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# CUBES AND CUBE ROOTS

## INTRODUCTION

1729 is the natural number following 1728 and preceding 1730. 1729 is known as the Hardy-Ramanujan number after a famous anecdote of the British mathematician G. H. Hardy regarding a hospital visit to the Indian mathematician Srinivasa Ramanujan. In Hardy's words:

I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

The quotation is sometimes expressed using the term "positive cubes", since allowing negative perfect cubes (the cube of a negative integer) gives the smallest solution as 91 (which is a divisor of 1729):

$91 = 6^3 + (-5)^3 = 4^3 + 3^3$

Of course, equating "smallest" with "most negative", as opposed to "closest to zero" gives rise to solutions like –91, –189, –1729, and further negative numbers. This ambiguity is eliminated by the term "positive cubes".

The numbers such as : $1729 = 1^3 + 12^3 = 9^3 + 10^3$

that are the smallest number that can be expressed as the sum of two cubes have been dubbed "taxicab numbers".

## CUBES

The cube of a number is the number raised to the power 3. Thus cube of $2 = 2^3 = 2 \times 2 \times 2 = 8$, cube of $5 = 5^3 = 5 \times 5 \times 5 = 125$

**Perfect cube :** We know that $2^3 = 8$, $3^3 = 27$, $6^3 = 216$, $7^3 = 343$, $10^3 = 1000$

The numbers 8, 27, 216, 343, 1000, ... are called perfect cubes. A natural number is said to be a perfect cube, if it is the cube of some natural number. That is,

A number x is a perfect cube if there exists a integer y such that $y^3 = x$.

**Test :** A given natural number is a perfect cube if in its prime factorization; every prime occurs three times or a multiple of three times, i.e., if it is expressible as the product of triples of the same prime.

**Prime factorisation of the numbers and their cubes :**

| **Prime factorisation of a number** | **Prime factorisation of its cube** |
|---|---|
| $4 = 2 \times 2$ | $4^3 = 64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^3 \times 2^3$ |
| $6 = 2 \times 3$ | $6^3 = 216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$ |
| $15 = 3 \times 5$ | $15^3 = 3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5 = 3^3 \times 5^3$ |
| $12 = 2 \times 2 \times 3$ | $12^3 = 1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 2^3 \times 3^3$ |

**Note :** Each prime factor appears three times in its cubes

**Example 1 :**

Find the value of the following cubes :

(i) $7^3$ (ii) $15^3$ (iii) $(-4)^3$ (iv) $\left(\frac{2}{3}\right)^3$ (v) $(0.4)^3$ (vi) $(1.2)^3$ (vii) $\left(-\frac{3}{4}\right)^3$

**Sol.** (i) $7^3 = 7 \times 7 \times 7 = 343$ (ii) $15^3 = 15 \times 15 \times 15 = 3375$

(iii) $(-4)^3 = (-4)(-4)(-4) = -64$ (iv) $\left(\frac{2}{3}\right)^3 = \frac{2\times2\times2}{3\times3\times3} = \frac{8}{27}$

(v) $(0.4)^3 = 0.4 \times 0.4 \times 0.4 = 0.064$ (vi) $(1.2)^3 = 1.2 \times 1.2 \times 1.2 = 1.728$

(vii) $\left(-\frac{3}{4}\right)^3 = \frac{(-3)\times(-3)\times(-3)}{4\times4\times4} = \frac{-27}{64}$

**Example 2 :**

In the following, identify the numbers which are perfect cubes.

(i) 1728 (ii) 540

**Sol.** (i) Resolving 1728 into prime factors, we get

Thus, $1728 = \underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times \underline{3 \times 3 \times 3}$

$\therefore$ 1728 is a perfect cube.

(ii) Resolving 540 into prime factors, we get

Thus, $540 = \underline{2 \times 2} \times \underline{3 \times 3 \times 3} \times 5$

Hence 2 and 5 do not appear in groups of three.

Hence, 540 is not a perfect cube.

| | |
|---|---|
| 2 | 540 |
| 2 | 270 |
| 3 | 135 |
| 3 | 45 |
| 3 | 15 |
| 5 | 5 |
| | 1 |

| | |
|---|---|
| 2 | 1728 |
| 2 | 864 |
| 2 | 432 |
| 2 | 216 |
| 2 | 108 |
| 2 | 54 |
| 3 | 27 |
| 3 | 9 |
| 3 | 3 |
| | 1 |

**Example 3 :**

Find out if 576 is a perfect cube.

**Sol.** Using prime factorization, 576 can be written as

$$576 = \underbrace{2\times2\times2}\times\underbrace{2\times2\times2}\times3\times3$$

$$= 2^3 \times 2^3 \times 3^2.$$

Here, 3 appears only twice, not thrice. So, 576 is not a perfect cube.

Now, to make 576 a perfect cube, it has to be multiplied once more by 3.

So, $576 \times 3 = 2^3 \times 2^3 \times 3^2 \times 3$

i.e., $1728 = 2^3 \times 2^3 \times 3^3$

In this case, 3 is the smallest number with which 576 is to be multiplied to make it a perfect cube.

| | |
|---|---|
| 2 | 576 |
| 2 | 288 |
| 2 | 144 |
| 2 | 72 |
| 2 | 36 |
| 2 | 18 |
| 3 | 9 |
| 3 | 3 |
| | 1 |

## PROPERTIES OF CUBES

Observe the cubes of these numbers.

| | |
|---|---|
| $1^3 = 1$ | $11^3 = 1331$ |
| $2^3 = 8$ | $12^3 = 1728$ |
| $3^3 = 27$ | $13^3 = 2197$ |
| $4^3 = 64$ | $14^3 = 2744$ |
| $5^3 = 125$ | $15^3 = 3375$ |
| $6^3 = 216$ | $16^3 = 4096$ |
| $7^3 = 343$ | $17^3 = 4913$ |
| $8^3 = 512$ | $18^3 = 5832$ |
| $9^3 = 729$ | $19^3 = 6859$ |
| $10^3 = 1000$ | $20^3 = 8000$ |

1. For numbers with their unit's digit as 1, their cubes also will have the unit's diglt as 1.
For example, $1^3 = 1; 11^3 = 1331; 21^3 = 9261; 31^3 = 29791$

2. The cubes of the numbers with 1, 4, 5, 6, 9, 0 as unit digits will have the same unit digits.
For example, $14^3 = 2744$, $25^3 = 15625$, $16^3 = 4096$, $19^3 = 6859$, $20^3 = 8000$

3. The cube of numbers ending in unit digit 2 will have a unit digit 8 and the cube of the numbers ending in unit digit 8 will have a unit digit 2 (observe $8 + 2 = 10$).

4. The cube of the numbers with unit digits as 3 will have a unit digit 7 and the cube of numbers with unit digit 7 will have a unit digit 3 (observe $7 + 3 = 10$).

5. The cubes of even numbers are all even, and the cubes of odd numbers are all odd.
For example, (i) Cube of 2 is 8 (2 and 8 both are even)
(ii) Cube of 7 is 343. (7 and 343 both are odd)

6. Cubes of negative integers are negative.
Examples : (ii) $(-2)^3 = (-2) \times (-2) \times (-2) = -8$ (iv) $(-4)^3 = (-4) \times (-4) \times (-4) = -64$

7. For any rational number $\frac{a}{b}$, we have $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$

Examples : $\left(\frac{3}{5}\right)^3 = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{3 \times 3 \times 3}{5 \times 5 \times 5} = \frac{3^3}{5^3}$

$$\left(\frac{-5}{7}\right)^3 = \frac{-5}{7} \times \frac{-5}{7} \times \frac{-5}{7} = \frac{(-5) \times (-5) \times (-5)}{7 \times 7 \times 7} = \frac{(-5)^3}{7^3}$$

8. (i) 1 and 8 are the only cubes in one digit.
(ii) 27, 64 are the only cubes with 2 digits.
(iii) 125, 216, 343, 512, 729 are the only cubes in three-digit numbers.
(iv) All the above are cubes of one-digit numbers from 1 to 9.
(v) The smallest two digit number is 10 and its cube is 1000 which is the smallest 4-digit number.
(vi) The cubes of 10 to 31 have four digits and they are below 10,000.
(vii) The cubes of 32 to 46 are having five digits and they are between 10,000 and 100,000.
(viii) The cubes of 47 to 99 are having six digits between 100,000 and 1000,000.
(ix) The cubes of three-digit numbers can have 7, 8 or 9 digits.

**Sum of consecutive odd numbers.**

| | Number of odd numbers | Triangular Number series |
|---|---|---|
| $1 = 1 = 1^3$ | 1 | 1 |
| $3 + 5 = 8 = 2^3$ | 2 | $1 + 2 = 3$ |
| $7 + 9 + 11 = 27 = 3^3$ | 3 | $1 + 2 + 3 = 6$ |
| $13 + 15 + 17 + 19 = 64 = 4^3$ | 4 | $1 + 2 + 3 + 4 = 10$ |
| $21 + 23 + 25 + 27 + 29 = 125 = 5^3$ | 5 | $1 + 2 + 3 + 4 + 5 = 15$ |
| | ⋮ | ⋮ |
| | 'n' odd numbers | $\frac{n(n+1)}{2}$ |
| | | $\left(\text{Check : If } n = 5 \text{ then } \frac{5 \times 6}{2} = 15\right)$ |

**Sum of cubes of consecutive odd numbers**

$$1^3 = 1 = 1^2 = \left(\frac{1\times 2}{2}\right)^2 = 1 = 1^2$$

$$1^3 + 2^3 = 9 = (1+2)^2 = \left(\frac{2\times 3}{2}\right)^2 = 9 = 3^2$$

$$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2 = \left(\frac{3\times 4}{2}\right)^2 = 36 = 6^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2 = \left(\frac{4\times 5}{2}\right)^2 = 100 = 10^2$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = (1+2+3+4+5)^2 = \left(\frac{5\times 6}{2}\right)^2 = 225 = 15^2$$

$$1^3 + 2^3 + 3^3 + ...... + n^3 = (1+2+3+....+n)^2 = \left[\frac{n(n+1)}{2}\right]^2$$

**Example 4 :**

Find $10^3$.

**Sol.** To find $10^3$, we need to find the 10 odd numbers which are to be added to get $10^3$.

For that, first find the 9th triangular number, using the formula

$$\frac{n\,(n+1)}{2} = \frac{9\times 10}{2} = 45.$$

Now, double it and add 1

$45 \times 2 + 1 = 91.$

Thus, 91 is the first odd number.

$\therefore\ 10^3 = 91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109 = 1000$

On cross checking, we see that $10 \times 10 \times 10 = 1000$

## SELF CHECK-1

**Q.1** Find the cubes of the following numbers : (i) 4.9 (ii) $\frac{4}{5}$ (iii) $1\frac{4}{7}$ (iv) $\frac{-8}{11}$

**Q.2** Which of the following numbers are perfect cubes ?
(i) 32 (ii) 243 (iii) 1728 (iv) 15625 (v) 106480

**Q.3** Which of the following are the cubes of even natural numbers ?
(i) 343 (ii) 2197 (iii) 17576

**Q.4** What is the smallest number by which 8640 must be divided so that the quotient is a perfect cube ?

**Q.5** Find the smallest number by which 2560 must be multiplied so that the product is a perfect cube.

## ANSWERS

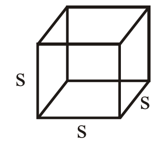
**(1)** (i) 117.649 (ii) $\frac{64}{125}$ (iii) $\frac{1331}{343}$ (iv) $\frac{-512}{1331}$

**(2)** (i) No (ii) No (iii) Yes (iv) Yes (v) No

**(3)** (i) No (ii) No (iii) Yes **(4)** 5 **(5)** 25

## CUBE ROOTS

Suppose we know that the volume of the cube shown in figure is 216 cubic inches. To find the length of each side, we substitute 216 for V in the formula $V = s^3$ and solve for s.




$V = s^3$

$216 = s^3$

To solve for s, we must find a number whose cube is 216. Since 6 is such a number, the sides of the cube are 6 inches long. The number 6 is called a cube root of 216, because $6^3 = 216$.

Here are more examples of cube roots:

* 3 is a cube root of 27, because $3^3 = 27$.
* –3 is a cube root of –27, because $(-3)^3 = -27$.
* 12 is a cube root of 1,728, because $12^3 = 1,728$.
* –12 is a cube root of –1,728, because $(-12)^3 = -1,728$.
* $\frac{1}{3}$ is a cube root of $\frac{1}{27}$, because $\left(\frac{1}{3}\right)^3 = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27}$
* $-\frac{1}{3}$ is a cube root of $-\frac{1}{27}$, because $\left(-\frac{1}{3}\right)^3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = -\frac{1}{27}$
* 0 is a cube root of 0, because $0^3 = 0$

In general, we have the following definition.

The number b is a cube root of a if $b^3 = a$.

All real numbers have one real cube root. As the preceding examples show, a positive number has a positive cube root, a negative number has a negative cube root, and the cube root of 0 is 0.

**Cube root notation :** The cube root of a is denoted by $\sqrt[3]{a}$.

By definition, $\sqrt[3]{a} = b$ if $b^3 = a$

**Cube root of a negative perfect cube :** If a is a positive integer then, –a is a negative integer.

We know that, $(-a)^3 = -a^3$. So, $\sqrt[3]{-a^3} = -a$. In general, $\sqrt[3]{-x} = -\sqrt[3]{x}$

**Cube root of product of integers :** For any two integers a and b, we have $\sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b}$

**Cube root of rational number :** For any rational number $\frac{a}{b}$, we have $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

**Example 5 :**

Find each cube root : (a) $\sqrt[3]{8}$ (b) $\sqrt[3]{343}$ (c) $\sqrt[3]{-8}$ (d) $\sqrt[3]{-125}$

**Sol.** (a) $\sqrt[3]{8} = 2$, because $2^3 = 8$

(b) $\sqrt[3]{343} = 7$, because $7^3 = 343$

(c) $\sqrt[3]{-8} = -2$, because $(-2)^3 = -8$

(d) $\sqrt[3]{-125} = -5$, because $(-5)^3 = -125$

## FINDING THE CUBE ROOT THROUGH PRIME FACTORIZATION METHOD

**Step 1 :** Resolve the given number into prime factors.
**Step 2 :** Form groups of threes of like factors.
**Step 3 :** Take out one factor from each group and multiply.

**Example 6 :**

Find the cube roots of 729.

**Sol.** Resolving 729 into prime factors, we get

| | |
|---|---|
| 3 | 729 |
| 3 | 243 |
| 3 | 81 |
| 3 | 27 |
| 3 | 9 |
| 3 | 3 |
| | 1 |

Thus, $729 = \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3}$

$\therefore \sqrt[3]{729} = 3 \times 3 = 9$

**Example 7 :**

Evaluate : $\sqrt[3]{2744}$

**Sol.** By prime factorisation, we have

| | |
|---|---|
| 2 | 2744 |
| 2 | 1372 |
| 2 | 686 |
| 7 | 343 |
| 7 | 49 |
| 7 | 7 |
| | 1 |

$2744 = 2 \times 2 \times 2 \times 7 \times 7 \times 7$
$= (2 \times 2 \times 2) \times (7 \times 7 \times 7)$

$\therefore \sqrt[3]{2744} = (2 \times 7) = 14$

## CUBE ROOT OF A CUBE NUMBER

If you know that the given number is a cube number then following method can be used.

**Step 1 :** Take any cube number say 857375 and start making groups of three digits starting from the right most digit of the number.

| $\underline{857}$ | $\underline{375}$ |
|---|---|
| ↓ | ↓ |
| second group | first group |

We can estimate the cube root of a given cube number through a step by step process.
We get 375 and 857 as two groups of three digits each.

**Step 2 :** First group, i.e., 375 will give you the one's (or unit's) digit of the required cube root.
The number 375 ends with 5. We know that 5 comes at the unit's place of a number only when it's cube root ends in 5. So, we get 5 at the unit's place of the cube root.

**Step 3 :** Now take another group, i.e., 857.
We know that $9^3 = 729$ and $10^3 = 1000$. Also, $729 < 857 < 1000$.
We take the one's place, of the smaller number 729 as the ten's place of the required cube root.

So, we get $\sqrt[3]{857375} = 95$.

**Example 12 :**

Find the cube-root of 1,85,193.

**Sol.** (1) Form the groups of three digits from the right of the given number as $\overline{185}\ \overline{193}$.

(2) The group on the left, i.e., 185 lies between $5^3$ (125) and $6^3$ (216), i.e., $5^3 < 185 < 6^3$.
Hence, the tens digit of the cube-root is 5.

(3) The group on the right, i.e., 193 ends with 3. We know that if a perfect cube ends with 3, the cube-root ends with 7. Hence, the units digit of the cube-root is 7. Thus, the required cube-root is 57, i.e., $\sqrt[3]{1,85,193} = 57$.

(4) Verify that $(57)^3 = 57 \times 57 \times 57 = 1,85,193$

**Example 13 :**

Find the cube root of $\frac{12167}{68921}$

**Sol.** Find $\sqrt[3]{\frac{12167}{68921}} = \frac{\sqrt[3]{12167}}{\sqrt[3]{68921}}$

(a) $\overline{12}\,\overline{167}$ : As 7 is the unit digit of the cube, 3 will be the unit digit of cube root.

(b) 8 is the biggest cube below 12. ($8 = 2^3$) so cube root of 12167 is 23.

(c) $\overline{68}\,\overline{921}$ : As 1 is the unit digit of the cube, 1 will be the unit digit of the cube root.

(d) 64 is the largest cube below 68 ($64 = 4^3$)

(e) So the cube root of 68921 is 41.

$$\therefore \sqrt[3]{\frac{12167}{68921}} = \frac{\sqrt[3]{12167}}{\sqrt[3]{68921}} = \frac{23}{41}$$

**CUBE ROOT THROUGH PATTERN**

$1^3 = 1$
$2^3 - 1^3 = 8 - 1 = 7$
$3^3 - 2^3 = 27 - 8 = 19$
$4^3 - 3^3 = 64 - 27 = 37$
$5^3 - 4^3 = 125 - 64 = 61$
$6^3 - 5^3 = 216 - 125 = 91$
$7^3 - 6^3 = 343 - 216 = 127$

Also,

$$1 = 1^3$$
$$1 + 7 = 2^3$$
$$1 + 7 + 19 = 3^3$$
$$1 + 7 + 19 + 37 = 4^3$$
$$1 + 7 + 19 + 37 + 61 = 5^3$$
$$1 + 7 + 19 + 37 + 61 + 91 = 6^3$$

Thus, to find the cube root of a perfect cube, we subtract from it successively 1, 7, 19, 27, 61, 91, 127, ....
The number of times we have to perform subtraction to arrive at 0 gives the cube root of the given number.
For example,

$343 - 1 = 342$, $342 - 7 = 335$, $335 - 19 = 316$, $316 - 37 = 279$, $279 - 61 = 218$,
$218 - 91 = 127$, $127 - 127 = 0$

Since, we have subtracted seven times to get 0.

Hence, $\sqrt[3]{343} = 7$

This method is suitable for finding the cube roots of small numbers.

**SHORT-CUT METHOD FOR FINDING THE CUBE OF A TWO DIGIT NUMBER**

To find the cube of a 2-digit number ab, where a is the digit and b is the unit digit, we use the identity :

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

We make make four columns and write $a^3$, $3a^2b$, $3ab^2$ and $b^3$ respectively in these columns.

**Example 8 :**

Find the value of $(29)^3$ by the short-cut method.

**Sol.** Here, a = 2 and b = 9.

| | | | |
|---|---|---|---|
| $a^2$ | $a^2$ | $b^2$ | $b^2$ |
| $\times a$ | $\times 3b$ | $\times 3a$ | $\times b$ |
| $a^3$ | $3a^2 \times b$ | $3a \times b^2$ | $b^3$ |

| | | | |
|---|---|---|---|
| 4 | 4 | 81 | 81 |
| × 2 | × 27 | × 6 | × 9 |
| 8 | 108 | 486 | 72$\underline{9}$ |
| +16 | +55 | +72 | |
| 2$\underline{4}$ | 16$\underline{3}$ | 55$\underline{8}$ | |

$\therefore\ (29)^3 = 24389$

## SELF CHECK-2

**Q.1** Evaluate : (i) $\sqrt[3]{343}$ (ii) $\sqrt[3]{-1331}$ (iii) $\sqrt[3]{\dfrac{-64}{343}}$

**Q.2** Show that : $\sqrt[3]{343} \times \sqrt[3]{-2744} = \sqrt[3]{343 \times (-2744)}$

**Q.3** Evaluate : $\sqrt[3]{700 \times 5 \times 2 \times 49}$

## ANSWERS

**(1)** (i) 7 (ii) –11 (iii) $\dfrac{-4}{7}$ **(3)** 70

## USEFUL TIPS

**1.** Numbers like 1729, 4104, 13832, are known as Hardy – Ramanujan Numbers. They can be expressed as sum of two cubes in two different ways.

**2.** Numbers obtained when a number is multiplied by itself three times are known as cube numbers. For example 1, 8, 27, ... etc.

**3.** If in the prime factorisation of any number each factor appears three times, then the number is a perfect cube.

**4.** In a cube number, every group of three digits from the right represents 1 digit of its cube root. The left-most group may have one, two or three digits. But it represents the left-most digit in the cube root.

**5.** (a) Cubes of all odd natural numbers are odd.
(b) Cubes of all even natural numbers are even.

**6.** Cube root of a negative number is negative, i.e. $\sqrt[3]{-a^3} = -a$

**7.** Cube root of product of two integers : $\sqrt[3]{ab} = \sqrt[3]{a}.\sqrt[3]{b}$

**8.** Cube root of a rational number : $\sqrt[3]{\dfrac{a}{b}} = \dfrac{\sqrt[3]{a}}{\sqrt[3]{b}}\ (b \neq 0)$