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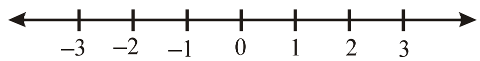
NUMBER SYSTEMS

INTRODUCTION

In earlier classes, you have learnt addition, subtraction, multiplication and division with the help of number line. Let us first review that number line.



To obtain (+10) + (+5) start from (+10) and count 5 points to the right you come to +15.

$\therefore$ (+10) + (+5) = +15

To obtain (+7) – (+4) start from (+7) and count 4 points to the left you come to +3.

$\therefore$ (+7) – (+4) = 3

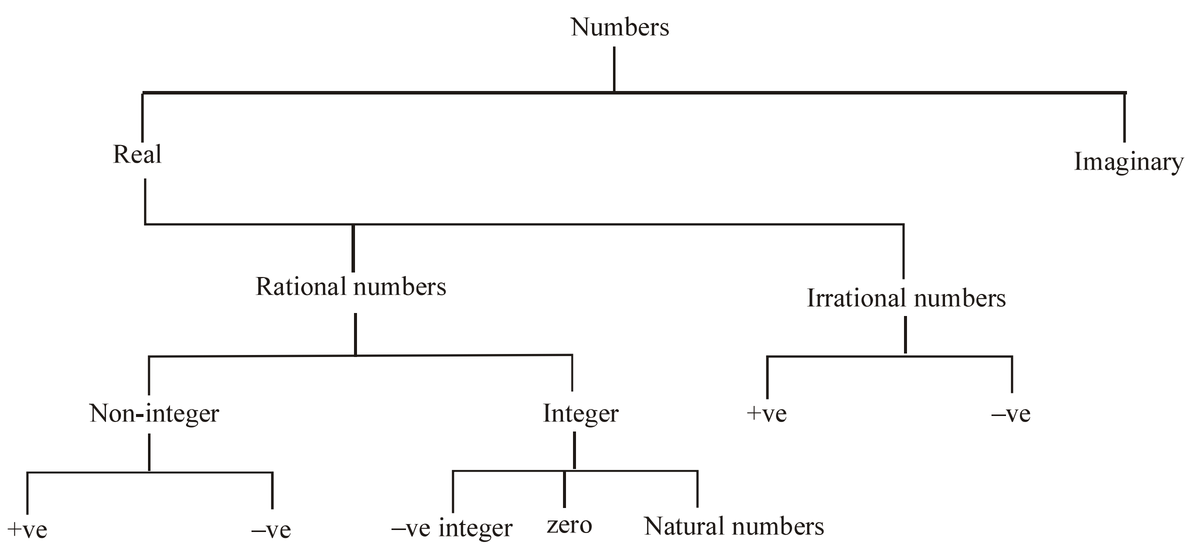
and similarly you can multiply and divide simple numbers but now let say we wish to calculate – 8 ÷ 3 it will be not be possible to operate on a line that contains only integers (– ∞,......–3, –2, –1, 0, 1, 2, 3.......∞) therefore we need a new system in which we should be able carry out all types of multiplication, division and hence rational number, irrational number were created.

Number tree :

Numbers are basically of 2 types : Real no. and Imaginary no.

Real Numbers :

These are the numbers which can represent actual physical quantities in a meaningful way. These can be represented on the number line. Number line is geometrical straight line with arbitrarily define zero (origin).



NATURAL NUMBERS

Set of all non-fractional numbers from 1 to N = {1, 2, 3, 4.............}.

If we add 1 to any natural number, we get its successor. Thus, every natural number has its successor. Consequently, the set of all natural numbers is an infinite set, given by N = {1, 2, 3, 4, 5, ...}

Clearly, the sum of two natural numbers is always a natural number. We express this property by saying that the set N of all natural numbers is closed for addition.
However, if we subtract a natural number from another natural number, the result is not always a natural number. e.g. $3 \in N. 7 \in N$ but $(3-7) \notin N$. Thus, the set N is not closed for subtraction.

Even natural numbers : Natural numbers which are exactly divisible by 2 are called even natural numbers.
E = {2, 4, 6, 8, 10, 12, ...} is the set of all even natural numbers.

Odd natural numbers : Natural numbers which are not even, are called odd natural numbers.
O = {1, 3, 5, 7, 9, 11,...........} is the set of all odd natural numbers.

Whole numbers : If 0 (zero) is adjoined to the set N, we obtain a new set W = {0, 1, 2, 3, 4, 5, ...} called the set of whole numbers. Clearly, every natural number is a whole number.

Integers : All natural numbers, negatives of natural numbers and 0, together form the set Z or I of all integers.
$Z = \{........., -3, -2, -1, 0, 1, 2, 3,..........\} = N \cup \{0\} \cup N^-$
$Z^+ = \{1, 2, 3, 4, 5, 6, ...\} = N$ is the set of all positive integers.
$Z^- = \{-1, -2, -3, -4, ...\}$ is the set of all negative integers.

Prime numbers : All natural numbers that have one & itself as their factors are prime numbers i.e. prime numbers are exactly by 1 and themselves Eg. 2, 3, 5, 7, 11, 13, 19, 23 identification of prime number.
Step 1 : Find approximate square root of given no.
Step 2 : Divide the given no. by prime numbers less than approximately square root of number. If given number is not divisible by any of these prime numbers then the number is prime otherwise not eg. : 571.
Sol. : Approximately square root = 24
Prime nos. < 24 are 2, 3, 5, 7, 11, 13, 17, 19 & 23. 571 is not divisible by any of these prime nos. so 571 is a prime number.
Composite numbers : Natural numbers having more than two factors are called composite numbers.
e.g. 4, 6, 8, 9, 10, 12, 14, 15, 16, etc. Note that 1 is neither prime nor composite.

Twin Primes : Prime numbers differing by 2 are called twin primes. e.g., 5 and 7, 11 and 13, 17 and 19 etc.

Prime Triplet : The set {3, 5, 7} of three consecutive primes is called the prime triplet.

Co-primes : Every pair of two natural numbers having no common factor, other than 1, is called a pair of co-primers. e.g., (5, 6), (6, 11), (14, 17), (16, 21), (20, 27), etc.

RATIONAL NUMBERS

These are real numbers which can be expressed in the form of p/q where p and q are integers and $q \neq 0$
eg. 2/3, 37/15, –17/19.
All natural numbers, whole numbers and integers are rational.

Fractions :

Common fraction	:	Fractions whose denominator is not 10.
Decimal fraction	:	Fractions whose denominator is 10 or any power of 10.
Proper fraction	:	Numerator < Denominator
Improper fraction	:	Numerator > Denominator, mixed fraction.

Examples :

(i) 0 can be written as $\frac{0}{1}$, which is rational. $\therefore$ 0 is a rational number.

(ii) Every integer a can be written as $\frac{a}{1}$, which is rational.

$\therefore$ Every integer is a rational number.

(iii) The square root of every perfect square number is rational.

e.g., $\sqrt{4}=2$, which is rational similarly, $\sqrt{9}, \sqrt{16}, \sqrt{25}$ etc. are all rational.

(iv) Every terminating decimal is a rational number.

e.g., $0.7=\frac{7}{10}$, which is rational ; $0.375=\frac{375}{1000}$, which is rational

(v) Every recurring decimal is a rational number

Let us consider the recurring decimal 0.333 ....

Let $x = 0.3333$ .... (1)

Then, $10x = 3.3333$ .... (2)

On subtracting (1) from (2), we get, $9x = 3 \Leftrightarrow x = \frac{3}{9} = \frac{1}{3}$ $\therefore$ 0.333......... $= \frac{1}{3}$, which is rational.

Irrational number : Every non-terminating and non-repeating decimal number is known as an irrational number e.g. 0.101001000100001....

Example : $\sqrt{2}, \sqrt{5}, \pi$ etc.

Properties of Rational numbers : If a, b, c are three rational numbers.

(i) Communicative property of addition : $a + b = b + a$

(ii) Associative property of addition : $(a + b) + c = a + (b + c)$

(iii) Additive inverse : $a + (-a) = 0$

0 is identity element, – a is called inverse of a.

(iv) Communicative property of multiplication $a \cdot b = b \cdot a$

(v) Associative property of multiplication $(a.b).c = a.(b.c)$

(vi) Multiplication inverse $a. 1/a = 1$

1 is called multiplication identity & 1/a is called multiplicative inverse of a or reciprocal of a.

(vii) Distributive property : $a.(b + c) = a.b + a.c$

Operations on rational numbers :

For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we have

(i) Addition : $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ **(ii) Subtraction :** $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

(iii) Multiplication : $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ **(iv) Division :** $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$, when $c \neq 0$

Density property of Rational numbers :

Between any two different rational numbers, there are infinitely many rational numbers.

To find many rational numbers between two given distinct rational numbers.

Method : Let the given rational numbers be a and b.

Then $q_1 = \frac{1}{2}(a+b)$, $q_2 = \frac{1}{2}(q_1+b)$, $q_3 = \frac{1}{2}(q_2+b)$, $q_4 = \frac{1}{2}(q_3+b)$, and so on.

In this manner we can find as many rational numbers as we please between two given distinct rational numbers.

Alternatively to find n rational numbers between a and b.

1st find $d = \frac{b-a}{n+1}$

1st rational number will be a + d, 2nd rational number will be a + 2d, 3rd rational number will be a + 3d.
and so on nth will be a + nd.

Example 1 :

Insert three rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$.

Sol. A rational number between $\frac{1}{3}$ and $\frac{1}{2}$ is $\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{12}$, We have, $\frac{1}{3} < \frac{5}{12} < \frac{1}{2}$

A rational number between $\frac{1}{3}$ and $\frac{5}{12}$ is $\frac{1}{2}\left(\frac{1}{3}+\frac{5}{12}\right) = \frac{3}{8}$

A rational number between $\frac{5}{12}$ and $\frac{1}{2}$ is $\frac{1}{2}\left(\frac{5}{12}+\frac{1}{2}\right) = \frac{11}{24}$. Clearly, we have $\frac{1}{3} < \frac{3}{8} < \frac{5}{12} < \frac{11}{24} < \frac{1}{2}$

Hence, three rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$ are $\frac{3}{8}, \frac{5}{12}$ and $\frac{11}{24}$.

Example 2 :

Find three rational no's between a and b (a < b).

Sol. $a < b$; $a + a < b + a$; $2a < a + b$

$a < \frac{a+b}{2}$. Again, $a < b$; $a + b < b + b$; $a + b < 2b = \frac{a+b}{2} < b$ $\quad \therefore\ a < \frac{a+b}{2} < b$

Hence 1st rational no. between a and b is $\frac{a+b}{2}$.

for next rational number $\frac{a+\frac{a+b}{2}}{2} = \frac{\frac{2a+a+b}{2}}{2} = \frac{3a+b}{4}$ $\quad \therefore \quad a < \frac{3a+b}{4} < \frac{a+b}{2} < b$

next $\frac{\frac{a+b}{2}+b}{2} = \frac{a+b+2b}{2\times 2} = \frac{a+3b}{4}$ $\quad \therefore \quad a < \frac{a+3b}{4} < \frac{a+b}{2} < \frac{a+3b}{4} < b$

Example 3 :

Find 7 rational numbers between 1/3 and 1/2.

Sol. **step 1 :** Gap = $\frac{1}{2}-\frac{1}{3}=\frac{3-2}{6}=\frac{1}{6}$

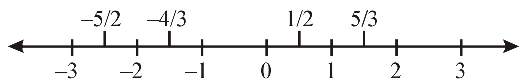
step 2 : Dividing by (7 + 1), i.e., 8 we get $\frac{1}{6}\div 8=\frac{1}{6}\times\frac{1}{8}=\frac{1}{48}$

step 3 : Then the seven rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$ are $\frac{1}{3}+1\times\frac{1}{48}$

$\frac{1}{3}+2\times\frac{1}{48},\frac{1}{3}+3\times\frac{1}{48},\frac{1}{3}+4\times\frac{1}{48},\frac{1}{3}+5\times\frac{1}{48},\frac{1}{3}+6\times\frac{1}{48},\frac{1}{3}+7\times\frac{1}{48}$ i.e., $\frac{17}{48},\frac{18}{48},\frac{19}{48},\frac{20}{48},\frac{21}{48},\frac{22}{48},\frac{23}{48}$

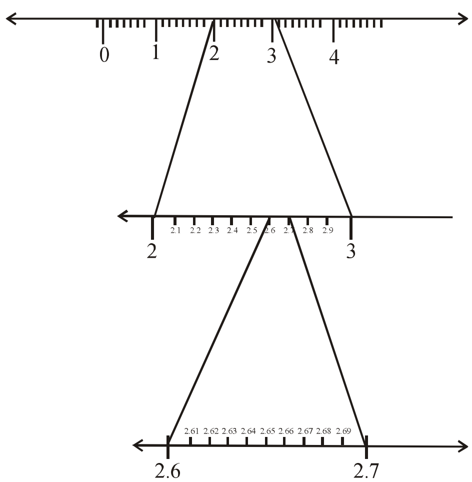
Representation of rational number on a number line :

Rational numbers can be represented by points on the number line.



Number Line

Even we can represent terminating decimal fraction as 2.65 (process of magnification)

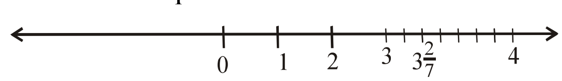


Example 4 :

Represent $3\frac{2}{7}$ on the number line.

Sol. In order to represent $3\frac{2}{7}$ on the number line, take 3 unit lengths between 0 and 3 and divide the unit length between 3 and 4 into seven equal parts and take the end of 2nd part on it.

This point represents the rational number $3\frac{2}{7}$.



Decimal representation of rational numbers :

Every rational number has a property that it when expressed in decimal form can be represented either in the form of terminating decimals e.g. $\frac{1}{5}=0.2$, $\frac{1}{2}=0.5$, $\frac{3}{4}=0.75$ etc. or in repeating (or recurring) decimals, e.g.

$\frac{1}{3} = 0.3333....,\ \frac{7}{6} = 1.16666....$ etc.

If the repeating decimals consist of only one repeating digit, we express it by putting a dot on the repeating digit, e.g. $\frac{7}{6} = 1.16666.... = 1.1\dot{6}$ $= 0.454545$ .... $= 0.\overline{45}$

If however, the number of digits in the repeating part is more than one, we put a dot on the first digit as well as on the last digit of the repeating part e.g. $\frac{13}{7} = 1.857142857142........ = 1.\dot{8}5714\dot{2}$

We may also express it by putting a line on the repeating part. This line is called vinculum.

e.g., $\frac{13}{7} = 1.857142857142........ = 1.\overline{857142}$

Conversely, every terminating decimal as well as every repeating decimal can be converted into the form (p/q), where p and q are integers and $q \neq 0$.

Conversion of Recurring decimal into a fraction :

(1) Long method :

Step 1 : Take the mixed recurring decimal and put it equal to x.

Step 2 : Count the number of non-recurring digits after the decimal point. Let it be n.

Step 3 : Multiply both sides of x by 1n so that only the repeating decimal is on the right hand side of the decimal point.

Step 4 : Multiply both sides of x by 10^{n+m} where m is the number of repeating digits in the decimal part.

Step 5 : Subtract the number in step 3 from number in step 4.

Step 6 : Divide both sides of the resulting equation by the coefficient of x

Step 7 : Write the rational number thus obtained in the simplest $\frac{p}{q}$ form.

Example 5 :

Express $0.\overline{47}$ to $\frac{p}{q}$ form.

Sol. $0.\overline{47}$. Let $x = 0.\overline{47}$ i.e. $x = 0.474747$ .... (1)

Multiply both sides of (1) by 100.

$100x = 47.474747$ .... (2)

Subtract (1) from (2)

$$100x = 47.474747$$
$$x = 0.474747$$
$$99x = 47 \Rightarrow x = \frac{47}{99}$$

(2) Simple method :

Step 1 : To obtain numerator subtract the number formed by non-repeating digits from the complete number after decimal. (Consider repeated digits only once)

Step 2 : To obtain denominator take number of 9's = No. of repeating digits & after than put no. of 0's = no. of non-repeating digits.

Ex. (i) $0.\overline{45} = \frac{45-0}{99} = \frac{45}{99} = \frac{5}{11}$ (ii) $0.46\overline{573} = \frac{46573-46}{99900} = \frac{46527}{99900}$

Division : General representation of result $0.46\overline{573} = \frac{46573-46}{99900} = \frac{46527}{99900}$

Divident = (Divisor × Quotient) + Remainder

Divisible test :

Number	**Test**
2	Unit digit should be 0 or even
3	The sum of digits of no. should be divisible by 3
4	The no formed by last 2 digits of given no. should be divisible by 4.
5	Unit digit should be 0 or 5.
6	No should be divisible by 2 and 3 both.
8	The number formed by last 3 digits of given no. should be divisible by 8.
9	Sum of digits of given no. should be divisible by 9
11	The difference between sums of the digits at even and at odd places should be zero or multiple of 11.
25	Last 2 digits of the number should be 00, 25, 50 or 75.

Rule for 7 :

Eg. : 167/13, Divisor → 13) 167 (12 ← Quotient

$\frac{13}{27}\frac{26}{11}$ ← Remainder

Double the last digit of given number & subtract from previous no. The result should be zero or by 17.

Eg. : 413 Last digit = 3, previous no. = 41

41 – (3 × 2) = 35 (divisible by 7) i.e. 413 is divisible by 7.

This rule can also be used for nos. having more than 3 digits.

Eg. : 126 Last digit = 6 previous no. = 12

12 – (6 × 2) = 0 i.e., 126 is divisible by 7

Eg. 6545 Last digit = 5, previous no. 654

654 – (5 × 2) = 644, 64 – (4 × 2) = 56 divisible by 7 i.e. 6545 is divisible by 7

Rule for 13 : Four times the last digit and add to previous no. the result should be divisible by 13.

Eg. : 234 (4 × 4) + 23 = 39 (divisible by 13) i.e. 234 is divisible by 13

Rule for 17 :

Five times the last digit of the no. and subtract from previous no. the result should be either 0 or divisible by 17.

Eg. : 357 (7 × 5) – 35 = 0 i.e. 357 is divisible by 17.

Rule for 19 :

Double the last digit of given no. and add to previous no. The result obtained should be divisible by 19.

Eg. : 589 (9 × 2) + 58 = 76 (divisible by 19) i.e. the no. is divisible by 19.

Rule : A rational number has the terminating decimal representation, , if its denominator has 2's or 5's or both as factors. **Ex. :** In $\frac{705}{304}$, denominator = 304 = 2 × 2 × 2 × 2 × 19

Thus the denominator has a factor other than 2's and 5's.

So, $\frac{705}{304}$ does not have the terminating decimal representation.

<u>SELF CHECK</u>

Q.1 Write the natural number whose successor is 5.

Q.2 Express each of the following decimals as a rational number in the simplest p/q form.

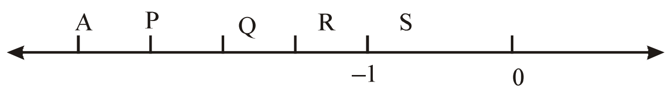
(i) 0.222................ (ii) $0.\overline{142857}$

Q.3 Insert a 4 rational numbers between – 5 and 2.

Q.4 Find five rational number between 3/5 and 4/5

Q.5 Every integer is a rational number. (True/False)

Q.6 From the adjoining number line, write down three rational numbers P, Q, R between –1 and –2, such that AP = PQ = QR = RB.



Q.7 Write down three rational number, P, Q and R between 1 and 2 such that PQ = QR, is the solution unique?

Q.8 Find three rational numbers between 0 and 0.2

Q.9 Express 12/125 as decimal fraction.

Q.10 Write the terminating decimal numeral for the given rational number.

(i) $\frac{29}{50}$ **(ii)** $\frac{123}{4000}$

<u>ANSWERS</u>

(1) 4 **(2)** $\frac{2}{9}$ **(3)** $\frac{1}{7}$ **(4)** $\frac{31}{50}, \frac{32}{50}, \frac{33}{50}, \frac{34}{50}, \frac{35}{50}$

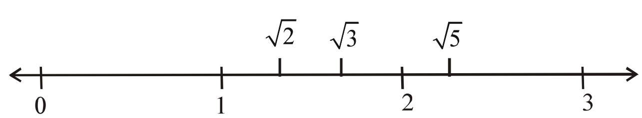
(5) True **(6) –1.25, –1.50, –1.75, Yes** **(7) 1.25, 1.50, 1.75, No**

(8) 0.1, 0.05, 0.15 **(9) 0.096** **(10) (i) 0.6** **(ii) 0.53125**

IRRATIONAL NUMBERS

Irrational numbers are represented by non-terminating non-repeating decimals.

In other word, numbers, which are not rational but which can be represented by points on the number line with the rational numbers, are called the Irrational number.

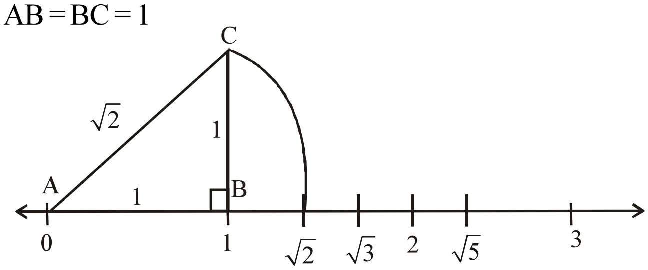


For example, To understand the existence of irrational numbers on the number line, let us study the adjoining diagram, in which AB = BC = 1

$\therefore$ $AC^2 = AB^2 + BC^2$

$\therefore$ $AC^2 = 1^2 + 1^2 = 1 + 1 = 2$

$\therefore$ $AC = \sqrt{2}$



But as shown in the diagram

$AC = AP = \sqrt{2}$. Therefore, $AP = \sqrt{2}$

Thus, the length of the line segment AP is $\sqrt{2}$ which is an irrational number and P lies on the number line.

If we find the square root of 2 by the division method, you will find that $\sqrt{2}$ can be represented by the non-repeating decimal 1.4142143...... and so it is irrational.

Irrational numbers occur as the square roots of certain number. They also result from many other mathematical processes. For example the ratio of the circumference of any circle to its diameter is the irrational number $\pi = 3.14159$.............

Properties of Irrational number :

(i) Negative of an irrational number is an irrational number. Eg. $-\sqrt{3}, -\sqrt[4]{5}$ are irrational.

(ii) Sum and difference of a rational and an irrational number is an irrational number.

(iii) Sum and difference of two irrational numbers is not necessarily an irrational number.

Example : (i) Two no's are 4 and $\sqrt[3]{3}$.

Sum = $4 + \sqrt[3]{3}$, is an irrational number. Difference = $4 - \sqrt[3]{3}$, is an irrational number.

(ii) Two irrational numbers are $\sqrt{3}, 2\sqrt{3}$

Sum = $\sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$, is an irrational. Difference = $2\sqrt{3} - \sqrt{3} = \sqrt{3}$, is an irrational.

(iii) 2 is a rational and $\sqrt{3}$ an irrational. $2 \times \sqrt{3} = 2\sqrt{3}$

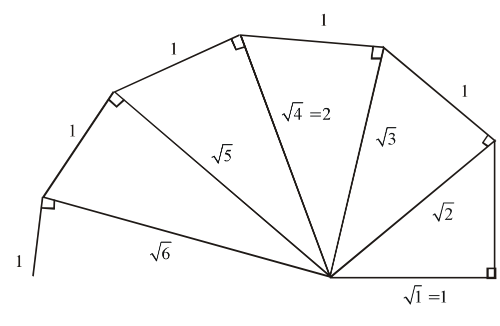
(iv) $\frac{4}{3} \times \sqrt{3} = \frac{4}{3}\sqrt{3} = \frac{4}{\sqrt{3}}$ is an irrational.

(v) $2\sqrt{3} \times 3\sqrt{2} = 2 \times 3\sqrt{3 \times 2} = 6\sqrt{6}$ an irrational number.

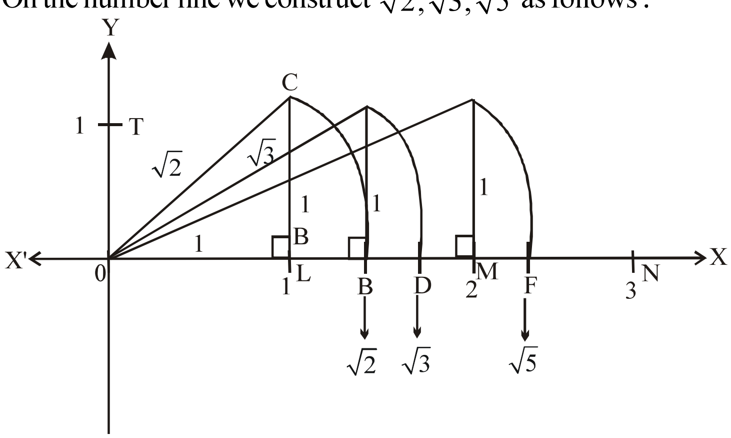
(vi) $(2+\sqrt{3})\,(2-\sqrt{3}) = (2)^2 - (\sqrt{3})^2 + 2(2) \times (\sqrt{3}) = 4 + 3 + 4\sqrt{3} = 7 + 4\sqrt{3}$

The Wheel of Theodorus :

Theodorus of Cyrene, discovered the construction of irrational numbers that occur due to square roots of number.



On the number line we construct $\sqrt{2}, \sqrt{3}, \sqrt{5}$ as follows :



Let the number line be X'OX. O represent zero, L represent 1 unit, M represents 2 unit, and N represents 3 units. Thus, OL = 1, OM = 2, ON = 3.

Let OY be perpendicular to OX at O. Let OT = 1 unit. LA is perpendicular to OL, and LA = 1 unit. Join OA.

By Pythagoras theorem, $OA = \sqrt{OL^2 + LA^2} = \sqrt{1+1} = \sqrt{2}$ with O as centre and OA as radius draw on arc cutting the number line OX at B. Thus, $OB = OA = \sqrt{2}$.

At B draw the perpendicular BC equal to 1 unit. Join OC

By Pythagoras theorem, $OC = \sqrt{OB^2 + BC^2} = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{2+1} = \sqrt{3}$. With O as centre and OC as radius draw on arc cutting the number line OX at D. Thus, $OD = OC = \sqrt{3}$.

At M draw the perpendicular ME equal to 1 unit.

Join ME. By Pythagoras Theorem, $OE = \sqrt{OM^2 + ME^2} = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$

With O as centre and OE as radius draw an arc cutting the number line OX at F.

Thus, OF = OE = $\sqrt{5}$.

Hence, $OB = \sqrt{2},\ OD = \sqrt{3},\quad OF = \sqrt{5}$

Example 6 :

Prove that $(\sqrt{2} + \sqrt{3})$ is irrational.

Sol. If possible, let $(\sqrt{2} + \sqrt{3})$ be rational.

Then, $(\sqrt{2} + \sqrt{3})$ is rational $\Rightarrow (\sqrt{2} + \sqrt{3})^2$ is rational

$$\Rightarrow (2 + 3 + 2\sqrt{2} \times \sqrt{3}) \text{ is rational} \Rightarrow 5 + 2\sqrt{6} \text{ is rational}$$

But, $\sqrt{6}$ is irrational $2\sqrt{6} \Rightarrow 5 + 2\sqrt{6}$ is irrational

Thus, we arrive at a contradiction. Hence $(\sqrt{2} + \sqrt{3})$ is irrational.

Example 7 :

Find two irrational numbers between 2 and 2.5

Sol. Given two distinct positive rational numbers a and b such that ab is not a perfect square, then the irrational number $\sqrt{ab}$ lies between a and b.

Irrational number between 2 and 2.5 = $\sqrt{2 \times 2.5} = \sqrt{5}$

$$2 < \sqrt{5} < 2.5$$

Irrational number between 2 and $\sqrt{5}$ = $\sqrt{2 \times \sqrt{5}} = 2^{1/2}.5^{1/4}$.

Hence, $2 < 2^{1/2}.5^{1/4} < \sqrt{5} < 2.5$

Two irrational numbers between 2 and 2.5 are $\sqrt{2}.\sqrt[4]{5}$ and $\sqrt{5}$

Example 8 :

Find two irrational number between $\sqrt{2}$ and $\sqrt{3}$

Sol. **1st method :** $\sqrt{\sqrt{2}\times\sqrt{3}} = \sqrt{\sqrt{6}} = \sqrt[4]{6}$

Irrational number between $\sqrt{2}$ and $\sqrt[4]{6}$; $\sqrt{\sqrt{2}\times\sqrt[4]{6}} = \sqrt[4]{2}\times\sqrt[8]{6}$

2nd method : As, $\sqrt{2} = 1.414213562.........$ & $\sqrt{3} = 1.732050808.........$

As $\sqrt{3} > \sqrt{2}$ and $\sqrt{2}$ has 4 in the 1st place of decimal while $\sqrt{3}$ has 7 in the 1st place of decimal.

$\therefore$ 1.501001000100001.........., 1.601001000100001.............

Example 9 :

Find two rational number between 0.2323323332........... and 0.252552555255552........

Sol. 1st place is same 2, 2nd place is 3 and 5, 3rd place is 2 in both., 4th place is 3 and 5.
Let a no. = 0.25 , it falls between the two irrational number. Also a number = 0.2525

SELF CHECK

Q.1 Prove that : (i) $(\sqrt{3}+\sqrt{5})$ is an irrational number. (ii) $(\sqrt{3}-\sqrt{2})$ is an irrational number.

Q.2 Give two rational numbers lying between the numbers
a = 0.515115111511115.................... and b = 0.53533353335...............

Q.3 Classify the following numbers as rational or irrational :

(i) $\sqrt{23}$ (ii) 0.3796 (iii) 1.101001000100001.........

Q.4 Check whether $7\sqrt{5}, \frac{7}{\sqrt{5}}, \sqrt{2}+21, \pi - 2$ are irrational number or not.

ANSWERS

(2) 0.5152, 0.532 **(3) (i) irrational** **(ii) rational** **(iii) irrational**
(4) irrational number

RATIONALISATION

If a given number is transformed into n equivalent form, such that the denominator is rational number then the process is known as Rationalisation. **For example :**

(i) $\frac{1}{\sqrt{3}}$ is equivalent to $\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{(\sqrt{3})^2} = \frac{\sqrt{3}}{3}$ (ii) $\frac{\sqrt{5}}{\sqrt{6}}$ is equivalent to $\frac{\sqrt{5}}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{5\times 6}}{(\sqrt{6})^2} = \frac{\sqrt{30}}{6}$

Rationalising factor : We know, $(a + b)(a - b) = a^2 - b^2$

$\therefore (\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2 = 5 - 2 = 3$, a rational number.

Definition : When the product of two irrational expressions is rational, then each is said to be the rationalising factor of the other.

For example : $(\sqrt{2}-1)$ and $(\sqrt{2}+1)$ are rationalising factors of each other, since

$(\sqrt{2}-1)(\sqrt{2}+1) = (\sqrt{2})^2 - 1 = 2 - 1 = 1$ [a rational number]

Similarly, $(\sqrt{3}-\sqrt{2})$ and $(\sqrt{3}+\sqrt{2})$ are rationalising factors of each other.

Following identities are useful in rationalisation– Let a and b be positive real numbers. Then

(i) $\sqrt{ab}=\sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ (iii) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$

(iv) $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$ (v) $(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d})=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $(\sqrt{a}+\sqrt{b})^2=a+2\sqrt{ab}+b$

Example 10 :

Rationalise the denominator of $\frac{5}{\sqrt{3}-\sqrt{5}}$

Sol. $\frac{5}{\sqrt{3}-\sqrt{5}}=\frac{5}{\sqrt{3}-\sqrt{5}}\times\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}}=\frac{5(\sqrt{3}+\sqrt{5})}{3-5}=\left(-\frac{5}{2}\right)(\sqrt{3}+\sqrt{5})$

Example 11 :

Rationalise the denominator of $\frac{a^2}{\sqrt{a^2+b^2}+b}$

Sol. $\frac{a^2}{\sqrt{a^2+b^2}+b}+\frac{\sqrt{a^2+b^2}-b}{\sqrt{a^2+b^2}-b}=\frac{a^2\left(\sqrt{a^2+b^2}-b\right)}{\left(\sqrt{a^2+b^2}\right)^2-(b)^2}=\frac{a^2\left(\sqrt{a^2+b^2}-b\right)}{a^2+b^2-b^2}=(\sqrt{a^2+b^2}-b)$

Note : (i) $\sqrt{-2}\neq-\sqrt{2}$, is not an irrational number. **(ii)** $\sqrt{-2}\times\sqrt{-3}\neq(\sqrt{-2\times-3}=\sqrt{6})$

Instead $\sqrt{-2},\sqrt{-3},\sqrt[3]{-2}$ are called imaginary numbers.

$\sqrt{-2}=i\sqrt{2}$, where i (= iota) = $\sqrt{-1}$

$\therefore$ $i^2=-1$ & $i^3=i^2\times i=(-1)\times i=-i$. Also $i^4=i^2\times i^2=(-1)\times(-1)=1$

SURDS

Any irrational number of the form $\sqrt[n]{a}$ is given a special name –Surd. Where 'a' is called radicand, it should always be a rational number. Also the symbol $\sqrt[n]{\ }$ is called the radical sign and the index n is called order of the surd. $\sqrt[n]{a}$ is read as nth root of 'a' and can also be written as $a^{1/n}$. Example : $\sqrt[3]{4}$, $(2+\sqrt{3})$, $\sqrt[3]{\sqrt{3}}$ etc.

Types of surd : Quadratic surd : Surd of order 2. eg. : $\sqrt{2},\sqrt{13},\sqrt{15}$.........

Cubic surd : surd of order 3. Eg. : $\sqrt[3]{3},\sqrt[3]{15}$

Biquadratic surd : Surd of order 4. Eg. $\sqrt[4]{8}$

Simplest form of a surd : Eg. (i) $\sqrt[3]{135}$ it's simplest form is $3\sqrt[3]{5}$ (ii) $\sqrt[4]{1875}$ it's simplest form is $5\sqrt[4]{3}$

So a surd is in simplest form if it has **(1)** no fraction under the radical sign **(2)** no factor with n^{th} power of rational under the radical sign of index n **(3)** order of surd is smallest possible.

Eg. $\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt{2}$ simplest form

Mixed and Pure surd :

Pure surd : A surd which has unity only as its rational factor, the other factor being irrational, is called pure surd.

Eg. : $\sqrt{3}, \sqrt{15}, \sqrt{2}, \sqrt[4]{3}, \sqrt[3]{135}, \sqrt[4]{1875}, \sqrt[6]{8}$

Mixed surd : A surd which as a factor other than unity, the other factor being irrational, is called a mixed surd.

Eg. : $2\sqrt{3}, 2\sqrt[3]{5}, 3\sqrt[3]{5}, 5\sqrt[4]{3}$

Like and Unlike surds : Two or more surds are called like if they have or can be reduced to have the same irrational or surd factor. Eg. : $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}$ are like

Two or more surds are called unlike, if they are not similar (i.e., radicand as well as index are different)

Eg. : $\sqrt{5}, \sqrt{3}, 2\sqrt[3]{4}, \sqrt[4]{6}$

Equiradical and Non-equiradical surds :

Surds of the same order are called equiradical surds. Eg. : $\sqrt{2}, \sqrt{a}, \sqrt{5}$

Surds of different orders are known as non-equiradical surds. Eg. : $\sqrt{2}, \sqrt[3]{4}, \sqrt[4]{a}$

Simple surds and compound surds :

A surd consisting of a single term is called a simple surd and an algebraic sum of two or more surds is called a compound surd. Thus $\sqrt{3}, 3\sqrt[4]{5}, \frac{7}{2}\sqrt[3]{6}$ etc. are simple surds where as $\sqrt{3} + \sqrt{5} - \sqrt[3]{4}, \sqrt{3}, \sqrt[3]{5}$ etc. are compound surds.

Monomial, Binomial and Trinomial surds : Single surd is called monomial. Eg. : $\sqrt[3]{2}, \sqrt{2}, \frac{4}{2}\sqrt[3]{3}$

Algebraic sum of two simple surds or a rational number and a simple surd is known as a binomial surd.

Eg. : $2 + \sqrt{3}, \sqrt{3} + \sqrt{2}, 2 + \sqrt[3]{3}, \sqrt[3]{2} + \sqrt[3]{3}$

Algebraic sum of three simple surds or the sum of a rational number and two simple surds is known as a trinomial surd. Eg. $1 + \sqrt{2} + \sqrt{3}, 2 + \sqrt{3} - \sqrt{5}, \sqrt{3} + \sqrt{5} - 4\sqrt[3]{2}, \sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}$

Laws of surds :

(i) $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$ Eg. (a) $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$ (ii) $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$

(ii) $\sqrt[n]{a} \times \sqrt[n]{a^n} = \sqrt[n]{ab}$ [Here order is same] Eg. (a) $\sqrt[3]{2} \times \sqrt[3]{6} = \sqrt[3]{2 \times 6} = \sqrt[3]{12}$

But $\sqrt[3]{3} \times \sqrt[4]{6} \neq \sqrt{3 \times 6}$ because order is not same 1^{st} make their order same and then you can multiply.

(iii) $\sqrt[n]{a} \div \sqrt[n]{b} \neq \sqrt[n]{\frac{a}{b}}$ **(iv)** $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[m]{\sqrt[n]{a}}$ Example $= \sqrt{\sqrt{\sqrt{2}}} = \sqrt[8]{8}$

(v) Product of a rational number with an irrational number is not always irrational.
(vi) Product of a non-zero rational number with an irrational number is always irrational.
(vii) Product of an irrational with an irrational is not always irrational.
(viii) **Addition and subtraction of surds :**
Addition and subtraction of surds are possible only when order and radicand are same i.e. only for like surds.

(ix) **Comparison of surds :** It is clear that if x > y > 0 and n > 1 is a +ve integer then $\sqrt[n]{x} > \sqrt[n]{y}$

Eg. : $\sqrt[3]{16} > \sqrt[3]{12} > \sqrt[5]{25}$ and so on.

(x) If the product of two surds is a rational number then each of them is called the rationalizing factor (R.F.) of the other.

Unit digit in exponential expressions :

(a) When there is 2 in unit's place of any number.
Since in 2^1 unit's place of any number.
Since in 2^1 unit digit is 2, in 2^2 unit digit is 4, in 2^3 unit digit is 8, in 2^4 unit digit is 6
after than the unit digits repeats.

(b) When there is 3 in unit's place of any number
Since in 3^1 unit digit is 3, in 3^2 unit digit is 9, in 3^3 unit digit is 7, in 3^4 unit digit is 1
after that the unit digits repeats.

(c) When there is 4 in unit's place of any number. Since in 4^1 unit digit is 4
in 4^2 unit digit is 6
after that the unit digits repeats.

(d) When there is 5 in unit's place of any number
Since in 5^1 unit digit is 5, in 5^2 unit digit is 5 & so on

(e) When there is 6 in unit's place of any number
Since in 6^1 unit digit is 6, in 6^2 unit digit is 6 & so on

(f) When there is 7 in unit's place of any number
Since in 7^1 unit digit is 7, in 7^2 unit digit is 9, in 7^3 unit digit is 3, in 7^4 unit digit is 1 after that the unit digits repeats.

(g) When there is 8 in unit's place of any number
Since in 8^1 unit digit is 3, in 8^2 unit digit is 4, in 8^3 unit digit is 4, in 8^4 unit digit is 6
after that the unit's digits repeats after a group of 4.

(h) When there is 9 in unit's place of any number
Since in 9^1 unit's digit is 9, in 9^2 unit's digit is 1
after that the unit's digits repeats after a group of 2.

(i) When there is zero in unit's place of any number
There will always be zero in unit's place.
after that the unit's digits repeats after a group of 4.

Example : (a) In $(23)^{13}$ unit digit is 3 **(b)** In $(34)^{14}$ unit digit is 6 **(c)** In $(97)^{99}$ unit digit is 3

Laws of Exponent for real numbers :

Positive Integral power : For any real number a and a positive integer 'n' we define a^n as
$a^n = a \times a \times a \times \ldots\ldots\ldots\ldots \times a$ (n times)
a^n is called the n^{th} power of a. The real number 'a' is called the base and 'n' is called the exponent of the n^{th} power of a. Eg. : $2^3 = 2 \times 2 \times 2 = 8$

For any non-zero real number 'a' we define $a^0 = 1$ Thus, $3^0 = 1$, 5^0 . i, $(3/4)^0 = 1$ and so on.
Negative Integral power : For any non-zero real number 'a' and a positive integer 'n' we define $a^{-n} = 1/a^n$
Thus we have defined a^n for all integral values of n, positive, zero or negative. a^n is called the n^{th}.
Law of integral exponents :
These laws are similar to the laws of integer exponents of rational numbers.
First law : If 'a' is any real number and m, n are positive integers, then $a^m \times a^n = a^{m+n}$.

Second law : If 'a' is non-zero real number and m, n are positive integers, then $\dfrac{a^m}{a^n} = a^{m-n}$.

Case I : When m > n $\quad a^m \div a^n = a^{m-n}$.
Case II : When m = n $\quad a^m \div a^n = a^0 = 1$

Case III : When m < n $\quad a^m \div a^n = a^{-(n-m)} = \dfrac{1}{a^{n-m}}$

Third law : If 'a' is any real number and m, n are positive integers, then $(a^m)^n = a^{mn} = (a^n)^m$
Fourth law : If a, b are real numbers and m, n are positive integers, then

(i) $(ab)^n = a^n b^n$ $\qquad$ (ii) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} = (b \neq 0)$

Rational exponents of a real number :
It is a positive real number and 'n' is a positive integer, then the principal n^{th} root of a is the unique positive real number × such that $x^n = a$.
The principal n^{th} root of a positive real number a is denote by $a^{1/n}$ or $\sqrt[a]{n}$
Rational power (Exponents) : For any positive real number a and a rational number p/q, where q > 0, we define $a^{p/q} = (a^p)^{1/q}$ i.e., $a^{p/q}$ is the principal q^{th} root of a^p.

Laws of rational exponents : The following laws hold the rational exponents

(i) $a^m a^n = a^{m+n}$ $\qquad$ **(ii)** $a^m \div a^n = a^{m-n}$ $\qquad$ **(iii)** $(a^m)^n = a^{mn}$

(iv$a^{-n} = 1/a^n$ $\qquad$ **(v)** $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$ **i.e.,** $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

(vi) $(ab)^m = a^m b^m$ $\qquad$ **(vii)** $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\qquad$ **(viii)** $a^{bn} = a^{b \times b \times b ....n \text{ times}}$

Where a, b, are positive real numbers and m, n are rational numbers.

Example 12 :

Simplify : $13^{\frac{1}{5}}.17^{\frac{1}{5}}$

Sol. $13^{\frac{1}{5}}.17^{\frac{1}{5}} = (13 \times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$

Example 13 :

Simplify : $\pi^{3/4}.\pi^{1/2}$

Sol. $\pi^{3/4}.\pi^{1/2} = \pi^{\frac{3}{4}+\frac{1}{2}} = \pi^{\frac{5}{4}}$

Example 14 :

Simplify : $\left(\frac{2^a}{2^b}\right)^{a+b}\left(\frac{2^b}{2^c}\right)^{b+c}\left(\frac{2^c}{2^a}\right)^{c+a}$

Sol. $\left(\frac{2^a}{2^b}\right)^{a+b}\left(\frac{2^b}{2^c}\right)^{b+c}\left(\frac{2^c}{2^a}\right)^{c+a} = 2^{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)} = 2^0 = 1$

ACTIVITY

The square roots of natural numbers :

Obtain a line segment corresponding to the square roots of natural numbers using graduated strips.

(i) You should know Pythagoras theorem, i.e., in any right angled triangle the square of the hypotenuse side is equal to the sum of the squares of the base and the perpendicular.

(ii) Expressing a given number as the sum of the squares of the two numbers.

Material required : Two wooden strips, nails, thread

Procedure :

1. Take a wooden strip and make a scale on it (call this strip as A).

2. Make a hole on each mark.

3. Put a thread attached on the zeroth position on scale A.

4. Take another wooden strip and make a scale on it. Fix nails on it (call this strip as B).

5. Now fix the scale B on horizontal scale A, scale a is fixed on scale B at point O.

6. For determining the line corresponding to $\sqrt{2}$

Insert scale B, in the hole 1 on scale A. Tie the thread to number one on scale B. This forms a triangle with base and height as one unit. Using Pythagoras theorem, the length of the thread is $\sqrt{2}$. Measure the length of the thread on scale A.

Observation :

1. The students observe that the length corresponding to $\sqrt{2}$, is approximately 1.41 cm.

2. They also understand that, to determine the corresponding length for $\sqrt{13}$, they should insert scale B into scale A at 3 and tie the thread to 2 on scale B.

3. By using Phythagoras theorem, the length of the thread is $\sqrt{(3^2+2^2)} = \sqrt{13}$.

They can measure it on scale A, which is 3.6 cm.

Learning Outcomes :

1. The students learn to find corresponding line segment for square roots of natural numbers.

2. They can see these irrational numbers represented geometrically.

ADDITIONAL EXAMPLES

Example 1 :

Prove that $2+\sqrt{3}$ is irrational.

Sol. Let $2+\sqrt{3}$ be a rational number equals to r. $\therefore\ 2+\sqrt{3}=r$; $\sqrt{3}=r-2$

Here, L.H.S. is an irrational no. while R.H.S. $r-2$ is rational. $\therefore$ L.H.S. $\neq$ R.H.S..

Hence is contradicts our assumption that $2+\sqrt{3}$ is rational. $\therefore\ 2+\sqrt{3}$ is irrational.

Example 2 :

Simplify : $\sqrt[6]{8a^5b}\times\sqrt[3]{4a^2b^2}$

Sol. $\sqrt[6]{8^3a^{15}b^3}\times\sqrt[6]{4^2a^4b^4}=4a^3b\sqrt[6]{2ab}$

Example 3:

Find a rational number between 1/5 and 7/10.

Sol. You may choose any number, one-half, one-third, one-fourth, one-fifth etc. of the way from 1/5 to 7/10.

Suppose you choose the number one fourth of the way from 1/5 to 7/10.

$\frac{1}{5}+\frac{1}{4}\left(\frac{7}{10}-\frac{1}{5}\right)=\frac{1}{5}+\frac{1}{4}\left(\frac{1}{2}\right)=\frac{1}{5}+\frac{1}{8}=\frac{13}{40}$ $\therefore$ One rational number between $\frac{1}{5}$ and $\frac{7}{10}$ is $\frac{13}{40}$.

Example 4 :

Which of the following rational numbers can be represented as terminating decimals ? (i) 7/20 (ii) –27/40

Sol. **(i)** In 7/20, $20=2^2\times 5$, i.e., it has factors in 2's and 5's.

$\therefore$ 7/20 has terminating decimal representation.

(ii) In –27/40, $40=2^3\times 5$ i.e., it has factors in 2's and 5's.

$\therefore$ –27/40 has terminating decimal representation.

Example 5 :

Find the value of $2.\overline{6}-0.\overline{9}$

Sol. Let $x=2.\overline{6}-0.\overline{9}$ .... (1) and $y=0.\overline{9}=0.9\overline{9}$ .... (3)

then $10x=26.\overline{6}$ .... (2) $10y=9.\overline{99}$ .... (4)

Subtracting (1) from (2) Subtracting (3) from (4)

$9x=24$ $9y=9$

$\therefore\ x=\frac{24}{9}=\frac{8}{3}$ $\therefore\ y=\frac{9}{9}=1$

$\therefore\ 2.\overline{6}-0.\overline{9}=x-y=\frac{8}{3}-1=\frac{8-3}{3}=\frac{5}{3}$

Example 6 :

Rationalise the denominator and simplify : $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

Sol. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}} = \dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}\times\dfrac{\sqrt{3}-\sqrt{6}}{\sqrt{3}-\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}\times\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}\times\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}$

$= \dfrac{3\sqrt{2}(\sqrt{3}-\sqrt{6})}{3-6}-\dfrac{4\sqrt{3}\,(\sqrt{6}-\sqrt{2})}{6-2}+\dfrac{\sqrt{6}\,(\sqrt{2}-\sqrt{3})}{2-3}$

$= -\sqrt{2}(\sqrt{3}-\sqrt{6})-\sqrt{3}\,(\sqrt{6}-\sqrt{2})-\sqrt{6}\,(\sqrt{2}-\sqrt{3}) = -\sqrt{6}+\sqrt{12}-\sqrt{18}+\sqrt{6}-\sqrt{12}+\sqrt{18}=0$

Example 7 :

Express $1.272727.....=1.\overline{27}$ in the form p/q, where p and q are integers and $q \neq 0$.

Sol. Let x = 1.272727.......... Since two digits are repeating, we multiply x by 100 to get

100x = 127.2727 .... So, 100x = 126 + 1.272727......... = 126 + x

Therefore, 100x – x = 126, i.e., 99x = 126 i.e., $x=\dfrac{126}{99}=\dfrac{14}{11}$. We can check the reverse that $\dfrac{14}{11}=1.\overline{27}$

Example 8 :

Recall, π is defined as the ratio of the circumference (say c) of a circle to its diameter (say d). That is, $\pi=\dfrac{c}{d}$.

This seems to contradict the fact that π is irrational. How will you resolve this contradiction ?

Sol. There is no contradiction. Remember that when you measure a length with a scale or any other device, you only get an approximate rational value. So, you may not realise that either c or d is irrational.

Example 9 :

Simplify : $5\sqrt[3]{250}+7\sqrt[3]{16}-14\sqrt[3]{54}$

Sol. $5\sqrt[3]{250}+7\sqrt[3]{16}-14\sqrt[3]{54} = 5\sqrt[3]{125\times 2}+7\sqrt[3]{8\times 2}-14\sqrt[3]{27\times 2}$

$= 5\times 5\sqrt[3]{2}+7\times 2\sqrt[3]{2}-14\times 3\times\sqrt[3]{2} = (25+14-42)\sqrt[3]{2} = -3\sqrt[3]{2}$

Example 10 :

Divide $\sqrt{24}$ by $\sqrt[3]{200}$

Sol. $\dfrac{\sqrt{24}}{\sqrt[3]{200}}=\dfrac{\sqrt[6]{(24)^3}}{\sqrt[6]{(200)^2}}=\sqrt[6]{\dfrac{216}{625}}$