$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by the formula $\frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$. Now we

can express in the form of a determinant as $= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Note:-

- (a) The area is always a positive quantity
 (b) The area of the triangle whose vertices are three collinear points is zero.

Example – 1

Find the area of the triangle formed by the points $(3, 8), (-4, 2)$ and $(5, 1)$.

Solution:-

The area of the triangle formed by

$$\Delta = \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix} = \frac{1}{2} [3(2-1) - 8(-4-5) + 1(-4-10)]$$

$$= \frac{1}{2} [3 + 72 - 14] = \frac{61}{2} \text{ sq. units}$$

Example – 2

Show that the points $A(a, b+c), B(b, c+a)$ and $C(c, a+b)$ are collinear.

Solution:-

To prove three points collinear, we have to prove the area formed by them is zero.

$$= \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} a-b & b-a & 0 \\ b-c & c-b & 0 \\ c & a+b & 1 \end{vmatrix} \quad (\because R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3)$$

Expanding by C_3

$$= \frac{1}{2} \{0 - 0 + 1((a-b)(c-b) - (b-c)(b-a))\}$$

$$= \frac{1}{2} \{0 - 0 + ac - ab - bc + b^2 - b^2 + ab + bc - ca\}$$

$$= \frac{1}{2} \times 0 = 0 \quad \text{Hence, collinear}$$

Example – 3

Find the value of K if the area of the triangle is 4 sq. Units and vertices are $(K, 0), (4, 0), (0, 2)$.

$$\text{Solution:- ATQ} \quad \frac{1}{2} \begin{vmatrix} K & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$$

$$\Rightarrow -0 + 0 - 2(K-4) = 8 \quad (\text{By expanding } C_2)$$

$$\Rightarrow -2K + 8 = 8$$

$$\Rightarrow K = 0$$

Example – 4

Find the equation of the line joining (1, 2) and (3, 6) using determinant

Solution:- Let P(x,y) be any point on the line given A(1, 2), B (3, 6).

As A, P, B points are collinear

$$\frac{1}{2} \begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1(y-6) - 2(x-3) + 1(6x-3y) = 0$$

$$\Rightarrow y - 6 - 2x + 6 + 6x - 3y = 0$$

$$\Rightarrow 4x - 2y = 0$$

$$\Rightarrow 2x - y = 0$$

Which is equal to the line

Home Work:-

NCERT (Exercise – 4.3)

Minors and co-factors:-

Definition of minor:- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i th row and j th column in which element a_{ij} lies.

Minor of an element a_{ij} is denoted by M_{ij}

Example:- Find minor of a_{12} , a_{23} , a_{22} , a_{31} of $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution:-

$$\text{Minor of } a_{12} = M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6$$

$$\text{Minor of } a_{23} = M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$$

$$\text{Minor of } a_{22} = M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 21 = -12$$

$$\text{Minor of } a_{31} = M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$$

Definition of Co-factor:-

Co-factor of an element a_{ij} , denoted by A_{ij} or C_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}

Example:- Find the minor and cofactors of elements of $\Delta = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$

Solution:-

Minor of $a_{11} = M_{11} = 4$

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} M_{11} = 1 \times 4 = 4$

Minor of $a_{12} = M_{12} = 3$

Cofactor of $a_{12} = C_{12} = (-1)^{1+2} M_{12} = (-1) \cdot 3 = -3$

Minor of $a_{21} = M_{21} = -2$

Cofactor of $a_{21} = A_{21} = (-1)^{2+1} M_{21} = (-1) \cdot (-2) = 2$

Minor of $a_{22} = M_{22} = 1$

Cofactor of $a_{22} = A_{22} = (-1)^{2+2} M_{22} = 1 \times 1 = 1$

Note:-

If elements of a row (or column) are multiplied with cofactors of any other, row (column) then their sum is zero.

Example:- $\Delta = a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$

$$= a_{11}(-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad (\text{Because } R_1 \text{ and } R_2 \text{ are identical})$$

Note:- If elements of a row (or column) are multiplied with cofactors of the same row (or column) then their sum is $|A|$.

Example:-

Find the minor and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify it

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0.$$

Answer:-

$$M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20, A_{11} = (-1)^{1+1} = -20$$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46, A_{12} = (-1)^{1+2} (-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30, A_{13} = (-1)^{1+3} \cdot 30 = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4, A_{21} = (-1)^{2+1} (-4) = -4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19, A_{22} = (-1)^{2+2} (-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13, A_{23} = (-1)^{2+3} (13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12, A_{31} = (-1)^{3+1} (-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22, A_{32} = (-1)^{3+2} (-22) = 22$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18, A_{33} = (-1)^{3+3} (18) = 18$$

So, $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2(-12) + (-3)22 + 5(18) = -24 - 66 + 90 = 0$

Homework:- Exercise – 4.4 NCERT Book

Adjoint and Inverse of a Matrix:-

Adjoint of a matrix:- The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij} . Adjoint of a matrix is denoted by $\text{adj}(A)$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Example – 1 Find $\text{adj} A$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Answer:- We have $A_{11} = (-1)^{1+1} 4 = 4, A_{12} = (-1)^{1+2} 1 = -1$

$A_{21} = (-1)^{2+1} \cdot 3 = -3, A_{22} = (-1)^{2+2} 2 = 2$

Hence $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

$\Rightarrow \text{adj}(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Note:- For a square matrix of order 2 given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The adj A can also be obtained by interchanging a_{11} and a_{22} and be changing the signs of a_{12} and a_{21}

i.e $\text{adj}A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Example:- Find the adjoint of the matrix. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Solution:-

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 1(3 - 0) = 3$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -1(2 + 10) = -12$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 1(0 + 6) = 6$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(-1 - 0) = 1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1(1 + 4) = 5$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = (-1)(0 - 2) = 2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = 1(-5 - 6) = -11$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = (-1)(5 - 4) = -1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 1(3 + 2) = 5$$

So, $\text{adj}(A) = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

Theorem – 1

If A be any given square matrix of order n, then

$$A(\text{adj}(A)) = \text{adj}(A). A = |A|.I$$

Where I is the identity matrix of order ‘n’.

Proof:-

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{Now, } A \cdot \text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\begin{aligned} R_1 &\rightarrow \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & 0 & 0 \\ 0 & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ 0 & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix} \\ = R_2 &\rightarrow \begin{bmatrix} a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ 0 & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix} \\ R_3 &\rightarrow \begin{bmatrix} a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} R_1 &\rightarrow \begin{bmatrix} a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & 0 & 0 \\ 0 & a_{21}(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ 0 & a_{31}(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix} \\ = R_2 &\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ R_3 &\rightarrow \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I$$

Similarly, we can prove $\text{adj}(A) \cdot A = |A| \cdot I$

Hence, $A \cdot \text{adj}(A) = |A| \cdot I = \text{adj}(A) \cdot A$

Note:- A square matrix A is said to be singular matrix if $|A| = 0$. If $|A| \neq 0$ then A is said to be a non-singular matrix.

Example:- If $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ Here $|A| = 4 - 4 = 0$

So, A is a singular matrix

$$\text{If } A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Here, $|A| = 12 - 10 = 2 \neq 0$ so, A is a non-singular matrix

Theorem:- If A and B are non-singular matrices of the same order then AB and BA are also the non-singular matrix of the same order.

Example:- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}_{2 \times 2}$

Here $|A| \neq 0$ and $|B| \neq 0$

$$AB = \begin{bmatrix} 1 \times 2 + 2 \times 3 & 1 \times 4 + 2 \times 1 \\ 3 \times 2 + 4 \times 3 & 3 \times 4 + 4 \times 3 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 18 & 16 \end{bmatrix}$$

Here $|AB| \neq 0$ i.e. $8 \times 16 - 18 \times 6 = 128 - 108 = 20 \neq 0$

$$BA = \begin{bmatrix} 2 \times 1 + 3 \times 4 & 2 \times 2 + 4 \times 4 \\ 3 \times 1 + 1 \times 3 & 3 \times 2 + 1 \times 4 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 6 & 10 \end{bmatrix}$$

Here $|BA| \neq 0$ i.e. $14 \times 10 - 20 \times 6$

$$= 140 - 120 = 20 \neq 0$$

Hence, not singular

Theorem:-

The determinant of the product of matrices is equal to the product of their respective determinant is

$|AB| = |A| \cdot |B|$. where A and B are square matrices of the same order.

Example:- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$|A| = 2 - 6 = -4 \text{ and } |B| = 4 - 3 = 1$$

$$\text{So, RHS} = |A| |B| = -4 \times 1 = -4$$

$$\text{Now, } AB = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + 2 \times 3 & 1 \times 1 + 2 \times 2 \\ 3 \times 2 + 2 \times 3 & 3 \times 1 + 2 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ 12 & 7 \end{bmatrix}$$

$$\Rightarrow |AB| = 56 - 60 = -4 = \text{LHS}$$

$$\text{So, } |AB| = |A| |B|$$

Remember:- We know that $(\text{adj}A) \cdot A = |A| \cdot I$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$$

Taking determinant in both sides

$$|\text{adj}(A)| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow |\text{adj}(A)| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow |\text{adj}(A)||A| = |A|^3 \times 1$$

$$\Rightarrow |\text{adj}(A)| = |A|^2$$

In general, if A is a non-singular matrix of order 'n' then $|\text{adj}(A)| = |A|^{n-1}$

Theorem – 4 A square matrix A is invertible if and only if A is a non singular matrix

Proof:- Let A be the invertible matrix of order n and I be the identity matrix of order n.

Hence, there exists a square matrix B of order n such that $AB = BA = I$

Now, $AB = I \Rightarrow |AB| = |I|$

$$\Rightarrow |A| \cdot |B| = 1 \quad (\because |AB| = |A| \cdot |B| \text{ and } |I| = 1)$$

Here, $|A| \neq 0$, hence A is non-singular

Conversely, let A be non-singular

Then $|A| \neq 0$

Now $A(\text{adj}A) = \text{adj}(A) \cdot A = |A|I$

$$\Rightarrow A \left(\frac{1}{|A|} \text{adj}A \right) = \left(\frac{1}{|A|} \text{adj}(A) \right) A = I$$

Or $AB = BA = I$, Where $B = \frac{1}{|A|} \text{adj}(A)$

This A is invertible and $A^{-1} = \frac{1}{|A|} \text{adj}A$

Example:- If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A \text{ adj } A = |A| I$. Also, find A^{-1}

Answer:- We have $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$

Cofactors

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 1(16 - 9) = 7$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1(4-3) = -1$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1(3-4) = -1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -1(12-9) = -3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1(4-3) = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 1(9-12) = -3$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -1(3-3) = 0$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1(4-3) = 1$$

Here $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$

$$\Rightarrow \text{adj}(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $A \cdot (\text{adj}(A)) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix} \text{ Changing your Tomorrow } \blacksquare$$

$$\Rightarrow A \cdot \text{adj}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Question – 1 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:- $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$$\Rightarrow |AB| = 14 - 25 = -11 \neq 0$$

So, $(AB)^{-1}$ exists

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = \frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Now, $|A| = -8 - 3 = -11 \neq 0$

$|B| = 3 - 2 = 1 \neq 0$

So, A^{-1}, B^{-1} both exist and are given by

$$A^{-1} = \frac{-1}{11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Now, $B^{-1}.A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$$= -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

$\therefore (AB)^{-1} = B^{-1}.A^{-1}$

Question -2

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = 0$, where I is 2×2 the identity matrix and O is 2×2 the null matrix. Using this equation, find A^{-1} .

Solution:- We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$\therefore A^2 - 4A + I = 0 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now, $A^2 - 4A + I = 0$

$\Rightarrow A.A - 4A = -I$

Multiplying A^{-1} both sides (post multiplication)

$\Rightarrow A.A(A^{-1}) - 4A.A^{-1} = -IA^{-1}$

$\Rightarrow AI - 4I = -(A^{-1})$

$\Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Question – 3 For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$

Solution:- $A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now $A^2 + aA + bI = 0$

Post multiplying A^{-1} both sides

$$\Rightarrow (A.A)A^{-1} + a.A.A^{-1} + bI.A^{-1} = 0.A^{-1}$$

$$\Rightarrow A.(A.A^{-1}) + aI + bA^{-1} = 0$$

$$\Rightarrow AI + aI + bA^{-1} = 0$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{b}(A + aI) \dots\dots\dots (1)$$

Now $A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

We have from equation (1)

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \left[\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right]$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-(3+a)}{b} & \frac{2}{b} \\ \frac{1}{b} & \frac{1+a}{b} \end{bmatrix}$$

By equality of matrix

$$-\frac{1}{b} = -1 \Rightarrow b = 1 \text{ and } \frac{-3-a}{b} = 1 \Rightarrow -3-a = 1$$

$$\Rightarrow a = -4$$

Question – 4 For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11I = 0$ hence finds A^{-1} .

Solution:- Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$\text{So, } A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\text{Now, } A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -2 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\text{Now, } A^3 - 6A^2 + 5A + 11I = 0$$

$$\Rightarrow A.A.A - 6A.A + 5A + 11I = 0$$

Post multiplying A^{-1} both sides

$$\Rightarrow A.A.A.A^{-1} - 6A.A.A^{-1} + 5A.A^{-1} + 11I.A^{-1} = 0$$

$$\Rightarrow A.A.I - 6A.I + 5I + 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$= \frac{-1}{11}(A^2 - 6A + 5I)$$

$$= -\frac{1}{11} \left[\begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix} - 6 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\Rightarrow A^{-1} = \frac{-1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}$$

Question – 5

Let A be a non-singular square matrix of order 3×3 . Then $|\text{adj}A|$ is equal to

- (a) $|A|$ (b) $|A|^2$ (c) $|A|^3$ (d) $3|A|$

Solution:- Here $n = 3$

We have $|\text{adj}(A)| = |A|^{3-1}$

$$\Rightarrow |\text{adj}(A)| = |A|^2$$

Question – 6

If A is an invertible matrix of order 2 then $\det(A^{-1})$ is equal to

- (a) $\det(A)$ (b) $\frac{1}{\det(A)}$ (c) 1 (d) 0

Solution:- A is an invertible matrix

$$\therefore A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

As matrix A is of order 2

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc \text{ and } \text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj}(A) = \begin{bmatrix} \frac{d}{|A|} & -\frac{b}{|A|} \\ -\frac{c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2} (ad - bc)$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2} \times |A| = \frac{1}{|A|}$$

$$\text{So, } \det(A^{-1}) = \frac{1}{\det(A)}$$

Theorem:- If A and B are non-singular square matrices of the same order, then

$$\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A)$$

Proof:-

Since A and B are non-singular square matrices of the same order. Hence, AB exists

$$\therefore |AB| = |A| \cdot |B| \neq 0 \quad (\because |A| \neq 0, |B| \neq 0)$$

$$\text{We know that } (AB)(\text{adj}(AB)) = |AB| \cdot I_n \dots\dots\dots (1)$$

$$\text{Also } (AB)(\text{adj}B \cdot \text{adj}A) = A(B \cdot \text{adj}B) \cdot \text{adj}A$$

$$= A \cdot (|B| \cdot I_n) \cdot \text{adj}A$$

$$= |B| (A \cdot \text{adj}A)$$

$$= |B| (|A| I_n)$$

$$= |AB| I_n$$

$$\Rightarrow (AB)(\text{adj}B \cdot \text{adj}A) = |AB| I_n \dots\dots\dots (2)$$

From equation (1) and (2) we get

$$(AB)(\text{adj}AB) = (AB) \cdot (\text{adj}B \cdot \text{adj}A)$$

Pre multiplying $(AB)^{-1}$ both sides

$$\Rightarrow (AB)^{-1} ((AB)(\text{adj}AB)) = (AB)^{-1} ((AB)(\text{adj}B \cdot \text{adj}A))$$

$$\Rightarrow I \text{adj}(AB) = I(\text{adj}B \cdot \text{adj}A)$$

$$\Rightarrow \text{adj}(AB) = \text{adj}B \cdot \text{adj}A$$

Theorem:- If A is an invertible square matrix then $\text{adj } A^T = (\text{adj } A)^T$

Proof:- Since A is an invertible matrix $|A| \neq 0 \Rightarrow |A^T| \neq 0$

$\Rightarrow A^T$ is invertible $(\because |A^T| = |A|)$

We have

$A \cdot \text{adj } A = |A| \cdot I_n$

$\Rightarrow [A \cdot \text{adj}(A)]^T = [|A| I_n]^T$

$\Rightarrow (\text{adj } A)^T \cdot A^T = |A| \cdot I_n \dots\dots\dots (1)$

Again $(\text{adj } A^T) \cdot A^T = |A| \cdot I_n$

$\Rightarrow (\text{adj } A^T) \cdot A^T = |A| I_n \dots\dots\dots (2)$

From equation (1) and (2)

$(\text{adj } A^T) \cdot (A^T) = (\text{adj } A)^T (A^T)$

$\Rightarrow \text{adj } A^T = (\text{adj } A)^T$ (By canceling A^T both sides)

Theorem:- Prove that adjoint of a symmetric matrix is also a symmetric matrix.

Proof:- Let A be a symmetric matrix $\Rightarrow A = A^T$

We know $(\text{adj } A)^T = (\text{adj } A^T)$

$\Rightarrow (\text{adj } A)^T = (\text{adj } A)$

\therefore adj A is a symmetric matrix

Theorem:- If A is a non-singular square matrix, then $\text{adj}(\text{adj } A) = |A|^{n-2} A$

Proof:-

We have $B(\text{adj } B) = |B| \cdot I_n$

Replacing B by adj (A)

We have $(\text{adj } A)[\text{adj}(\text{adj}(A))] = |\text{adj } A| \cdot I_n$

$\Rightarrow (\text{adj } A)[\text{adj}(\text{adj } A)] = |A|^{n-1} \cdot I_n$

Premultiplying A both the side

$\Rightarrow A \cdot (\text{adj } A)[\text{adj}(\text{adj } A)] = A \cdot |A|^{n-1} I_n$

$\Rightarrow (A \cdot \text{adj } A)(\text{adj } \text{adj } A) = |A|^{n-1} A I_n$

$$\Rightarrow |A|_n (\text{adj adj } A) = |A|^{n-1} \cdot A$$

$$\Rightarrow |A| (I_n (\text{adj adj } A)) = |A|^{n-1} \cdot A$$

$$\Rightarrow |A| (\text{adj adj } A) = |A|^{n-1} \cdot A$$

$$\Rightarrow \text{adj adj } A = |A|^{n-2} \cdot A$$

Theorem:- If A is a non-singular matrix of order n, then $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

Proof:- We know that $\text{adj}(\text{adj } A) = |A|^{n-2} A$

$$\Rightarrow |\text{adj}(\text{adj } A)| = ||A|^{n-2} A|$$

$$\Rightarrow |\text{adj}(\text{adj } A)| = |A|^{n(n-2)} |A|$$

$$\Rightarrow |\text{adj}(\text{adj } A)| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

Problem:-

01. If a is an invertible matrix of the order 3×3 such that $|A| = 2$, then find $\text{adj}(\text{adj } A)$.

Answer:- $\text{adj}(\text{adj } A) = |A|^{3-2} A = |A| \cdot A = 2A$

02. If A is a square matrix of order 3 such that $|A| = 2$, then write the value of $|\text{adj adj } A|$

Answer:- $|\text{adj adj } A| = |A|^{(n-1)^2} = 2^{(3-1)^2} = 2^4 = 16$

03. If $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, then find, $|\text{adj adj } A|$

Answer:- Try it. (1)

Question – 1 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ then verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:- $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$$\Rightarrow |AB| = 14 - 25 = -11 \neq 0$$

So, $(AB)^{-1}$ exists

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Now, $|A| = -8 - 3 = -11 \neq 0$

$$|B| = 3 - 2 = 1 \neq 0$$

So, A^{-1}, B^{-1} both exist and are given by $A^{-1} = -\frac{1}{11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\begin{aligned} \text{Now, } B^{-1} \cdot A^{-1} &= -\frac{1}{11} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} \\ &= -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore (AB)^{-1} = B^{-1} \cdot A^{-1}$$

Question – 2

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = 0$, where I is 2×2 the identity matrix and O is 2×2 the identity matrix and O is 2×2 the null matrix. Using this equation, find A^{-1} .

Solution:- We have $A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$$\begin{aligned} \therefore A^2 - 4A + I &= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

Now, $A^2 - 4A + I = 0$

$$\Rightarrow A \cdot A - 4A = -I$$

Multiplying A^{-1} both sides (post multiplication)

$$\Rightarrow A \cdot A(A^{-1}) - 4A \cdot A^{-1} = -IA^{-1}$$

$$\Rightarrow AI - 4I = -(A^{-1})$$

$$\Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Question – 3 For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$

Solution:- $A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now $A^2 + aA + bI = 0$

Post multiplying A^{-1} both sides

$$\Rightarrow (A.A)A^{-1} + a.A.A^{-1} + bI.A^{-1} = 0.A^{-1}$$

$$\Rightarrow A.(A.A^{-1}) + aI + bA^{-1} = 0$$

$$\Rightarrow AI + aI + bA^{-1} = 0$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = \frac{-1}{b}(A + aI) \dots \dots \dots (1)$$

$$\text{Now } A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

We have from (1)

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \left[\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right]$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-(3+a)}{b} & \frac{2}{b} \\ \frac{1}{b} & \frac{1+a}{b} \end{bmatrix}$$

By equality of matrix $-\frac{1}{b} = -1 \Rightarrow b = 1$ and $\frac{-3-a}{b} = 1 \Rightarrow -3-a = 1 \Rightarrow a = -4$

Question No. - 4 For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I = 0$ find A^{-1} .

Solution:- Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$\text{So, } A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\text{Now, } A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24+42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$\therefore A^3 - 6A + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -169 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \text{Now, } A^3 - 6A^2 + 5A + 11I = 0$$

$$\Rightarrow A.A.A.A^{-1} - 6A.A.A^{-1} + 5A.A^{-1} + 11I.A^{-1} = 0$$

$$\Rightarrow A.A.I - 6A.I + 5I + 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$= \frac{-1}{11}(A^2 - 6A + 5I)$$

$$= \frac{-1}{11} \left[\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \right] - 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{-1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}$$

Question No. – 05

Let A be a non-singular square matrix of order 3×3 . Then $|\text{adj}A|$ is equal to

- (a) $|A|$ (b) $|A|^2$ (c) $|A|^3$ (d) $3|A|$

Solution:- We have $|\text{adj}(A)| = |A|^{3-1}$

$$\Rightarrow |\text{adj}(A)| = |A|^2$$

Question No. – 06

If A is an invertible matrix of order 2 then $\det(A^{-1})$ is equal to

- (a) $\det(A)$ (b) $\frac{1}{\det(A)}$ (c) 1 (d) 0

Solution:- As a is an invertible matrix

$$\therefore A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

As matrix a is of order 2

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc \text{ and } \text{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj}(A) = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2} (ad - bc)$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|^2} \times |A| = \frac{1}{|A|}$$

$$\text{So, } \det(A^{-1}) = \frac{1}{\det(A)}$$

Theorem:-

If A and B are non-singular square matrices of the same order, then $\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A)$

Proof:- Since A and B are non-singular square matrices of the same order.

Hence, AB exists

$$\therefore |AB| = |A| \cdot |B| \neq 0 \quad (\because |A| \neq 0, |B| \neq 0)$$

We know that $(AB) \cdot (\text{adj}(AB)) = |AB| \cdot I_n \dots\dots\dots (1)$

Also $(AB)(\text{adj}B \cdot \text{adj}A) = A(B \cdot \text{adj}B) \cdot \text{adj}A$

$$= A \cdot (|B| \cdot I_n) \cdot \text{adj}A$$

$$= |B| (A \cdot \text{adj}A)$$

$$= |B| (|A| I_n)$$

$$= |A| |B| I_n$$

$$= |AB| I_n$$

$$\Rightarrow (AB)(\text{adj}B \cdot \text{adj}A) = |AB| I_n \dots\dots\dots (2)$$

From equation (1) and (2) we get

$$(AB)(\text{adj}AB) = (AB) \cdot (\text{adj}B \cdot \text{adj}A)$$

Pre multiplying (AB) both side

$$\Rightarrow (AB)^{-1} ((AB)(\text{adj} AB)) = (AB)^{-1} ((AB)(\text{adj}B \cdot \text{adj}A))$$

$$\Rightarrow \text{adj}(AB) = \text{adj}B \cdot \text{adj}A$$

$$\Rightarrow \text{adj}(AB) = \text{adj}B \cdot \text{adj}A$$

Theorem:- If A is an invertible square matrix then $\text{adj} A^T = (\text{adj} A)^T$

Proof:- Since A is an invertible matrix

$$\therefore |A| \neq 0 \Rightarrow |A^T| = 0$$

$$\Rightarrow A^T \text{ is invertible} \quad (\because |A^T| = |A|)$$

$$\text{We have } A \cdot \text{adj}A = |A| \cdot I_n$$

$$\Rightarrow [A \cdot \text{adj}(A)]^T = [|A| I_n]^T$$

$$\Rightarrow (\text{adj}A)^T \cdot A^T = |A| \cdot I_n \dots\dots\dots (1)$$

$$\text{Again } (\text{adj} \cdot A^T) \cdot A^T = |A^T| \cdot I_n$$

$$\Rightarrow (\text{adj} \cdot A^T) \cdot A^T = |A| \cdot I_n \dots\dots\dots (2)$$

From equation (1) and (2)

$$(\text{adj} A^T) \cdot (A^T) = (\text{adj}A)^T (A)^T$$

$$\Rightarrow \text{adj} A^T = (\text{adj} A)^T \quad \text{By canceling } A^T \text{ both sides}$$

Theorem:- Prove that adjoint of a symmetric matrix is also a symmetric matrix.

Proof:- Let A be a symmetric matrix $\Rightarrow A = A^T$

We know $(\text{adj}A)^T = (\text{adj}A^T)$

$$\Rightarrow (\text{adj}A)^T = (\text{adj}A)$$

$\therefore \text{adj}A$ is a symmetric matrix.

Theorem:- If A is a non-singular square matrix, then $\text{adj}(\text{adj}A) = |A|^{n-2} A$

Proof:- We have $B(\text{adj}B) = |B| I_n$ (for n order b matrix)

Replacing b by $\text{adj}(A)$

We have $(\text{adj}A)[\text{adj}(\text{adj}(A))] = |\text{adj}A| \cdot I_n$

$$\Rightarrow (\text{adj}A)[\text{adj}(\text{adj}A)] = |A|^{n-1} I_n$$

Pre-multiplying A both the side

$$\Rightarrow A \cdot (\text{adj}A)[\text{adj}(\text{adj}A)] = A \cdot |A|^{n-1} \cdot I_n$$

$$\Rightarrow (A \cdot \text{adj}A)(\text{adj}(\text{adj}A)) = |A|^{n-1} A I_n$$

$$\Rightarrow |A| I_n (\text{adj}(\text{adj}A)) = |A|^{n-1} \cdot A$$

$$\Rightarrow |A| (I_n (\text{adj}(\text{adj}A))) = |A|^{n-1} A$$

$$\Rightarrow |A| (\text{adj}(\text{adj}A)) = |A|^{n-1} \cdot A$$

$$\Rightarrow \text{adj}(\text{adj}A) = |A|^{n-2} \cdot A$$

Theorem:-

If A is a non-singular matrix of order n , then $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

Proof:-

We know that $\text{adj}(\text{adj}A) = |A|^{n-2} A$

$$\Rightarrow |\text{adj}(\text{adj}A)| = ||A|^{n-2} A|$$

$$\Rightarrow |\text{adj}(\text{adj}A)| = |A|^{n(n-2)} |A|$$

$$\Rightarrow |\text{adj}(\text{adj}A)| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

Problems:-

01. If A is an invertible matrix of the order 3×3 such that $|A| = 2$, then find $\text{adj}(\text{adj}A)$

$$\text{Answer:- } \text{adj}(\text{adj} A) = |A|^{3-2} A = |A| \cdot A = 2A$$

02. If a is a square matrix of order 3 such that $|A| = 2$, then write the value of $|\text{adjadj}A|$

$$\text{Answer:- } |\text{adjadj}A| = |A|^{(n-1)^2} = 2^{(3-1)^2} = 2^4 = 16$$

03. If $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, then find, $|\text{adj} \text{adj} A|$

Answer:- Try it (1)

Applications of determinants and Matrices:-

We shall discuss the application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent System:-

A system of equations is said to be consistent if its solution (one or more) exists.

Inconsistent system:-

A system of equations is said to be inconsistent if its solution does not exist.

Remark:-

In this chapter, we restrict ourselves to the system of linear equations having unique solutions only i.e only one solution.

The solution of a system of linear equations using the inverse of the matrix:-

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the system of equations can be written as $AX = B$

$$\text{i.e } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case – 1 If A is a non-singular matrix, then the inverse exists. Now

$$AX = B$$

$$\Rightarrow A^{-1}(AX) = A^{-1}B \quad (\text{By pre-multiplying by } A^{-1})$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

This method of solving the system of equations is known as the Matrix method.

Case – 2 If A is a singular matrix, then $|A|=0$ in this case, we calculate $(adjA) \cdot B$

If $(adjA)B \neq 0$ (0 being zero matrices) then the solution does not exist and the system of equations is called inconsistent.

If $(adjA)B = 0$, then the system may be either consistent or inconsistent according to the system have either infinitely many solutions or no solution.

Example – 1 Solve the system of equations $2x + 5y = 1, 3x + 2y = 7$

Solution:- Given $2x + 5y = 1 \quad 3x + 2y = 7$

$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\text{So, } AX = B$$

$$\Rightarrow A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B \dots\dots\dots (1)$$

$$\text{Here } |A| = 4 - 15 = -11 \neq 0$$

So A^{-1} exists

So has a unique solution

$$\text{Now } adj(A) = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{(-11)} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \quad \left(\because A^{-1} = \frac{adj(A)}{|A|} \right)$$

From equation (1) we get

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence, $x = 3, y = -1$

Example:- 2 Solve the system of equations $2x + 3y = 5$ $2x + 3y = 1$

Solution:- Here $|A| = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 6 - 6 = 0$

So, A^{-1} can not be found out \therefore the system of the equation has no solution.

Example:- 3

Solve the following system of equations by matrix method $3x - 2y + 3z = 8$, $2x + y - z = 1$,
 $4x - 3y + 2z = 4$

Solution:- Here $A = \begin{bmatrix} 3 & -2 & 3 \\ 4 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

Here $|A| = 3(2 - 3) + 2(4 + 4) + 3(-6 - 4) = -17 \neq 0$

So, A^{-1} can be found out

Co-factors:-

$A_{11} = -1$ $A_{12} = -8$ $A_{13} = -10$

$A_{21} = -5$ $A_{22} = -6$ $A_{23} = 1$

$A_{31} = -1$ $A_{32} = 9$ $A_{33} = 7$

$adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

So, $adj(A) = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$\therefore A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$ We have $AX = B$

$\Rightarrow A^{-1}.AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \frac{-1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ So, By equality of matrix $x = 1, y = 2$ and $z = 3$

Home Work:- Exercise – 4.6 (NCERT Book) Question number 01 to 14

Question No. – 01

The sum of three numbers is 6 if we multiply the third number by 3 and add the second number to it, we get 11. By adding first and third numbers, we get double the second number, represent it algebraically and find the numbers using the matrix method.

Solution:- Let first, 2nd, and 3rd numbers are x, y, and z respectively.

ATQ $x + y + z = 6$ $y + 3z = 11$ $x + z = 2y$

$\Rightarrow x - 2y + z = 0$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ -1 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

We have $AX = B$ (1)

Now, $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$

Now to find out adj A

Co-factors

$A_{11} = 1(1+6) = 7$ $A_{12} = -(0-3) = 3$ $A_{13} = -1$
 $A_{21} = -3$ $A_{22} = 0$ $A_{23} = 3$
 $A_{31} = 2$ $A_{32} = -3$ $A_{33} = 1$

So, $\text{adj } A = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

From equation (1) $A^{-1}AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ So, $x = 1$, $y = 2$ and $z = 3$

Problem – 2 If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ find A^{-1} . Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11; 3x + 2y - 4z = -5; x + y - 2z = -3$$

Solution:- $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2) = 0 - 6 + 5 = -1 \neq 0$$

Therefore, A is a non-singular matrix so A^{-1} exists

$$C_{11} = 0; C_{12} = 2; C_{13} = 1; C_{21} = -9; C_{23} = -5$$

$$C_{31} = 2; C_{32} = 23; C_{33} = 13$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

The given system can be expressed as $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

Now, $AX = B \Rightarrow X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 68 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

On equating, we get $x = 1, y = 2$ and $z = 3$

Problem – 3

The cost of 4 kg onion, 3 kg wheat, and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat, and 2 kg rice is Rs. 90. The cost of 6 kg onion, 2 kg wheat, and 3 kg rice is Rs. 70. Find the cost of each item kg by matrix method.

Solution:-

Let the cost of per kg = Rs. X

Cost of wheat per kg = Rs. Y

And the cost of rice per kg = Rs. Z

According to question $4x + 3y + 2z = 60$; $2x + 4y + 6z = 90$ and $6x + 2y + 3z = 70$

The given system of equations can be expressed as $AX = B$ where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 0 + 90 - 40 = 50 \neq 0$$

\Rightarrow The system has a unique solution

$$C_{11} = 0, C_{12} = 30, C_{13} = -20, C_{21} = -5, C_{22} = 0, C_{23} = 10, C_{31} = 10, C_{32} = -20, C_{33} = 10$$

$$\text{adj } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} 0 - 450 + 700 \\ 1800 + 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$X = 5, y = 8 \text{ and } z = 8$$

Thus, the cost of 1 kg onion = Rs. 5

Cost of 1 kg Wheat = Rs. 8

Cost of 1 kg Rice = Rs. 8

Problem – 4

If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ find A^{-1} and hence solve the system of linear equations

$$x + 2y + z = 4, -x + y + z = 0, x - 3y + z = 2$$

Solution:- We have $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) + 1(2+3) + 1(2-1) = 10 \neq 0$$

So A is invertible

Let C_{ij} be the co-factors of elements a_{ij} in $A[a_{ij}]$, then

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$

$$\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{10} \begin{bmatrix} -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Now, the given system of equations is expressible as

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Or, $A^T X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Now, $|A^T| = |A| = 10 \neq 0$, so the given system of equations is consistent with a unique solution given

by

$$X = (A^T)^{-1} B = (A^{-1})^T B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}^T \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16+0+2 \\ 8+0-4 \\ 8+0+6 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 2/5 \\ 7/5 \end{bmatrix}$$

$$\Rightarrow x = 9/5, y = 2/5 \text{ and } z = 7/5$$

Hence, $x = 9/5, y = 2/5$ and $z = 7/5$ is the required solution.

Problem – 5 Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of

equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

Solution:- Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$.

Then the given product is $CA = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$\Rightarrow CA = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I_3$$

$$\Rightarrow \frac{1}{8}CA = I_3$$

$$\Rightarrow \left(\frac{1}{8}C\right)A = I_3$$

$$\Rightarrow A^{-1} = \frac{1}{8}C$$

$$\Rightarrow A^{-1} = \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \dots\dots\dots (1)$$

The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

Or $AX = B$, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

The solution of this system of equations is given by

$$X = A^{-1}B$$

$$\Rightarrow X = \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 3, y = -2 \text{ and } z = -1$$

Problem:- 06 Solve the following system of equations $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$; $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$; $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

Solution:- Putting $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$ in the given system of equations we get

$$2a + 3b + 10c = 4 \quad 4a - 6b + 5c = 1 \quad 6a + 9b - 20c = 2$$

The system of the equation can be expressed as

$$AX = B \text{ where } A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120 - 45) - 3(-80 - 30) + 10(36 + 36) = 1200 \neq 0$$

∴ The system has a unique solution

$$C_{11} = 75, C_{12} = 110, C_{13} = 72, C_{21} = 150, C_{22} = -100, C_{23} = 0, C_{31} = 75, C_{32} = 30, C_{33} = -24$$

$$\text{Adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow a = \frac{1}{2} \Rightarrow x = 2$$

$$b = \frac{1}{3} \Rightarrow y = 3$$

$$c = \frac{1}{5} \Rightarrow z = 5$$

Questions:-

01. If $\Delta = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}$ find (i) M_{23} (ii) C_{32}

02. The area of a triangle with vertices $(k, 0), (1, 1),$ and $(0, 3)$ is 5 sq units. Find the value of k

03. Find the area of a triangle, whose vertices are (0,3), (-1, 4), (2, 6)

04. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, find the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$

05. Find the value of p such that the matrix $\begin{bmatrix} -1 & 2 \\ 4 & p \end{bmatrix}$ is singular

06. Write the value of $\begin{vmatrix} \sin 20^\circ & -\cos 20^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix}$

07. If A is a square matrix satisfying $A^T A = I$, write the value of $|A|$

08. If A and B are square matrices of the same order such that $|A| = 3$ and $AB = I$, then write the value of $|B|$.

09. If A is a skew-symmetric matrix of order 3, write the value of $|A|$.

10. If $A = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find the value of $|A| + |B|$

11. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ find $|AB|$

12. Find x, if $\begin{vmatrix} -1 & 2 \\ 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & x \\ x & -4 \end{vmatrix}$

13. If matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find $|A|$

14. Write down the adjoint of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

15. If $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find $(AB)^{-1}$

16. Is the following system of equations consistent? $2x - 3y = 4$ $3x + 2y = 1$

17. Find the number of solutions of the system of equations $3x + 2y = 5$ $6x + 4y = 3$

18. If A is a scalar matrix of the order 3×3 such that $a_{22} = 5$ then find $\det A$.

19. Find the value of x, such that the points (0, 2), (1, x), (3, 1) are collinear

20. For two given square matrices A and B of the same order, such that $|A| = 20$ and $|B| = -20$, find $|AB|$

21. For the adjoint of a matrix $A = \begin{bmatrix} 3 & 1 \\ -5 & 4 \end{bmatrix}$

22. Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ -6 & -18 \end{bmatrix}$, if possible

23. If A is of order 5×5 and $|A| = I$, find $|A|$

24. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $\text{adj}(\text{adj } A)$

25. Write the value of the determinant $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

26. Find the value of the determinant $\begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

27. Find the value of the determinant $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

28. Find the value of $\begin{vmatrix} 3 & 1 & 8 \\ -4 & 2 & 6 \\ -5 & 3 & 24 \end{vmatrix}$

Problems on Determinants:-

Problem – 1 Prove, using properties of determinants $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:- LHS = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ (On applying $R_2 \rightarrow R_2 - R_1$ $R_3 \rightarrow R_3 - R_1$)

$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$ (On taking common $(b-a)$ from R_2 and $(c-a)$ from R_3)

$= (b-a)(c-a)[1\{(c+a)-(b+a)\} - 0 + 0]$ (On expanding along C_1)

$= (b-a)(c-a)(c-b)$

$= (a-b)(b-c)(c-a) = \text{RHS}$

Problem – 2 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution:- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \quad (\text{On applying } C_2 \rightarrow C_2 - C_1 \quad C_3 \rightarrow C_3 - C_1)$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+a^2+ab & c^2+a^2+ac \end{vmatrix} \quad (\text{On taking common } (b-a) \text{ from } C_2 \text{ and } (c-a) \text{ from } C_3)$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & b^2-c^2+ab-ac & c^2+a^2+ac \end{vmatrix} \quad (\text{On applying } C_2 \rightarrow C_2 - C_3)$$

$$= (b-a)(c-a)(b^2-c^2+ab-ac) \begin{vmatrix} 1 & 0 & 0 \\ a & 0 & 1 \\ a^3 & 1 & c^2+a^2+ac \end{vmatrix} \quad (\text{On taking common } (b^2-c^2+ab-ac) \text{ from } C_2)$$

$$= (b-a)(c-a)(b-c)(b+c+a) [1\{0-1\} - 0+0] \quad (\text{On expanding along } R_1)$$

$$= (a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$$

Problem – 3

Prove that $\begin{vmatrix} x=y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 3(x+y+z)^3$

Solution:-

$$\begin{vmatrix} x=y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$= \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix} \quad (\text{On applying } C_1 \rightarrow C_1 + C_2 + C_3)$$

$$= (2x+2y+2z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \quad (\text{On taking common } (2x+2y+2z) \text{ from } C_1)$$

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix} \quad (\text{On applying } R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 - R_1)$$

$$= 2(x+y+z) [1\{(x+y+z)^2 - 0\} - 0+0] \quad (\text{On expanding along } C_1)$$

$$= \text{RHS}$$

Problem – 4 Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solutions:- $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ (On applying $R_1 \rightarrow R_1 + R_2 + R_3$)

$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ (On taking common $(a+b+c)$ from C_1)

$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$ (On applying $C_2 \rightarrow C_2 - C_1$ $C_3 \rightarrow C_3 - C_1$)

$= (a+b+c) [1\{(a+b+c)^2 - 0\} - 0 + 0]$ (On expanding along R_1)

$= (a+b+c)^3 = \text{RHS}$

Problem – 5 Prove, using properties of determinates $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Solution:- LHS = $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$ (On applying $R_1 \rightarrow R_1 - R_2 - R_3$)

$= -2x \begin{vmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$ (On taking common $(-2x)$ from R_1)

$= -2x \begin{vmatrix} 0 & 0 & 1 \\ z & z & x \\ y & -y & x+y \end{vmatrix}$ (On applying $C_2 \rightarrow C_2 - C_3$)

$= -2x [0 - 0 + 1(-yz - yz)]$ (On expanding along R_1)

$= 4xyz = \text{RHS}$

Homework:- NCERT Exercise – 4.2 Question number 10 (ii) and question number 11

Problem – 6 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution:- LHS = $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ $R_1 \rightarrow R_1 - R_2 - R_3$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad (\text{Expanding along } R_1, \text{ we obtain})$$

$$\Delta = 0 - (-2c) \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b) \begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c(ab + b^2 - bc) - 2b(bc - a^2 - ac)$$

$$= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc = 4abc = \text{RHS}$$

Problem – 5 If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then show that $1 + xyz = 0$

Solution:- $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{By property -5})$$

$$\Rightarrow (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

For 1st determinant using $(C_2 \leftrightarrow C_2 \text{ and then } C_1 \leftrightarrow C_2)$

For 2nd determinant taking x, y, z common from R_1, R_2 and R_3 respectively

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz) = 0 \quad R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2 - x^2 \\ 0 & z-x & z^2 - x^2 \end{vmatrix} = 0$$

Taking out a common factor $(y-x), (z-x)$ common from R_2 and R_3 respectively.

$$\Rightarrow (1 + xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} = 0$$

$$\Rightarrow (1 + xyz)(y-x)(z-x)(z-y) = 0, \text{ Expanding by } C_1$$

Since $\Delta = 0$ and x, y, z are all different i.e $x - y \neq 0, y - z \neq 0, z - x \neq 0$

$$\therefore \Delta = (1 + xyz) = 0$$

Problem – 6 Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + bc + ca + ab$

Solution:- LHS $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ (Taking a, b, c common from R₁, R₂ and R₃ respectively)

$= abc \begin{vmatrix} \frac{1}{a}+1 & 1 & 1 \\ 1 & \frac{1}{b}+1 & 1 \\ 1 & 1 & \frac{1}{c}+1 \end{vmatrix}$ (Applying R₁ → R₁ + R₂ + R₃)

$= abc \begin{vmatrix} 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ 1 & \frac{1}{b} + 1 & 1 \\ \frac{1}{c} & 1 & \frac{1}{c} + 1 \end{vmatrix}$ (Taking $\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$ common from)

$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b} + 1 & 1 \\ \frac{1}{c} & 1 & \frac{1}{c} + 1 \end{vmatrix}$ $C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$

$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c} + 1 \end{vmatrix}$ Expanding by R₁ by we get

$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + bc + ca + ab = \text{RHS}$

Home Work:- NCERT Exercise 4.2 question number 12

Problem – 07

Prove by using properties of determinants $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2+b^2)^3$

Solution:- From LHS, taking b and a common from C₁ and C₂ respectively we get

$$= b.a \begin{vmatrix} \frac{1+a^2-b^2}{b} & 2b & -2b \\ 2a & \frac{1-a^2+b^2}{a} & 2-a \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix}$$

Now multiplying b and a with R₁ and R₂ respectively we get

$$= \begin{vmatrix} 1+a^2-b^2 & 2b^2 & -2b^2 \\ 2a^2 & 1-a^2+b^2 & 2a^2 \\ 2 & -2 & 1-a^2-b^2 \end{vmatrix} \quad \text{Applying } C_1 \rightarrow C_1 + C_2, C_2 \rightarrow C_2 + C_3$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b^2 \\ 1+a^2+b^2 & 1+a^2+b^2 & 2a^2 \\ 0 & -(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} \quad \text{Taking common } (1+a^2+b^2) \text{ from both } C_1 \text{ and } C_2 \text{ we get}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b^2 \\ 1 & 1 & 2a^2 \\ 0 & -1 & 1-a^2-b^2 \end{vmatrix}$$

Now expanding by C₁ we get

$$= (1+a^2+b^2)^2 \{ 1[(1-a^2-b^2)+2a^2] - 1(0-2b^2) + 0 \}$$

$$= (1+a^2+b^2)^2 (1+a^2+b^2) = (1+a^2+b^2)^3$$

Problem – 8 Prove by using properties of determinant that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Solution:- From LHS taking a, b, c common from R₁, R₂ and R₃ respectively

$$= abc \begin{vmatrix} \frac{a^2+1}{a} & b & c \\ a & \frac{b^2+1}{b} & c \\ a & b & \frac{c^2+1}{c} \end{vmatrix} \quad \text{Again a, b, c multiplying in } C_1, C_2 \text{ and } C_3 \text{ respectively}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad \text{Taking } (1+a^2+b^2+c^2) \text{ common from } C_1$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} \quad R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & b^2 & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2) \times 1 = 1+a^2+b^2+c^2 \quad (\text{By expanding by } R_1)$$

Problem – 9 If a, b, c are positive and unequal, show that the value of the determinant $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

is -ve

Solution:- Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \quad \text{Taking } (a+b+c) \text{ common from } C_1$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \quad R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad \text{Expanding by } C_1 \text{ we get}$$

$$= (a+b+c) [(c-b)(b-c) - (a-c)(a-b)]$$

$$= (a+b+c) (-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= -\frac{1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Which is -ve (since $a + b + c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$)

Problem – 10 If a, b, c are in AP find value of $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$

Solution:- Given a, b, c are in AP so $2b = a + c$ (1)

From LHS applying $R_1 \rightarrow R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 6y+10-6y-10 & 12y+16-12y-16 & 18y+a+c-18y-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{From (10)}$$

= 0

Problem – 11 Show that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+y)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$

Solution:- From LHS $R_1 \rightarrow xR_1, R_2 \rightarrow yR_2, R_3 \rightarrow zR_3$ and dividing by xyz we get

$$\frac{1}{xyz} \begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common x, y, z from $C_1, C_2,$ and C_3 respectively we get

$$= \frac{xyz}{xyz} \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} \quad C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$$

$$= \begin{vmatrix} (y+z)^2 & x^2 - (y+z)^2 & x^2 - (y+z)^2 \\ y^2 & (x+y)^2 - y^2 & 0 \\ z^2 & 0 & (x+y)^2 - z^2 \end{vmatrix} \quad \text{Taking common factor } (x+y+z) \text{ from } C_2 \text{ and } C_3$$

$$= (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x-(y+z) & x-(y+z) \\ y^2 & (x+z)-y & 0 \\ z^2 & 0 & (x+y)-z \end{vmatrix} \quad \text{Applying } R_1 \rightarrow R_1 - (R_2 + R_3)$$

$$= (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x-y+z & 0 \\ z^2 & 0 & x+y-z \end{vmatrix} \quad \text{Applying } C_2 \rightarrow \left(C_2 + \frac{1}{y}C_1\right) \text{ and } C_3 \rightarrow \left(C_3 + \frac{1}{z}C_1\right)$$

$$= (x+y+z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+y & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x+y \end{vmatrix} \quad \text{Expanding by } R_1 \text{ we get}$$

$$= (x+y+z)^2 2yz((x+z)(x+y) - yz) = (x+y+z)^2 2yz(x^2 + xy + xz) = (x+y+z)^3 2xyz$$

Problem – 12 Use properties of determinants to solve for x: $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$ and $x \neq 0$

Solution:- Let $\Delta = \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}$

$$= \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$= (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}$$

On taking common $(x+a+b+c)$ from C_1

$$= (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & x & a-c \\ 0 & 0 & x \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$= (x+a+b+c) [1(x^2 - 0) - 0 + 0]$$

$$= x^2(x+a+b+c)$$

Given that $\Delta = 0$

$$\Rightarrow x^2(x+a+b+c) = 0$$

$$\Rightarrow x+a+b+c = 0$$

$$\Rightarrow x = -a - b - c$$

Problem – 13 If $x = -4$ is the root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$, then find the other two roots

Solution:- Given $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = \begin{vmatrix} x-3 & 0 & 3-x \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$

On applying $R_1 \rightarrow R_1 - R_3$

$$= (x-3) \begin{vmatrix} 1 & 0 & -1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

On taking common $(x-3)$ from R_1

$$= (x-3) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 2 \\ 3 & 2 & x+3 \end{vmatrix}$$

On applying $C_2 \rightarrow C_3 + C_1$

$$= (x-3) [1\{x(x+3) - 4\} - 0 + 0]$$

$$= (x-3)(x^2 + 3x - 4)$$

$$=(x-3)(x-1)(x+4)$$

Given that $\Delta = 0$

$$\Rightarrow (x-3)(x-1)(x+4) = 0$$

$$\Rightarrow x = 3, 1, -1$$

Hence, the other two required roots are 3 and 1

Problem – 14

Prove the following using properties of determinants:
$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = (x^4-1) \begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

Solution:- We have
$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

$$\begin{vmatrix} a-ax^4 & c-cx^4 & p-px^4 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

On operating $R_1 \rightarrow R_1 - x^4$

$$= \begin{vmatrix} a(1-x^4) & c(1-x^4) & p(1-x^4) \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

$$= (1-x^4) \begin{vmatrix} a & c & p \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

On taking $(1-x^4)$ common from R_1

$$= (1-x^4) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

On performing $R_2 \rightarrow R_2 - x^2 R_1$

$$= -(1-x^4) \begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

$$= (x^4-1) \begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

$[R_1 \Leftrightarrow R_2]$

Problem – 15

Prove the following using properties of determinants:
$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc - a^3 - b^3 - c^3)$$

Solution:- Consider
$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By performing $R_1 \rightarrow R_1 + (R_2 + R_3)$

$$= 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By taking $2(a+b+c)$ common from R_1

$$= 2(a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a-b & a-c & b+c \\ b-c & b-a & c+a \end{vmatrix}$$

By performing $C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3$

$$= 2(a+b+c) [0 - 0 + 1 \{ (a-b)(b-a) - (a-c)(b-c) \}]$$

$$= 2(a+b+c) [-a^2 - b^2 + 2ab - ab + ac + bc - c^2]$$

$$= 2(a+b+c)(ab + bc + ca - a^2 - b^2 - c^2) = 2(3abc - a^3 - b^3 - c^3)$$

Problem – 16 Using properties of determinants, prove that
$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$$

Solution:- Consider
$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

By performing $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$= [a \{ a(7a+3b) - 3a(2a+b) \} - 0 + 0]$$

$$= a \{ 7a^2 + 3ab - 6a^2 - 3ab \} = a \{ a^2 \} = a^3$$

Problem – 17 Find the value of x satisfying the determinant equation
$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Solution:- Using $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -4 & -18 & -48 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(-2) \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 2 & 9 & 24 \end{vmatrix} = 0$$

On operating $C_2 \rightarrow C_2 - 2C_1, C_3 \rightarrow C_3 - 3C_1$

$$\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 5 & 18 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(18-15) - 1(18-6) + 2(5-2) = 0$$

$$\Rightarrow 3(x-2) - 12 + 6 = 0$$

$$\Rightarrow 3(x-2) = 6$$

$$\Rightarrow x-2 = 2$$

$$\Rightarrow x = 4$$

Problem – 18 Using properties of determinants prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha + \delta) \\ \sin\beta & \cos\beta & \cos(\beta + \delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$

Solution:- LHS = $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha + \delta) \\ \sin\beta & \cos\beta & \cos(\beta + \delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma + \delta) \end{vmatrix}$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix} - \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\sin\delta \end{vmatrix}$$

$$= \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix} - \sin\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta & \sin\beta \\ \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

(On taking common $\cos\delta$ from C_3 in the first determinant and $\sin\delta$ from C_3 in the second determinant)

$$\Rightarrow \cos\delta(0) + \sin\delta(0) = 0 = \text{RHS}$$

Problem – 19 Using properties of determinant prove that $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Solution:- LHS apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$\text{LHS} = \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 1 - a^2 & 1 - a & 0 \\ 3 - 2a - a^2 & 2 - 2a & 0 \end{vmatrix}$$

$$= (1 - a) \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 1 + a & 1 & 0 \\ (1 - a)(3 + a) & 2(1 - a) & 0 \end{vmatrix}$$

$$= (1 - a)^2 \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 1 + a & 1 & 0 \\ 3 + a & 2 & 0 \end{vmatrix}$$

$$= (1 - a)^2 \{1(2 + 2a - 3 - a) - 0 + 0\} \quad \text{Expanding in } C_3$$

$$= (1 - a)^2 (a - 1)$$

$$= (a - 1)^2 (a - 1) = (a - 1)^3 = \text{RHS}$$

Problem – 20 Show that the ΔABC is an isosceles triangle if the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

Solution:- We have $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 + \cos A & \cos B - \cos A & \cos C - \cos A \\ \cos^2 A + \cos A & \cos^2 B - \cos^2 A + \cos B - \cos A & \cos^2 C - \cos^2 A + \cos C - \cos A \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 + \cos A & \cos B - \cos A & \cos C - \cos A \\ \cos^2 A + \cos A & (\cos B - \cos A)(\cos B + \cos A + 1) & (\cos C - \cos A)(\cos C + \cos A + 1) \end{vmatrix}$$

$$= (\cos B - \cos A)(\cos C - \cos A) \begin{vmatrix} 1 & 0 & 0 \\ 1 + \cos A & 1 & 1 \\ \cos^2 A + \cos A & \cos B + \cos A + 1 & \cos C + \cos A + 1 \end{vmatrix}$$

On taking common $(\cos B - \cos A)$ from C_2 and $(\cos C - \cos A)$ from C_3

$$= (\cos B - \cos A)(\cos C - \cos A) \begin{vmatrix} 1 & 0 & 0 \\ 1 + \cos A & 0 & 1 \\ \cos^2 A + \cos A & \cos B - \cos C & \cos C + \cos A + 1 \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_3$

$$= (\cos B - \cos A)(\cos C - \cos A)(\cos B - \cos C) \begin{vmatrix} 1 & 0 & 0 \\ 1 + \cos A & 0 & 1 \\ \cos^2 A + \cos A & 1 & \cos C + \cos A + 1 \end{vmatrix}$$

On taking common $(\cos B - \cos C)$ from C_2

$$= (\cos B - \cos A)(\cos C - \cos A)(\cos B - \cos C) [0 - 1 - 0 + 0]$$

$$= (\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A)$$

Suppose that $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$

$$(\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A) = 0$$

$$\cos A = \cos B \quad \text{or} \quad \cos B = \cos C \quad \text{or} \quad \cos C = \cos A$$

$$A = B \quad \text{or} \quad B = C \quad \text{or} \quad C = A$$

Since $\triangle ABC$ is isosceles.

Problem – 21 Prove $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Solution:- $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \quad \text{On applying } C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad \text{On taking common } (a+b+c) \text{ from } C_1 \text{ and } C_2$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \text{On applying } R_3 \rightarrow R_3 - R_2 - R_1$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + \frac{1}{a}C_3, C_2 \rightarrow C_2 + \frac{1}{b}C_3$

$$= (a+b+c)^2 [0+0+2ab\{(b+c)(c+a)-ab\}]$$

$$= (a+b+c)^2 [2abc\{a+b+c\}] = 2abc(a+b+c)^3 = \text{RHS}$$

Multiple Choice Questions:-

01. The value of $\begin{vmatrix} 4^2 & 4^3 & 4^4 \\ 4^3 & 4^4 & 4^5 \\ 4^4 & 4^5 & 4^6 \end{vmatrix}$

- (a) 4^2 (b) 4^9 (c) 4^{12} (d) 0

02. If $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ then the values of x are

- (a) $\frac{2}{3}, \frac{11}{3}$ (b) $\frac{7}{3}, \frac{9}{3}$ (c) $\frac{1}{2}, \frac{11}{4}$ (d) $\frac{2}{5}, \frac{9}{5}$

03. The value of determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ is

- (a) 1 (b) -1 (c) 0 (d) 2

04. The value of $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ is

- (a) 0 (b) 2 (c) 7 (d) -2

05. The value of $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$ is

- (a) 102 (b) 0 (c) 18 (d) 4

Answer :-

1. (d) 02. (a) 03. (c) 04. (a) 05. (b)