6

SQUARES AND SQUARE ROOTS

INTRODUCTION

If a number is multiplied by itself, the product so obtained is called the square of that number.

For example,

$3 \times 3 = 3^2 = 9$	We say that 9 is the square of 3.
$16 \times 16 = 16^2$	We say that 256 is the square of 16.
$0.7 \times 0.7 = (0.7)^2 = 0.49$	We say that 0.49 is the square of 0.7
$\frac{4}{5} \times \frac{4}{5} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$	We say that $\frac{16}{25}$ is the square of $\frac{4}{5}$

The square root of a number n is that number which when multiplied by itself gives n as the product.

For example, $5 \times 5 = 25$, so 5 is the square root of 25. We write $\sqrt{25} = 5$

PERFECT SQUARE

The numbers 1, 4, 9, 16, 25, 36 are the squares of natural numbers 1, 2, 3, 4, 5, 6 respectively and are called perfect squares or square numbers.

A natural number is called a perfect square or a square number if it is the square of some natural number.

Test : A given number is a perfect square if its prime factors can be expressed in pairs of equal factors.

Example 1 :

Find the square $\frac{5}{8}$.

Sol. $\left(\frac{5}{8}\right)^2 = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$

Example 2 :

Find the smallest number by which 252 must be multiplied so that the product becomes a perfect square.

Sol. $252 = \underline{2 \times 2} \times \underline{3 \times 3} \times 7$

We see that, in prime factorisation of 252, there exists a number 7 which is unpaired.

Hence, to make a pair of 7, we will have to multiply 252 by 7. Hence, 7 is the required number.

Example 3 :

Is 196 a perfect square ? If so, find the number whose square is 196.

Sol. Resolving 196 into prime factors, we get

$196 = \underline{2 \times 2} \times \underline{7 \times 7}$

2	196
2	98
7	49
7	7
	1

Thus, 196 can be expressed as a product of pairs of equal factors.

$\therefore$ 196 is a perfect square.

Also, $196 = (2 \times 7) \times (2 \times 7) = (14 \times 14) = (14)^2$.

Hence, 14 is the number whose square is 196.

PROPERTIES OF PERFECT SQUARES

The squares of natural numbers obey several interesting properties. Some of them are given below :

Property 1 : A number ending in 2, 3, 7 or 8 is never a perfect square. Thus, none of the numbers 82, 73,57 and 88 is a perfect square.

Property 2 : A number ending in an odd number of zeroes is never a perfect square, i.e., 130, 24000, 35400000 etc. are not perfect squares. (Alt. The number of zeros at the end of a perfect square is always even. e.g. $300 \times 300 = 90000$)

Property 3 : Squares of even numbers are always even. e.g. $(12)^2 = 12 \times 12 = 144$

Property 4 : Squares of odd numbers are odd. e.g. $(15)^2 = 15 \times 15 = 225$

Property 5 : For every natural n, we have

$(n + 1)^2 – n^2 = (n + 1 + n) (n + 1 – n) = (n + 1) + n$

This will help us to write immediately the difference between two consecutive squares.

Thus, $3^2 – 2^2 = 3 + 2 = 5$, $11^2 – 10^2 = 11 + 10 = 21$, $15^2 – 14^2 = 15 + 14 = 29$ etc.

Between n^2 and $(n + 1)^2$ there are 2n numbers which is 1 less than the difference of two squares.

Thus, in general we can say that there are 2n non perfect square numbers between the squares of the numbers n and (n + 1).

Property 6 : The square of proper fraction is smaller than the fraction.

$$\left(\frac{2}{3}\right)^2 = \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{4}{9} \text{ and } \frac{4}{9} < \frac{2}{3}, \text{ since } (4 \times 3) < (9 \times 2)$$

Property 7 : The squares of a natural number n is equal to the sum of the first n odd numbers.

i.e., sum of first n odd natural numbers is n^2.

For example,

1 [one odd number] $= 1 = 1^2$

$1 + 3$ [sum of first two odd numbers] $= 4 = 2^2$

$1 + 3 + 5$ [sum of first three odd numbers] $= 9 = 3^2$

$1 + 3 + 5 + 7$ [...] $= 16 = 4^2$

$1 + 3 + 5 + 7 + 9$ [...] $= 25 = 5^2$

$1 + 3 + 5 + 7 + 9 + 11$ [...] $= 36 = 6^2$

Alternatively, If the number is a square number, it has to be the sum of successive odd numbers starting from 1.

For example,

(A) Consider the number 25. Successively subtract 1, 3, 5, 7, 9, ... from it

(i) $25 – 1 = 24$ (ii) $24 – 3 = 21$ (iii) $21 – 5 = 16$ (iv) $16 – 7 = 9$ (v) $9 – 9 = 0$

This means, $25 = 1 + 3 + 5 + 7 + 9$. Also, 25 is a perfect square.

(B) consider another number 38, and again do as above.

(i) $38 – 1 = 37$ (ii) $37 – 3 = 34$ (iii) $34 – 5 = 29$ (iv) $29 – 7 = 22$ (v) $22 – 9 = 13$

(vi) $13 – 11 = 2$ (vii) $2 – 13 = – 11$

This shows that we are not able to express 38 as the sum of consecutive odd numbers starting with 1. Also, 38 is not a perfect square.

Property 8 : The square of a natural number (other than 1) is a multiple of 3 or exceeds a multiple of 3 by 1.
e.g. (a) $3^2 = 9 = (3 \times 3)$ (b) $4^2 = 16 = (3 \times 5) + 1$

Property 9 : The square of a natural (other than 1) is a multiple of 4 or exceeds a multiple of 4 by 1.
(a) $3^2 = 9 = (4 \times 2) + 1$ (b) $5^2 = 25 = (4 \times 6) + 1$ (c) $6^2 = 36 = (4 \times 9)$.

Property 10 : Product of two consecutive even or odd natural numbers
$11 \times 13 = 143 = 122 - 1$
Also $11 \times 13 = (12 - 1) \times (12 + 1)$
Therefore, $11 \times 13 = (12 - 1) \times (12 + 1) = 12^2 - 1$
Similarly, $13 \times 15 = (14 - 1) \times (14 + 1) = 14^2 - 1$
$29 \times 31 = (30 - 1) \times (30 + 1) = 30^2 - 1$
$44 \times 46 = (45 - 1) \times (45 + 1) = 45^2 - 1$
So in general we can say that $(a + 1) \times (a - 1) = a^2 - 1$.

Property 11 : Pythagorean triplet :
A triplet (m, n, p) of three natural numbers m, n and p is called a Pythagorean triplet, if $m^2 + n^2 = p^2$.
It is easy to verify that for any natural number m greater than 1, $(2m, m^2 - 1, m^2 + 1)$ is a Pythagorean triplet.
For example, (6, 8, 10) is a Pythagorean triplet.
Since, $6^2 + 8^2 = 36 + 64 = 100$
and $10^2 = 100$

SOME INTERESTING PATTERN

1. Square pyramid

1	×	1	=	1
11	×	11	=	121
111	×	111	=	12321
1111	×	1111	=	1234321
11111	×	11111	=	123454321
111111	×	111111	=	12345654321
1111111	×	1111111	=	1234567654321
11111111	×	11111111	=	123456787654321
111111111	×	111111111	=	12345678987654321

2.
$7^2 = 49$
$67^2 = 4489$
$667^2 = 444889$
$6667^2 = 44448889$
$66667^2 = 4444488889$
$666667^2 = 444444888889$

3. Square mirrors :
$14^2 + 87^2 = 41^2 + 78^2$ $\quad$ $15^2 + 75^2 = 51^2 + 57^2$ $\quad$ $17^2 + 84^2 = 71^2 + 48^2$
$26^2 + 97^2 = 62^2 + 79^2$ $\quad$ $27^2 + 96^2 = 72^2 + 69^2$

4.
$11^2 = 121$
$101^2 = 10201$
$1001^2 = 1002001$
$100001^2 = 10000200001$
$1000001^2 = 100000020000001$

5.
$11^2 = 121$
$101^2 = 10201$
$10101^2 = 102030201$
$1010101^2 = 1020304030201$
$101010101^2 = 10203040504030201$

6.
$1^2 + 2^2 + 2^2 = 3^2$
$2^2 + 3^2 + 6^2 = 7^2$
$3^2 + 4^2 + 12^2 = 13^2$
$4^2 + 5^2 + 20^2 = 21^2$
$5^2 + 6^2 + 30^2 = 31^2$
$6^2 + 7^2 + 42^2 = 43^2$

Example 4 :

Find the Pythagorean triplet whose smallest member is 12.

Sol. For every natural number $m > 1$, $(2m, m^2 - 1, m^2 + 1)$ is a Pythagorean triplet.
Putting $2m = 12$, i.e., $m = 6$. We get the triplet (12, 35, 37)

Example 5 :

From observing the unit's digits, which of the following numbers cannot be perfect squares ?
(i) 3486 (ii) 4867 (iii) 8913 (iv) 64000

Sol. (i) Since 6 is in unit's digit, there is a chance that 3486 is a perfect square as 4×4 and 6×6 ends in 6.
(ii) and (iii) are surely not perfect squares as 7 and 3 are the unit digit of these numbers.
There cannot be any number with unit digits 2, 3, 7 or 8 as a perfect square.
(iv) We know that the numbers ending in an odd number of consecutive zeros are not perfect squares. Therefore, 64000 is not perfect squares.

Example 6 :

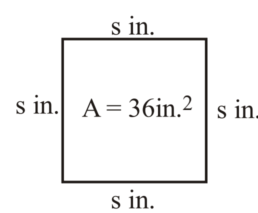
Without adding, find the sum (1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17).

Sol. We have,
$(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17)$ = sum of first 9 odd numbers $= 9^2 = 81$

SELF CHECK-1

Q.1 Find the smallest number by which 140 should be divided so that the quotient becomes a perfect square.
Q.2 Write down the unit digits of the squares of the following numbers. (i) 24 (ii) 78 (iii) 35
Q.3 Is 36 a perfect square ? If so, find the number whose square is 36.
Q.4 Find the value of (i) $(-5)^2$ and (ii) $(-3/7)^2$.
Q.5 State whether the following numbers are perfect square or not : (i) 7212 (ii) 563
Q.6 Express 49 as the sum of seven odd numbers.

SQUARE ROOTS



Suppose we know that the area of the square shown in figure is 36 square inches. To find the length of each side, we substitute 36 for A in the formula $A = s^2$ and solve for s.

$A = s^2$

$36 = s^2$

To solve for s, we must find a positive number whose square is 36. Since 6 is such a number, the sides of the square are 6 inches long. The number 6 is called a square root of 36, because 6 is the positive number that we square to get 36.

* 3 is a square root of 9, because $3^2 = 9$.
* – 3 is a square root of 9, because $(-3)^2 = 9$.
* 12 is a square root of 144, because $12^2 = 144$.
* – 12 is a square root of 144, because $(-12)^2 = 144$.
* $\frac{1}{3}$ is a square root of $\frac{1}{9}$, because $\left(\frac{1}{3}\right)^2 = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$
* $-\frac{1}{3}$ is a square root of $\frac{1}{9}$, because $\left(-\frac{1}{3}\right)^2 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{1}{9}$
* 0 is a square root of 0, because $0^2 = 0$.

In general, we have the following definition, The number b is a square root of a if $b^2 = a$

The symbol $\sqrt{\ }$, called a radical sign, is used to represent the positive (or principal) square root of a number.

Principal square root : If a > 0, the expression $\sqrt{a}$ represents the principal (or positive) square root of a. The principal square root of 0 is 0 : $\sqrt{0} = 0$. The expression under a radical sign is called a radicand.

The principal square root of a positive number is always positive. Although 3 and –3 are both square roots of 9, only 3 is the principal square root. The symbol $\sqrt{9}$ represents 3. To represent –3, we place a – sign in. front of the radical : $\sqrt{9} = 3$ and $-\sqrt{9} = -3$. Likewise, $\sqrt{144} = 12$ and $-\sqrt{144} = -12$

PROPERTIES OF SQUARE ROOT

1. **Property of equality :** If a and b are positive numbers, then if a = b, then $\sqrt{a} = \sqrt{b}$.
2. **The multiplication property :** We introduce the first of two properties of radicals with the following examples

   $\sqrt{4 \times 25} = \sqrt{100} = 10$ and $\sqrt{4}\sqrt{25} = 2 \times 5 = 10$. In each case, the answer is 10. Thus, $\sqrt{4 \times 25} = \sqrt{4}\sqrt{25}$.

   Likewise, $\sqrt{9 \times 16} = 144 = 12$ and $\sqrt{9}\sqrt{16} = 3 \times 4 = 12$

   In each case, the answer is 12. Thus, $\sqrt{9 \times 16} = \sqrt{9}\sqrt{16}$. These results suggest the multiplication property of radicals. If a and b are positive or zero, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$

   In words, the square root of the product of two nonnegative numbers is equal to the product of their square roots
3. **The division property :** To introduce the second property of radicals, we consider these examples.

   $\sqrt{\dfrac{100}{25}} = \sqrt{4} = 2$ and $\dfrac{\sqrt{100}}{\sqrt{25}} = \dfrac{10}{5} = 2$. Since the answer is 2 in each case, $\sqrt{\dfrac{100}{25}} = \dfrac{\sqrt{100}}{\sqrt{25}}$

   Likewise, $\sqrt{\dfrac{36}{4}} = \sqrt{9} = 3$ and $\dfrac{\sqrt{36}}{\sqrt{4}} = \dfrac{6}{2} = 3$. Since the answer is 3 in each case, $\sqrt{\dfrac{36}{4}} = \dfrac{\sqrt{36}}{\sqrt{4}}$

   These results suggest the division property of radicals. If a ≥ 0 and b > 0, then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

   In words, the square root of the quotient of two numbers is the quotient of their square roots.

TO FIND THE SQUARE ROOT OF A PERFECT SQUARE

Method 1 : By the prime factorization

Step 1 : Resolve the given number into prime factors.

Step 2 : Make pairs of similar factors.

Step 3 : Take the product of the prime factors choosing one factor out of every pair.

Remember : (i) We may also write the product of prime factors in exponential form and for finding the square root, we take half of the index value of each factor and then multiply.

(ii) Square root of a fraction, i.e., $\sqrt{\dfrac{p}{q}} = \dfrac{\sqrt{p}}{\sqrt{q}} = \dfrac{\text{Square root of the numerator}}{\text{Square root of the denominator}}$ e.g. $\sqrt{\dfrac{121}{49}} = \dfrac{\sqrt{121}}{\sqrt{49}} = \dfrac{11}{7}$

(iii) Square root of a decimal number = Square root of the corresponding decimal fraction.

e.g., $\sqrt{0.25} = \sqrt{\dfrac{25}{100}} = \dfrac{\sqrt{25}}{\sqrt{100}} = \dfrac{5}{10} = 0.5$

Example 7 :

Find the square root of 1764 by finding prime factors.

2	1764
2	882
3	441
3	147
7	49
7	7
	1

Sol. $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$

Hence, $\sqrt{1764} = \sqrt{\underline{2 \times 2} \times \underline{3 \times 3} \times \underline{7 \times 7}}$

$= 2 \times 3 \times 7 = 42$

Note that if negative factors are considered, we shall have

$\sqrt{1764} = -42$

Example 8 :

Evaluate : $\sqrt{11025}$

5	11025
5	2205
3	441
3	147
7	49
7	7
	1

Sol. By prime factorisation, we get

$11025 = \underline{5 \times 5} \times \underline{3 \times 3} \times \underline{7 \times 7}$

$\therefore \quad \sqrt{11025} = (5 \times 3 \times 7) = 105$

Example 9 :

Find the least perfect square which is divisible by 5, 6 and 8.

Sol. The least number that is divisible by 5, 6 and 8 is the L.C.M. of 5, 6 and 8 which is 120.

Prime factors of $120 = \underline{2 \times 2} \times 2 \times 3 \times 5$.

Clearly, to make it a perfect square, we will have to multiply it by $2 \times 3 \times 5 = 30$

$\therefore$ Required number $= 120 \times 30 = 3600$

Example 10 :

Find the least square number divisible by each one of 8, 9 and 10.

Sol. The least number divisible by each one of 8, 9, 10 is their LCM.

Now, LCM of 8, 9, 10 $= (2 \times 4 \times 9 \times 5) = 360$

2	8, 9, 10
	4, 9, 5

By prime factorisation, we get

$360 = \underline{2 \times 2} \times 2 \times \underline{3 \times 3} \times 5$

2	360
2	180
2	90
3	45
3	15
5	5
	1

To make it a perfect square it must be multiplied by (2×5), i.e. 10

Hence, required number $= (360 \times 10) = 3600$

Example 11 :

Find the value of : (i) $\sqrt{\dfrac{25}{64}}$ (ii) $\sqrt{\dfrac{121}{169}}$

Sol. (i) $\sqrt{\dfrac{25}{64}} = \dfrac{\sqrt{25}}{\sqrt{64}} = \dfrac{\sqrt{5 \times 5}}{\sqrt{8 \times 8}} = \dfrac{5}{8}$ (ii) $\sqrt{\dfrac{121}{169}} = \dfrac{\sqrt{121}}{\sqrt{169}} = \dfrac{\sqrt{11 \times 11}}{\sqrt{13 \times 13}} = \dfrac{11}{13}$

Method 2 : Square root of a perfect square by the long division method

The method of finding square roots by the prime factorization method is efficient only if the number has small prime factors. Sometimes it is difficult and time consuming to obtain prime factors of a given number. To overcome this difficulty, we use an alternative method called the 'Long Division Method'.

Steps of Long Division method :

Step 1 : Divide the digits into pairs of two digits starting with the digit at the units place. Each pair and the remaining one digit (if any) is called a period.

Step 2 : Think of a whole number whose square is equal to or just less than the first period. Take this number as the divisor and the quotient.

Step 3 : Subtract the product of the divisor and the quotient from the first period and bring down the next period to the right of the remainder. This becomes the new dividend.

Step 4 : Now, the new divisor is obtained by taking two times the quotient and annexing with a suitable digit which is also taken as the next digit of the quotient, chosen in such a way that the product of the new divisor and this digit is equal to or just less than the new dividend.

Step 5 : Repeat steps 2, 3 and 4 till all the periods have taken up. Now, the quotient so obtained is the required square root of the given number.

Example 12 :

Find the square root of 225.

Sol. Steps : (i) The digits are marked in pairs of digits called periods, from the one's digit.

(ii) The first digit from the left (the first period) is 2. $1^2 = 1$ is the biggest square less than 2. So, 1 is the first digit of the square root. Write 1 at the top. Multiply $1 \times 1 = 1$ and write 1 below 2.

(iii) From 2, subtract 1 and you get 1. Now bring the second period 25 down to the remainder to make it 125.

(iv) Now, for the second digit in the divisor, double the first quotient 1. This forms the initial part of the new divisor, in this case 2.

(v) The next digit of the divisor has to be obtained by the trial-and-error method. Find a number such that $2a \times a = 125$.

$$\begin{array}{r|ll} & 1 & 5 \\ \hline 1 & \overline{2} & \overline{25} \\ & 1 & \\ \hline 25 & 125 & \\ \times 5 & 125 & \\ \hline & \;\;0 & \end{array}$$

Remember 2 is in the tens place and 'a' is in the unit's place. Here the number is 5. The divisor becomes 25 and the multiplier is 5.

(vi) We get $25 \times 5 = 125$, leaving no remainder. So, $\sqrt{225} = 15$.

Example 13 :

Find the square root of 784 by the long-division method.

Sol. Marking periods and using the long-division method, we have

$\therefore \sqrt{784} = 28$

$$\begin{array}{r|ll} 2 & \overline{7} & \overline{84}\,(28 \\ & 4 & \\ \hline 48 & 384 & \\ & 384 & \\ \hline & \;\times & \end{array}$$

SQUARE ROOT OF NUMBERS IN DECIMAL FORM

The square root of a number in decimal form can be found by taking the following steps.

Step 1 : Make the number of decimal places even by affixing zero or zeroes, if necessary.

Step 2 : Mark off the periods as usual in the integral part.

Step 3 : Mark off the periods in the decimal part beginning with the first decimal place.

Step 4 : Start finding square root by the long division method.

Step 5 : Put the decimal point in the square root as soon as the integral part is exhausted.

Step 6 : Complete the process as usual.

For example, while finding the square roots of 0.00 00 2601, 492.84, 998.56, 252.70729 the periods will be formed as under :

$0.\overrightarrow{00}\,\overrightarrow{00}\,\overrightarrow{26}\,\overrightarrow{00}$, $\overrightarrow{4}\,\overrightarrow{92}.\overrightarrow{84}$, $\overrightarrow{9}\,\overrightarrow{88}.\overrightarrow{56}$, $\overrightarrow{2}.\overrightarrow{52}.\overrightarrow{70}\,\overrightarrow{72}\,\overrightarrow{90}$ (one zero is added)

The square root of a decimal will contain as many decimal places as there are periods, or half as many decimal places as the given number.

The operations in obtaining the square root of a decimal number are the same as for whole numbers.

Example 14 :

Evaluate : $\sqrt{6.4009}$

Sol. Using the division method :

$\therefore\ \sqrt{6.4009} = 2.53$

2	$\overline{6}.\overline{40}\ \overline{09}$ (2.53
	4
45	240
	225
503	1509
	1509
	×

TO FIND THE VALUE OF SQUARE ROOT CORRECT UP TO CERTAIN PLACES OF DECIMAL

Rule : If the square root is required correct up to two places of decimal, we shall find it up to 3 places decimal and then round it off up to two places of decimal.

Similarly, if the square root is required correct up to three places of decimal, we shall find it up to 4 places of decimal and then round it off up to three places of decimal, and so on.

Example 15 :

Find the square roots 9.3 correct to two places of decimals :

Sol. 9.3 can be written as 9.30000000

	3. 0 4 9 5
3	$\overline{9}.\ \overline{30}\ \overline{00}\ \overline{00}\ \overline{00}$
	9
60	30
	00
604	30 00
	24 16
6089	5 84 00
	5 48 01
60985	35 99 00
	30 49 25
	5 49 75

$\therefore\ \sqrt{9.3} = 3.0495 = 3.05$ correct to two places of decimal.

SHORT CUT METHODS FOR SQUARING A NUMBER

To find the square of a given integer, we only multiply the given integer with itself. For large numbers, multiplication may prove to be laborious and time consuming.

The method to find the square quickly without actual multiplication is based upon an old Indian method of multiplying two numbers, which we call 'column method'.

COLUMN METHOD

To find the square of a 2-digit number, we use the identify : $(a+b)^2 = a^2 + 2ab + b^2$

To square a 2-digit number ab (where a is the tens digit and b is the unit digit), we make three columns and write a^2, $2a \times b$ and b^2 respectively in these columns as follows (As an example, we will find square of 84)

Column I	Column II	Column III
a^2	$2a \times b$	b^2
$(8^2 = 64)$	$(2 \times 8 \times 4 = 64)$	$(4^2 = 16)$

We then go through the following steps :

Step 1 : Underline the units digit of b^2 (in column III) and add the tens digit of b^2, if any to $2a \times b$ in column II.

I	II	III
a^2	$2a \times b$	b^2
	64	1$\underline{6}$
64	+1	
	65	

Step 2 : Underline the units digit in column II and the remaining digits if any to a^2 in column I.

a^2	$2a \times b$	b^2
64	64	1$\underline{6}$
+6	+1	
70	6$\underline{5}$	

Step 3 : Underline the number in column I.

a^2	$2a \times b$	b^2
64	64	1$\underline{6}$
+6	+1	
$\underline{70}$	6$\underline{5}$	

The underlined digits give the required square, i.e., $84^2 = 7056$

Example 16 :

Find the square of (i) 47 (ii) 86

Sol. (i) Given number = 47

$\therefore$ a = 4 and b = 7

I	II	III
a^2	$(2 \times a \times b)$	b^2
16 +6 $\underline{22}$	56 +4 6$\underline{0}$	4$\underline{9}$

$\therefore$ $(47)^2 = 2209$

(i) Given number = 86

$\therefore$ a = 8 and b = 6

I	II	III
a^2	$(2 \times a \times b)$	b^2
64 +9 $\underline{73}$	96 +3 9$\underline{9}$	3$\underline{6}$

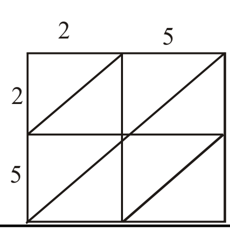
$\therefore$ $(86)^2 = 7396$

DIAGONAL METHOD

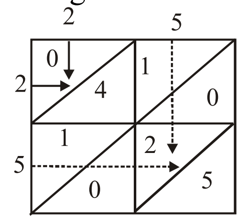
As the number of digits increases, the column method becomes difficult. In this case, we have use the following method which we call the 'Diagonal method'. This method is also an old Indian method of multiplying two numbers.

We illustrate this method for squaring 25.

First we form a square. Then we divide it into sub-squares. The number of sub-square in each row or column should be equal to the number of digits in the given number. Then draw the diagonals and write the digits of the number to be squared as shown in figure.

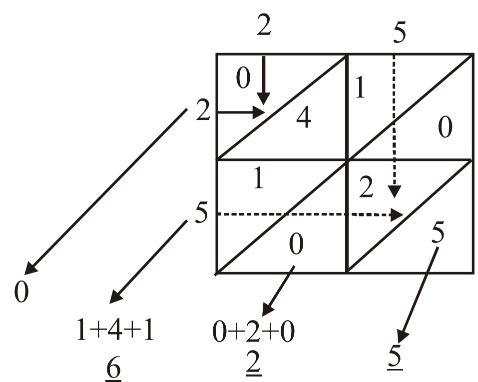


Now multiply each digits on the left of the square with each digit on top of the column one-by-one. Write the product in the corresponding sub-square. If the number so obtained is a single digit number, write it below the diagonal. If it is 2-digit number, write the tens digit above the diagonal and the units digit below the diagonal.



Starting below the lowest diagonal, sum the digits along the diagonals so obtained, underline the units digit of the sum and take carry, if any, to the diagonal above. Unit digits so underlined together with all the digits in the sum above the topmost diagonal give the square. Numbers in the empty places are taken as zero.

The diagonal method can be applied to find the square of any number irrespective of the number of digits.



$\therefore (25)^2 = 625$

Example 17 :

Find the square of 39 by using the diagonal method.

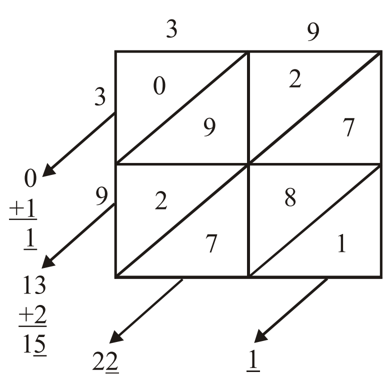
Sol. Step 1 : The given number contains two digits. So, draw a square and divide it into 4 subsquares Write down the digits 3 and 9, horizontally and vertically.

Step 2 : Multiply each digit on the left of the square with each digit on the top, one by one. Write the product in the corresponding subsquare.

Step 3 : Starting below the lowest diagonal, sum the digits diagonally.

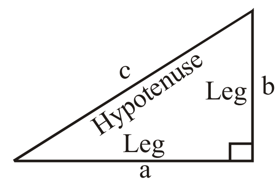
Step 4 : Underline all the digits in the sum above the topmost diagonal.

Step 5 : The underlined digits give the required square number.



$\therefore (39)^2 = 1521$

RIGHT TRIANGLES

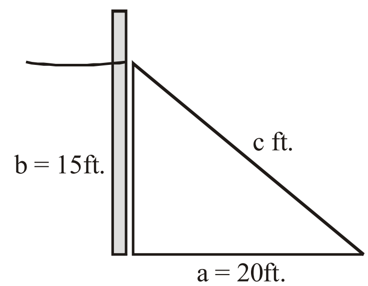


A triangle that contains a 90° angle is called a right triangle. The longest side of a right triangle is the hypotenuse, which is the side opposite the right angle. The remaining two sides are the legs of the triangle.
In the right triangle shown in figure, side c is the hypotenuse and sides a and b are legs.

THE PYTHAGOREAN THEOREM

The Pythagorean theorem provides a formula relating the lengths of the three sides a right triangle.
If the length of the hypotenuse of a right triangle is c and the lengths of the two legs are a and b, then $c^2 = a^2 + b^2$
Since the lengths of the sides of a triangle are positive numbers, we can use the square root property of equality and the Pythagorean theorem to find the length of the third side of any right triangle when the measures of two sides are given.

Example 17 :



A builder of a high ropes course wants to stabilize the pole shown in figure by attaching a cable to a stake anchored 20 feet from its base to a point 15 feet up the pole. How long will the cable be?

Sol. We can use the Pythagorean theorem, with a = 20 and b = 15, to find c.

$c^2 = a^2 + b^2$

$c^2 = 20^2 + 15^2$ (Substitute 20 for a and 15 for b)

$c^2 = 400 + 225$ ($20^2 = 400$ and $15^2 = 225$)

$c^2 = 625$ (Do the addition : 400 + 225 = 625)

$\sqrt{c^2} = \sqrt{625}$ (To find c, we isolate it by "undoing" the operation performed on it. Since c is squared, we take the positive square root of both sides.

$c = 25$ ($\sqrt{625} = 25$ and $\sqrt{c^2} = c$. because $c \, . \, c = c^2$)

The cable will be 25 feet long.

THE SQUARING PROPERTY OF EQUALITY

The equation $\sqrt{x} = 6$ is called a radical equation, because it contains a radical expression with a variable radicand. To solve this equation, we need to isolate x by undoing the operation performed on it. Recall that $\sqrt{x}$ represents the number that, when squared, gives x. Therefore, if we square $\sqrt{x}$, we will obtain x.

$(\sqrt{x})^2 = x$

Using this observation, we can eliminate the radical on the left-hand side of the equation $\sqrt{x} = 6$ by squaring that side. Intuition tells us that we should also square the right-hand side. This is a valid step, because if two numbers are equal, their squares are equal.
If a = b, then $a^2 = b^2$
We can now solve the equation by applying the squaring property of equality.

$\sqrt{x} = 6$ (The original equation to solve)

$(\sqrt{x})^2 = (6)^2$ (Square both sides of the equation to eliminate the radical)

$x = 26$ (Simplify each side: $(\sqrt{x})^2 = x$ and $(6)^2 = 36$)

Checking this result, we have

$\sqrt{x} = 6$; $\sqrt{36} \stackrel{?}{=} 6$ (Substitute 36 for x)

$6 = 6$ (Simplify the left-hand side: $\sqrt{36} = 6$. We obtain a true statement, so x = 36 is a solution)

Checking solutions : If we square both sides of an equation, the resulting equation mayor may not have the same solutions as the original one. For example, if we square both sides of the equation

1. $x = 2$ with solution 2, we obtain $(x)^2 = 2^2$, which simplifies to
2. $x^2 = 4$ with solutions 2 and –2, since $2^2 = 4$ and $(-2)^2 = 4$.

Equations 1 and 2 are not equivalent, because they have a different set of solutions. The solution –2 of Equation 2 does not satisfy equation 1. Because squaring both sides of an equation can produce an equation with solutions that don't satisfy the original one, we must always check each potential solution in the original equation.

SOLVING EQUATIONS CONTAINING ONE SQUARE ROOT

To solve an equation containing square root radicals, we follow these steps.

1. Whenever possible, isolate a single radical on one side of the equation.

2. Square both sides of the equation and solve the resulting equation.

3. Check the solution in the original equation. This step is required.

Example 18 :

Solve : $\sqrt{x+2} = 3$

Sol. We square both sides to eliminate the radical and proceed as follows:

$\sqrt{x+2} = 3$; $(\sqrt{x+2})^2 = (3)^2$ (Square both sides)

$x + 2 = 9$ (Simplify each side : $(\sqrt{x+2})^2 = x + 2$ and $3^2 = 9$)

$x = 7$ (Subtract 2 from both sides)

We check by substituting 7 for x in the original equation.

$\sqrt{x+2} = 3$; $\sqrt{7+2} \stackrel{?}{=} 3$ (Substitute 7 for x)

$\sqrt{9} \stackrel{?}{=} 3$ (Do the addition within the radical symbol)

$3 = 3$. The solution checks. Since no solutions are lost in this process, 7 is the only solution of the original eq.

ESTIMATING APPROXIMATE VALUE OF SQUARE ROOT

Let us find square root of 250.

We know that $100 < 250 < 400$ and $\sqrt{100} = 10$ and $\sqrt{400} = 20$. So $10 < \sqrt{250} < 20$

But still we are not very close to the square number.

We know that $15^2 = 225$ and $16^2 = 256$

Therefore, $15 < \sqrt{250} < 16$ and 256 is much closer to 250 than 225. So, $\sqrt{250}$ is approximately 16.

SELF CHECK-2

Q.1 Find the square root of 324.

Q.2 Evaluate $\sqrt{16384}$

Q.3 Evaluate : $\sqrt{42.25}$

Q.4 Solve : $\sqrt{x-4} = 9$

Q.5 Find the square root of $1\frac{56}{169}$

Q.6 $\sqrt{x-2} + 5 = 2$

EXTRA EDGE

1. For faster calculation : Let us consider Duplex combination method useful for squaring the number fastly. First find D. What is D ? To find D see the following examples and follow the procedure as explained

(i) For 2, Duplex (D) = $2^2 = 4$ (ii) For 8, D = $8^2 = 64$

(iii) For 34, D = $2 \times (3 \times 4) = 24$ (iv) For 79, D = $2 \times (7 \times 9) = 126$

(v) For 103, D = $2\ (1 \times 3) + 0^2 = 6$ (vi) For 346, D = $2\ (3 \times 6) + 4^2 = 52$

(vii) For 096, D = $2\ (0 \times 6) + 81 = 81$ (viii) For 1342, D = $2\ (1 \times 2) + 2\ (3 \times 4) = 28$

(ix) For 7358, D = $2\ (56) + 2\ (15) = 142$

(x) For 23564, D = $2\ (2 \times 4) + 2\ (3 \times 6) + 5^2 = 77$

(xi) For 123456, D = $2\ (1 \times 6) + 2\ (2 \times 5) + 2\ (3 \times 4) = 56$

Example 19 :

$207^2 = ?$

Sol. 207^2 = D for 2 / D for 20 / D for 207 / D for 07 / D for 7

= $2^2 / 2\ (2 \times 0) / 2\ (2 \times 7) + 0^2 / 2\ (0 \times 7) / 7^2$

= 4 / 0 / 28 / 0 / 49

= 4 / 0 / $_{2}8$ / 0 / $_{4}9$

= 4 / 0 + 2 / 8 / 0 + 4 / 9 = 42849

Example 20 :

$(897)^2 = ?$

Sol. $80_{16}4_{20}6_{13}0_{4}9 = 804609$

1. $7^2 = 49$; write down 9 and carry over 4.
2. $2 \times 9 \times 7 + 4$ (carried) = 130, write down 0 and carry over 13.
3. $2 \times 8 \times 7 + 9^2 + 13 = 206$, write down 6 and carry over 20.
4. $2 \times 8 \times 9 + 20 = 164$, write down 4 and carry over 16.
5. $8^2 + 16 = 80$; write it down.

Square of a number with unit digit as 5 :

We take a general representative of such number, say, A5

Now, $(A5)^2 = A^2 / 2 \times A \times 5 / 5^2$

$= A^2 / 10 \times A / 25 = A^2 / A0 / 25$ [$\because$ $10 \times A = A0$]

$= A^2 + A / 25 = A\ (A + 1) / 25$

Using the above equations :

(i) $(15)^2 = 1 \times (1 + 1) / 5^2 = 2/25 = 225$

(ii) $(25)^2 = 2 \times (2 + 1) / 5^2 = 6/25 = 625$

(iii) $(85)^2 = 8 \times (8 + 1) / 5^2 = 72/25 = 7225$

(iv) $(115)^2 = 11 \times (12) / 5^2 = 132/25 = 13225$

(v) $(225)^2 = 22 \times (23) / 5^2 = 506/25 = 50625$

Square of a number which is nearer to 10^x

We use the algebraic formula

$x^2 = (x^2 - y^2) + y^2 = (x + y)\ (x - y) + y^2$

(i) $(98)^2 = (98 + 2)\ (98 - 2) + 2^2 = 9600 + 4 = 9604$

(ii) $(103)^2 = (103 - 3)\ (103 + 3) + 3^2 = 10600 + 9 = 10609$

(iii) $(993)^2 = (993 + 7)\ (993 - 7) + 7^2 = 986000 + 49 = 986049$

(iv) $(1008)^2 = (1008 - 8)\ (1008 + 8) + 8^2 = 1016000 + 64 = 1016064.$

USEFUL TIPS

1. If a natural number m can be expressed as n^2, where n is also a natural number, then m is a square number.
2. All square numbers end with 0, 1, 4, 5, 6 or 9 at unit's place.
3. Square numbers can only have even number of zeros at the end.
4. Square root is the inverse operation of square.
5. If a number has 1 or 9 in the unit's place, then it's square ends in 1.
6. When a square number ends in 6, the number whose square it is, will have either 4 or 6 in unit's place.
7. If a natural number cannot be expressed as a sum of successive odd natural numbers starting with 1, then it is not a perfect square.
8. If a perfect square is a 3-digit or a 4-digit number, then its square root will have 2-digits.
9. If a perfect square is of n-digits, then its square root will have $\frac{n}{2}$ digits if n is even or $\frac{(n+1)}{2}$ if n is odd.
10. The numerator and the denominator of fraction $\frac{108}{92}$ are not perfect squares. At first sight, it seems that the square-root of $\frac{108}{92}$ is not possible. But, the reduced form of $\frac{108}{92}$ can be obtained as $\frac{108}{92}=\frac{12\times 9}{12\times 16}=\frac{9}{16}$.

    Now, both, the numerator as well as the denominator of $\frac{9}{16}$ are perfect squares.

    So the square-root of $\frac{9}{16}$ is $\frac{3}{4}$. Consequently the square-root of $\frac{108}{92}$ is $\frac{3}{4}$.

    Thus, to find the square-root of a fraction, first of all it should be converted to the reduced form.
11. The square-root of a mixed number can be obtained by expressing it as improper fraction in the form $\frac{p}{q}$.

    e.g., $\sqrt{1\frac{25}{144}}=\sqrt{\frac{169}{144}}=\frac{13}{12}=1\frac{1}{12}$ Note : $\sqrt{1\frac{25}{144}}\neq 1\frac{5}{12}$. Similarly, $\sqrt{1\frac{49}{576}}=\sqrt{\frac{625}{576}}=\frac{25}{24}=1\frac{1}{24}$
12. **27 magic**

    $2+7=3^2 : 27=3^3=3^2+3^2+3^2$ $27^2=729 : 7+2=3^2$

    $\frac{2}{7}=0.283714 : 2+8+3+7+1+4=27$ $27^3=19683 : 1+9+6+8+3=27$

    $27=14+13 : 27=14^2-13^2$ $2+7=6+3 : 27=6^2-3^2$