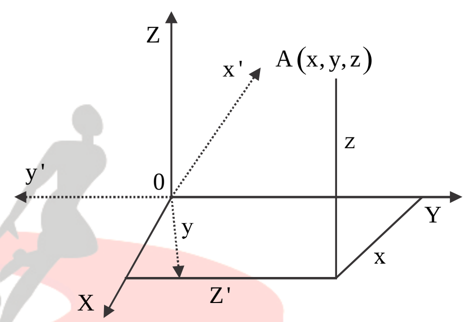
Chapter- 11

Three Dimensional Geometry

Introduction: -

Some basic concepts of three-dimensional geometry that we have discussed in class – XI

Let O be the origin and $X'OY, Y'OY and Z'O\,Z$ be three mutually perpendicular lines in space as shown in the figure. These three lines are called rectangular axes of coordinates named as x-axis, y-axis, and z-axis respectively.



Let A be any point in space such that

x=Perpendicular distance of A from YZ plane

y = Perpendicular distance of A from XZ plane

z = Perpendicular distance of A from XY plane

Then (x, y, z) are called co-ordinate of A we denote as $A(x, y, z)$

Direction Cosines (dcs) and direction rations (drs) of a line

If a line makes angles α, β and γ with $x, y,$ and z-axis respectively, then the angle $\alpha, \beta and \gamma$ is called direction angles of the line and cosines of these angles i.e $l = cos\,\alpha$, $m = cos\,\beta$and $n = cos\,\gamma$ are called direction cosines.

Remark:-

(1) If l, m, n are direction cosines of a line, then $-l, -m, -n$ are also dcs of that line

(2) For any line, there are two sets of dcs.

(3) if l, m, n are dcs of a line that $l^2 + m^2 + n^2 = 1$

Direction Ratio:-

Any three numbers which are parallel to dcs of a line are called direction ratio (drs) of that line

Let l, m, n be dcs of a line AB, then a, b, c are drs of AB

Where $a = \lambda l, b = \lambda m, c = \lambda n$, for some $\lambda \in R, \lambda \neq 0$

Remark:-

(1) Since λ being any real number so for any line, there are infinitely many sets of drs.

(2) By taking $\lambda = 1$, we get $a = l, b = m$, and $c = n$, for any line, a set of dcs is also a set of drs.

(3) The dcs of a line with drs as a, b, c are $l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$

Direction cosines and Ratio of a line through two points:-

Consider a line through two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

Then the line AB or $\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_1 - z_2)\hat{k}$

The drs of the line AB (same as drs of the vector $\overrightarrow{AB}$ can be taken as $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the dcs of the line is.

$$l = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2 + (z_2 - z_1)^2}} = \frac{x_2 - x_1}{AB}$$

$m = \frac{y_2 - y_1}{AB}$, $n = \frac{z_2 - z_1}{AB}$, where AB = distance between A and B

Example:- Find the distance of the point $P(p, q, r)$ from the x –axis.

Solution:- Let Q be the foot of perpendicular drawn from the point $P(p, q, r)$ on the x-axis

Then co-ordinate of Q are $P(p, 0,0)$

Hence the length of the perpendicular is

$$PQ = \sqrt{(p-p)^2 + (q-0)^2 + (r-0)^2} = \sqrt{q^2 + r^2}$$

Example:-

01. What is the dcs of a line, if the drs of the line are $2, -1, -2$?

02. Find the dcs of x, y, and z-axis

03. Find the drs of the line passing through the two points $(-2, 4, -5)$ and $(1, 2, 3)$ also find dcs of the line

04. If a line makes angles $90^0, 135^0, 45^0$ with the positive direction of x, y, and z-axis respectively find its dcs.

05. If the dcs of a line is $\frac{1}{a}, \frac{1}{a}, \frac{1}{a}$, then find values of '$a$'.

06. Find the dcs of a line which makes equal angles with the co-ordinate axis

07. if a line makes, α, β, γ with the positive direction of co-ordinate axes then prove that

(a) $sin^2 \alpha + sin^2 \beta + sin^2 \gamma = 2$ (b) $cos\, 2\, \alpha + cos\, 2\, \beta + cos\, 2\, \gamma = -1$

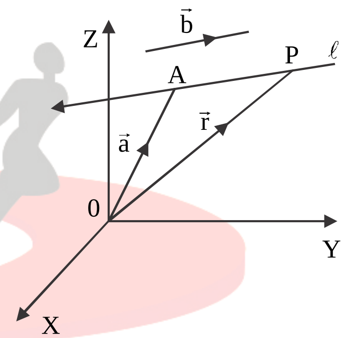
Equation of a line in space:-

Vector equation of a line through a given point where the position vector of the point is $\vec{a}$ and parallel to a given vector $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda\vec{b}; \lambda \in R$$

Here the line ℓ passes through point A whose P.V is $\vec{a}$ and parallel to $\vec{b}$.

Let $\vec{r}$ be the P.V of an arbitrary point P on the line



As $\overrightarrow{AP}$ is parallel to $\vec{b}$

$$\overrightarrow{AP} = \lambda\vec{b}$$

Also $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$\Rightarrow \lambda\vec{b} = \vec{r} - \vec{a}$$

$\Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$.. (b)

Note:-

If $\vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then a, b, c are drs of the line and conversely if a, b, c are drs of a line, then $\vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ will parallel to the line

Cartesian equation of a line:-

Let the line ℓ passes through the point $A(x_1, y_1, z_1)$ and drs of the line be a, b, c

Let $P(x, y, z)$ be any point on line then

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Substituting in equation (1) and equating coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we have

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c$$

$$\Rightarrow \lambda = \frac{x - x_1}{a}, \lambda = \frac{x - x_1}{b}, \lambda = \frac{z - z_1}{c}$$

Thus, the Cartesian equation of the line which passes through (x_1, y_1, z_1) and a, b, c as its drs is

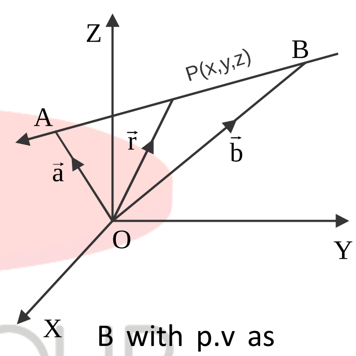
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Example:- Find the vector and cartesian equation of the line passing through $(1, 2, 3)$ and having drs $2, 1, 0$.

Equation of a line passing through the given points:-

Formula:-

The vector equation of the line passing through the given points whose position vector as $\vec{a}$ and $\vec{b}$ is



$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Let $\vec{r}$ be p.v of any point P on line which passes through two points A and B with p.v as $\vec{a}$ and $\vec{b}$.

Here $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OP} = \vec{r}$

$$\overrightarrow{AP} = \vec{r} - \vec{a}, \overrightarrow{AB} = \vec{b} - \vec{a}$$

As $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear

$$\overrightarrow{AP} = \lambda(\overrightarrow{AB})$$

$$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ .... (1)

Formula:- The Cartesian equation of the line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_2}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Here we have $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

Substituting in (1) we can get the result.

Example:- Find the vector and cartesian equation of the line which passes through the points $(-1, 0, 2)$ and $(3, 4, 6)$

Angle Between two lines:-

Let L_1 and L_2 be two lines with drs a_1, b_1, c_1 and a_2, b_2, c_2 respectively and θ be the acute angle between them

Then $cos\,\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$cos\,\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Here $\vec{b}_1 = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}\ \vec{b}_2 = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$ and θ is also the angle between $\vec{b}_1$ and $\vec{b}_2$

Note:-

(1) Two lines L_1 and L_2 are perpendicular if $\vec{b}_1.\vec{b}_2 = 0 \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

(2) Two lines are parallel if $\vec{b} = \lambda\vec{b}\ i.e \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(3) The angle θ between the two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $cos\,\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

(4) The angle θ between the two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is $cos\,\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Example:- Find the angle between the two lines $\vec{r}=\left(2\hat{\imath}-5\hat{\jmath}+\hat{k}\right)+\lambda\left(3\hat{\imath}+2\hat{\jmath}+6\hat{k}\right)$ and $\vec{r}=\left(7\hat{\imath}-6\hat{k}\right)+\mu\left(\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)$ (use both vector and Cartesian form)

Example:- If the equation of a line is $\frac{2x-1}{4}=\frac{3y+5}{2}=\frac{2-z}{3}$, then find drs of line and a point the line.

Example:- If the Cartesian equation of a line is $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$, write vector equation of the line

Example:- Find the value λ so that the lines $\frac{1-x}{3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{6-7}{7}$ are perpendicular to each other.

Example:- Show that the two lines given by

$\vec{r}=(3\hat{\imath}+8\hat{\jmath})+\lambda\left(2\hat{\imath}-\hat{\jmath}+\hat{k}\right)$ and $\vec{r}=\left(\hat{\imath}+\hat{\jmath}-\hat{k}\right)+\mu(-4i+2j-2k)$ are parallel to each other.

The shortest distance between two lines

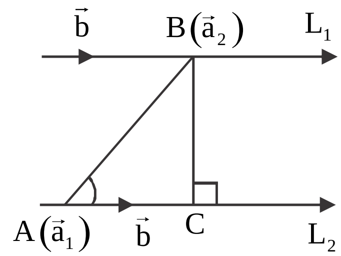
We have to determine the shortest distance between the two lines. Consider two lines L_1 and L_2 whose vector equations are given by $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$

Let $\vec{a}_1$ be the p.v of point A on L_1 and $\vec{a}_2$ be the p.v of point B on L_2. Then the following cases arise

Case – I (L_1 and L_2 are intersecting)

In this case, the distance between L_1 and L_2 is zero

Case – II (L_1 and L_2 are parallel lines)



In this case, L_1 and L_2 are coplanar and $\vec{b}_1 = \vec{b}_2 = \vec{b}$

Let C be the foot of the perpendicular from B on the line L_1 as shown.

Then $|\vec{BC}| = BC$ is the shortest distance between L_1 and L_2

Let $\angle BAC = \theta$ which angle between $\overrightarrow{AB}$ and $\vec{b}$

$$\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$$

$$sin\,\theta = \frac{BC}{AB} \Rightarrow BC = AB\,sin\,\theta$$

Now $\overrightarrow{AB} \times \vec{b} = |\overrightarrow{AB}||\vec{b}|\,sin\,\theta\,\hat{n}$

$$|\overrightarrow{AB} \times \vec{b}| = |\overrightarrow{AB}||\vec{b}|\,sin\,\theta$$

$$\Rightarrow |\overrightarrow{AB} \times \vec{b}| = BC|\vec{b}|$$

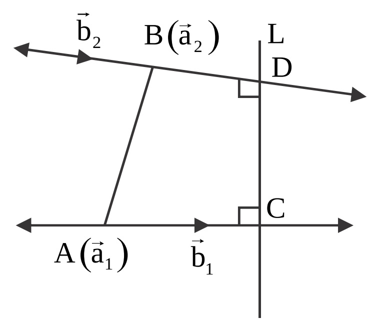
$$\Rightarrow BC = \frac{|\overrightarrow{AB} \times \vec{b}|}{|\vec{b}|} = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

Hence the shortest distance between two lines L_1 and L_2 = $\frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$

Case – III (L_1 and L_2 are skew lines i.e neither intersecting nor parallel)

In this case, L_1 and L_2 are non-coplanar and $\vec{b}_1 \neq \vec{b}_2$

Let the line of shortest distance L intersects the lines L_1 and L_2 in points C and D respectively as shown.



Then $|\overrightarrow{CD}| = CD$ is the shortest distance between L_1 and L_2

$$\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$$

Let $\hat{n}$ be a unit vector along $\overrightarrow{CD}$. Since $\overrightarrow{CD}$ is perpendicular to both $\vec{b}_1$ and $\vec{b}_2$

So $\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|}$

Since CD is the projection of $\overrightarrow{AB}$ on the line L

$$CD = \frac{\overrightarrow{AB}.\hat{n}}{|\hat{n}|} = \overrightarrow{AB}.\hat{n}$$

$$= \overrightarrow{AB}.\left(\frac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right)$$

$$\Rightarrow \frac{\overrightarrow{AB}.\left(\vec{b}_1 \times \vec{b}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|} = \frac{(\vec{a}_2 - \vec{a}_1).\left(\vec{b}_1 \times \vec{b}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}$$

Thus, the shortest distance between L_2 and L_2 = $\left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

Example:- Find the shortest distance between two lines $\vec{r} = \left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right) + \lambda\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right)$ and $\vec{r} = \left(2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}\right) + \mu\left(4\hat{\imath} + 6\hat{\jmath} + 8\hat{k}\right)$

Ans:- $\sqrt{\frac{5}{29}}$units

Example:- Show that the lines $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0}$ and $\frac{x-4}{2} = \frac{y-0}{0} = \frac{z+1}{3}$ intersect each other and find their point of intersection

Ans:- $(4, 0, -1)$

Example:- Find the image of the point $(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Ans:- (1, 0, 7)

Example:- Find the equation of the line passing through $(2, -1, 3)$ and perpendicular to the lines. $\vec{r} = \left(\hat{\imath} + \hat{\jmath} - \hat{k}\right) + \lambda\left(2\hat{\imath} - 2\hat{\jmath} + \hat{k}\right)$ and $\vec{r} = \left(2\hat{\imath} - \hat{\jmath} - 3\hat{k}\right) + \mu\left(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right)$

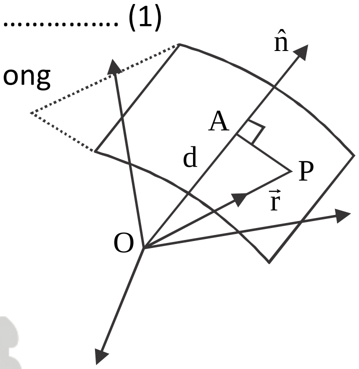
Ans:- $\vec{r} = \left(2\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) + \lambda\left(2\hat{\imath} + \hat{\jmath} - 2\hat{k}\right)$

Plane:-

Definition:- A plane is a surface such that if any two distinct points A and B are taken on it, then the line segment AB lies on the surface.

Equation of Plane:-

Normal Form:- The vector form equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r}.\hat{n} = d$ .... (1)



Here, OA is the normal drawn from O on the plane $\hat{n}$ is the unit vector along

$\overrightarrow{OA} = d\hat{n}, \vec{r} =$P.V of P.

By triangle law

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$\vec{r} = d\hat{n} + \overrightarrow{AP}$$
$$\overrightarrow{AP} = \vec{r} - d\hat{n}$$

Since $\overrightarrow{AP} \perp \hat{n}$

$$\Rightarrow \overrightarrow{AP}.\hat{n} = 0 \Rightarrow (\vec{r} - d\hat{n}).(\hat{n}) = 0$$
$$\Rightarrow \vec{r}.\hat{n} - d\hat{n}.\hat{n} = 0$$
$$\Rightarrow \vec{r}.\hat{n} - d = 0$$
$$\Rightarrow \vec{r}.\hat{n} = d$$

If l, m, n be the dcs of normal to plane drawn from origin then

$$\hat{n} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$$
$$\Rightarrow \vec{r}.\hat{n} = d$$
$$\Rightarrow lx + my + nz = d$$

Then the Cartesian equation of the plane in the normal form which is at a distance of d from the origin and l, m, n as dcs of normal drawn from the origin is $lx + my + nz = d$ .... (2)

Note:-

(a) The co-ordinate of the foot of perpendicular drawn from the origin to the plane is (ld, md, nd)

(b) From equation (1) and (2) It is clear that when $\vec{r}.\left(a\hat{\imath} + b\hat{\jmath} + c\hat{k}\right) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the cartesian equation of the plane where a, b, c are drs of the normal to plane and $a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \vec{N}$ (say) vector normal to the plane.

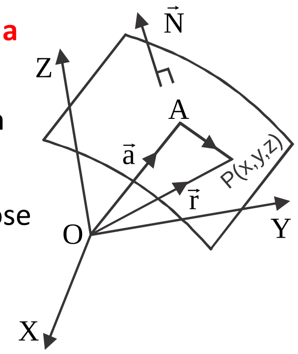
Example:- Find the vector equation of a plane that is at a distance of 18 units from the origin which is normal to vector $2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$.

Ans:- $\vec{r}.\left(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}\right) = 126$

Example:- Find the length of the perpendicular from origin to plane $x - 2y - 2z = 15$. Also, find dcs of the normal to the plane and coordinate of the foot of perpendicular.

Ans:- 5; dcs $\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}; \left(\frac{5}{3}, \frac{-10}{3}, \frac{-10}{3}\right)$

Equation of a plane perpendicular to a given vector and passing through a given point:-



Vector equation of the plane passing through a given point whose position vector $\vec{a}$ and perpendicular to $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$

Let $\vec{r}$ be the p.v. of any point P on a plane and A be a point on the plane whose p.v is $\vec{a}$.

$\overrightarrow{AP}$ lies on the plane so it is perpendicular to $\vec{N}$

$$\overrightarrow{AP}.\vec{N} = 0$$

$$\Rightarrow (\vec{r} - \vec{a}).\vec{N} = 0$$

Cartesian equation of the plane passing through the point $A(x, y, z)$ and a, b, c as drs of normal is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Here $\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}, \vec{N} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ the result can be obtained by putting these values in the vector equation.

Example:- Find the vector and Cartesian equation of the plane which passes through the point $(5,2,-4)$ and perpendicular to a line with drs $2, 3, -1$.

Equation of the plane passing through three given points which are non-collinear:-

Let A, B, C be three points in the plane with position vector $\vec{a}, \vec{b}, \vec{c}$ respectively and P be any arbitrary point with p.v $\vec{r}$. As A, B, C, P lies on the same plane so $\overrightarrow{AP}, \overrightarrow{AB} and \overrightarrow{AC}$ are coplanar then

$$\left[\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AC}\right] = 0$$

$$\Rightarrow \overrightarrow{AP}.\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0$$

$$\Rightarrow (\vec{r} - \vec{a}).\left[\left(\vec{b} - \vec{a}\right) \times (\vec{c} - \vec{a})\right] = 0$$

Then is the vector equation of the plane through three points whose p.v are $\vec{a}, \vec{b}$ and $\vec{c}$

The cartesian equation of the plane which passes through points

$A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

As $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}, \vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}, \vec{c} = x_3\hat{\imath} + y_3\hat{\jmath} + z_3\hat{k}$

Example:- Find the vector equation of the plane passing through $(2,2,-1), (3,4,2), (7,0,6)$

Example:- Prove that equation of the plane whose intercepts on the co-ordinates axes as a, b, c is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This is the intercept form of a plane.

Angle Between two planes:-

The angle between two planes is defined as the angle between their normals. Let θ be the angle between two planes $\vec{r}.\vec{N}_1 = d_1$ and $\vec{r}.\vec{N}_2 = d_2$ then θ will be the angle between $\vec{N}_1$ and $\vec{N}_2$

$$\cos\theta = \left|\frac{\vec{N}_1.\vec{N}_2}{|\vec{N}_1||\vec{N}_2|}\right|$$

If θ be the angle between planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

Here normal has drs a_1, b_1, c_1 & a_2, b_2, c_2 then $\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Note:-

(a) Two planes are perpendicular if $\vec{N}_1.\vec{N}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) Two planes are parallel if $\vec{N}_1 = \lambda\vec{N}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example:- Find the angle between two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$

Example:- Show that the planes $2x + 6y + 6z = 7$ and $\vec{r}.\left(3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}\right) = 8$ are perpendicular to each other.

The angle between a line and a plane:-

The angle between a line and a plane is defined as the complement of the angle between the line and normal to the plane. when θ is the angle between the line and a plane, then $\frac{\pi}{2} - \theta$ is the angle between the line and the normal to the plane.

Then the angle θ between the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and plane $Ax + By + Cz + D = 0$ is

$$\cos\left(\frac{\pi}{2} - \theta\right) = \left|\frac{aA + bB + cC}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}\right|$$

$$\Rightarrow \sin\theta = \left|\frac{aA + bB + cC}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}\right|$$

Example:- Find the angle θ between the line $\frac{x-2}{3}=\frac{y-3}{5}=\frac{z-4}{4}$ and the plane $2x-2y+z-5=0$

Example:- If the line $\frac{x-1}{2}=\frac{y+4}{1}=\frac{z-7}{2}$ is parallel to the plane $3x-2y+cz=14$ then find the value of c.

Ans:- C = -2

Example:- Find the co-ordinate of the point where the line joining points (1, -2, 3) and (2, -1, 5) cuts the plane $x-2y+3z=19$

Ans:- (2, -1, 5)

Co-planarity of two lines:-

Let the given lines be $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$. The 1st line passes through point A whose p.v is $\vec{a}_1$ and parallel to $\vec{b}_1$ and the 2nd line passes through point B whose p.v is $\vec{a}_2$ and parallel to $\vec{b}_2$.

As the two lines are coplanar then $\overrightarrow{AB}=\vec{a}_2-\vec{a}_1$ is perpendicular to $\vec{b}_1\times\vec{b}_2$.

i.e $\overrightarrow{AB}.(\vec{b}_1\times\vec{b}_2)=0$

Or $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=0$

And the equation of the plane containing them is $(\vec{r}-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=0$

In Cartesian form for two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

Here, $\vec{a}_1=x_1\hat{\imath}+y_2\hat{\jmath}+z_1\hat{k},\vec{b}_1=a_1\hat{\imath}+b_1\hat{\jmath}+c_1\hat{k}$

$$\vec{a}_2=x_2\hat{\imath}+y_2\hat{\jmath}+z_2\hat{k},\vec{b}_2=a_2\hat{\imath}+b_2\hat{\jmath}+c_c\hat{k}$$

$$(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=0$$

$$\Rightarrow[\vec{a}_2-\vec{a}_1\quad\vec{b}_1\quad\vec{b}_2]=0$$

$$\Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Which is the condition of coplanar and equation of the plane containing the given two lines is
$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

Note:- If the two lines are parallel then the two lines will be coplanar.

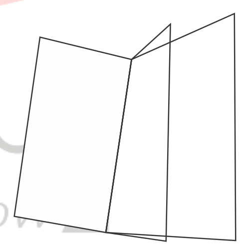
Example:- Show that the lines $\vec{r} = (\hat{\imath} + \hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k}) and \vec{r} = (4\hat{\jmath} + 2\hat{k}) + \mu(2\hat{\imath} - \hat{\jmath} + 3\hat{k})$ are coplanar. Also, find the equation of the plane containing them.

Example:- Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar.

Plane Passing through the intersection of two planes:-

Let $P1, P2$ be given intersecting planes and $\vec{r}.\vec{N}_1 = d_1,\ \vec{r}.\vec{N}_2 = d_2$ be their equations respectively.

Let $P(\vec{r})$ be any point on their line of intersection. It must satisfy both equations



Therefore, we get $\vec{r}.\vec{N}_1 = d_1,\ \vec{r}.\vec{N}_2 = d_2 \Rightarrow \vec{r}.\vec{N}_1 + \lambda\vec{r}.\vec{N}_2 = d_1 + \lambda d_2$ for all real λ

$$\Rightarrow \vec{r}.(\vec{N}_1 + \lambda\vec{N}_2) = d_1 + \lambda d_2$$

$\Rightarrow P$ satisfies the equation $\vec{r}.(\vec{N}_1 + \lambda\vec{N}_2) = d_1 + \lambda d_2$

But $\vec{r}.(\vec{N}_1 + \lambda\vec{N}_2) = d_1 + \lambda d_2$ represents a plane say $P3$, which is such that if the position vector $\vec{r}$ of any point satisfies the equation of the planes $P1$ and $P2$. It also satisfies the equation $P3$.

Note:- The equation of the plane passing through the intersection of the planes $\vec{r}.\vec{N_1} = d_1$ and $\vec{r}.\vec{N_2} = d_2$ is $\vec{r}.\left(\vec{N_1} + \lambda\vec{N_2}\right) = d_1 + \lambda d_2$ for the different real value of λ representing a family (system) of planes through the line of intersection of planes $\vec{r}.\vec{N_1} = d_1$ and $\vec{r}.\vec{N_2} = d_2$.

Similarly, the Cartesian equation of the plane through the line of intersection of planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2 = d_2$ is $(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2 - d_2) = 0$

Example:- Find the vector and Cartesian equation of the plane passing through the intersection of the planes $\vec{r}.\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 6$ and $\vec{r}.\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = -5$ and the point $(1, 1, 1)$

Ans:- $\vec{r}.\left(20i + 23\hat{\jmath} + 26\hat{k}\right) - 69 = 0$

Example:- Find the equation of the plane which is perpendicular to $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$

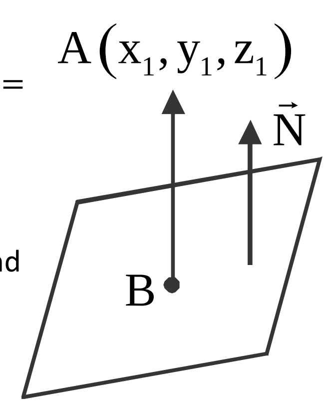
Ans:- $51x + 15y - 50z + 173 = 0$

Example:- Find the equation of the plane passing the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$, and parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$

Ans:- $7x + 9y - 10z - 27 = 0$

Example:- Show that the line of intersection of the planes $x + 2y + 3z = 8$ and $2x + 3y + 4z = 11$ is coplanar with the line $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3}$. Also, find the equation of the plane containing them.

The distance of a point from a plane:-



To find the distance of a point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz = d$

Let B be the foot of perpendicular drawn from A to the given plane.

Then AB is perpendicular to the plane.

AB is parallel to normal to the given plane. The direction ratio of AB and normal to the plane are proportional.

$\therefore$ drs of AB are ka, kb, kc i.e a, b, c

So the equation of AB is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Let $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=k$

Then general point on the line AB can be taken as $B(x_1+ak, y_1+bk, z_1+ck)$

As B lies on the given plane

Then, $a(x_1+ak)+b(y_1+bk)+c(z_1+ck)=d$

$$\Rightarrow K=-\frac{ax_1+by_1+cz_1-d}{a^2+b^2+c^2}$$

So $AB=\sqrt{(x_1+ak-x_1)^2+(y_1+bk-y_1)^2+(z_1+ck-z_1)^2}$

$=|k|\sqrt{a^2+b^2+c^2}$

$=\left|-\frac{ax_1+by_1+cz_1-d}{a^2+b^2+c^2}\right|\sqrt{a^2+b^2+c^2}$

$=\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$

Note:-

(a) The distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz=d$ is $\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$

(b) In vector form the distance of the point whose p.v is $\vec{a}$ from the plane $\vec{r}.\vec{N}=d$ is $\left|\frac{\vec{a}.\vec{N}-d}{|\vec{N}|}\right|$

Example:- Find the distance of the point $(2,5,3)$ from the plane $\vec{r}.\left(6\hat{\imath}-3\hat{\jmath}+2\hat{k}\right)=4$

Ans:- $\frac{13}{7}$ units

Example:- Find the distance between the planes $3x+4y-7=0$ and $6x+8y+6=0$.

Ans:- 2 units

Example:- Find the coordinate, of the foot of perpendicular and perpendicular distance from point $P(4,3,2)$ to the plane $x+2y+3z=2$. Also, find the image of P in the plane.

Ans:- (3, 1, -1), distance = $\sqrt{14}$ units, image $(2,-1,-4)$

Example:- Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{\imath}+3\hat{\jmath})+6=0$ and $\vec{r}.\left(3\hat{\imath}-\hat{\jmath}+4\hat{k}\right)=0$. Whose perpendicular distance from the origin is unity.

Ans:- $2x+y+2z=-3$ & $x-2y+2z=3$

The practice of Problems from NCERT example and Miscellaneous exercise:-

Problem – 01:- Find the distance of the point $(-2,3,-4)$ from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z+1=0$

Ans:- $\frac{17}{2}$ units

Problem – 2 Find the equation of the plane passing through the point (1, 1, -1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$

Ans:- $17x + 2y - 7z - 26 = 0$

Problem – 3 Find the equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$.

Ans:- $18x + 17y + 47 - 49 = 0$

Problem – 4 Find the equation of the plane through the point $(3, 0, -1)$ and parallel to lines $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z}{3}$ and $\vec{r} = (-i + 4j - 2k) + \lambda(2i - 3j + 4k)$

Ans:- $17x + 2y - 7z - 58 = 0$

Problem –05 Find the distance of the point $(2, 3, 4)$ from the plane $3x + 2y + 2z + 5 = 0$ measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$

Ans:- 7 units

Problem – 6 Prove that if a plane has the intercepts a, b, c, and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Previous year board problems:-

Problem:- A-line makes angles α, β, γ, and δ units the diagonals of a cube prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Problem:- Find the equation of the plane which passes through the point $(3, 2, 0)$ and contains the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$.

Ans:- $x - y + z - 1 = 0$

Problem:- Find the sum of the intercepts cut off by plane $2x + y - z = 5$ on the coordinate axes.

Ans:- $\frac{5}{2}$

Problem:- Find the value of k for which the following lines are perpendicular to each other. $\frac{x+3}{k-5} = \frac{y-1}{1} = \frac{5-7}{-2k-1}$; $\frac{x+2}{-1} = \frac{2-y}{-k} = \frac{z}{5}$. Hence find the equation of the plane containing the above lines.

Ans:- $k \ = -1, 4x + 31y + 7z = 54$

Problem:- If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines.

Ans:- $k = \frac{9}{2}; -5x + 2y + z + 6 = 0$

Problem:- Write the equation of the plane which is at a distance of $5\sqrt{3}$ units from origin and the normal to which is equally inclined to co-ordinate axes.

Ans:- $x + y + z = 15$

Problem:- Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 2$.

Ans:- $\vec{r}.(x + y + z) = a + b + c$