2

POLYNOMIALS

INTRODUCTION

An algebraic expression f (x) of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. Where $a_0, a_1, a_2, ...,a_n$ are real numbers and all the index of x are non-negative integers is called a polynomials in x and the highest index n is called the degree of the polynomial. Here $a_0 , a_1x , a_2x^2 , ... + a_nx^n$ are called the terms of the polynomial and $a_0, a_1, a_2......... a_n$ are called various coefficients of the polynomial f (x). A polynomial in x is said to be in standard form when the terms are written either increasing order or decreasing order of the indices of x in various terms.

A symbol which takes an various numerical values is known as a variable or a literal.

Like terms : Terms having same literal coefficients are called like terms, otherwise they are called unlike terms.

If $a_0, a_1, a_2, ...a_n$ are all integers then f (x) is said to be a polynomial over integers.

Degree of a Polynomial in two or more variables :

If a polynomial involves two or more variables, then the sum of the powers of all the variables in each term is taken up and the highest sum so obtained is the degree of the polynomial.

Examples :

(1) $\frac{3}{4}x^3 - \frac{1}{3}x^2 + 5x - 6$ is a polynomial over rationals.

(2) $6x^7 - 5x^4 + 2x + 3$ is a polynomial of degree 7.

(3) $2 + 5x^{3/2} + 7x^2$ is an expression but not a polynomial , since it contains a term in which power of x is 3/2, which is not a non-negative integer.

(4) $3ab^2 - 4\,a\sqrt{b} + 5b^3$ is an expression but not a polynomial, as it contains a term in which the sum of the powers of the variables is 3/2, which is not a non-negative integer.

(5) In $-9y^2$, the numerical coefficient is –9 and literal coefficient is y^2.

(6) $6a^2, -8b^2. - 4\,ab$ are unlike terms.

Different types of polynomials :

There are four types of polynomials based on degrees.

(i) **Linear polynomials :** A polynomial of degree one is called a linear polynomial. The general formula linear polynomial is ax + b, where a and b are any real constant and $a \neq 0$. Example : (3 + 5x) is a linear polynomial.

(ii) **Quadratic polynomials :** A polynomial of degree two is called a quadratic polynomial. The general form of quadratic polynomial is $ax^2 + bx + c$, where $a \neq 0$. Example : $2y^2 + 3y - 1$ is a quadratic polynomials.

(iii) **Cubic polynomials :** A polynomial of degree three is called a cubic polynomial. The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$. Example : $6x^3 - 5x^2 + 2x + 1$ is a cubic polynomial.

(iv) **Biquadratic (or quadratic) polynomials :** A polynomial of degree four is called a biquadratic (quadratic) polynomial. The general form of a biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$.

A polynomial of degree five or more than five does not have any particular name. Such a polynomial usually called a polynomial of degree five or six or .... etc. There are three types of polynomials based on number of terms. These are as follows :

(a) Monomial : A polynomial is said to be a monomial if it has only one term.
For Ex., $-3x^2$, $5x^3$, $10x$ are monomials.

(b) Binomial : A polynomial is said to be a binomial if it contains two terms.
For example, $(2x^2 + 5)$, $(3x^3 - 7)$, $(6x^2 + 8x)$ are binomials.

(c) Trinomials : A polynomial is said to be a trinomial if it contains three terms.

For example, $3x^3 - 8x + \frac{5}{2}$, $\sqrt{7}x^{10} + 8x^4 - 3x^2$, $5 - 7x + 8x^9$ etc. are trinomials.

Zero degree polynomial : Any non-zero number is regarded as a polynomial of degree zero or zero degree polynomial.
For example, $f(x) = a$, where $a \neq 0$ is a zero degree polynomial, since we can write $f(x) = a$ as $f(x) = ax^0$.
Zero polynomial : A polynomial whose coefficient are all zeros is called a zero polynomial i.e., $f(x) = 0$ but we cannot talk about the degree of a zero polynomial.

Example 1 :

Which of the following functions are polynomials ? (a) $5x^2 - 3x + 9$ (ii) $\frac{4}{3}x^7 - 5x^4 + 3x^2 - 1$

Sol. Both are polynomials.

Example 2 :

Write down the degree of the following polynomials : (a) $3x + 5$ (b) $3t^2 - 5t + 9t^4$. (c) $2 - y^2 - y^3 + 2y^8$

Sol. (a) The highest power term is 3x and its exponent is 1. $\therefore$ degree $(3x + 5) = 1$
(b) The highest degree term is $9t^4$ and its exponent is 4. So, degree of the given polynomial is 4.
(c) The highest power of the variable is 8. So, the degree of the polynomial is 8.

Example 3 :

Write the following polynomials in standard form : (i) $x^6 - 3x^4 + \sqrt{2}x + \frac{5}{2}x^2 + 7x^5 + 4$

Sol. The given polynomials in standard form are : (i) $x^6 + 7x^5 - 3x^4 + \frac{5}{2}x^2 + \sqrt{2}x + 4$

VALUE OF A POLYNOMIAL

The value of a polynomial $f(x)$ at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $f(\alpha)$. Consider the polynomial, $p(x) = 5x^3 - 2x^2 + 3x - 2$
If we replace x by 1 everywhere in p (x), we get

$$p(1) = 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2 = 5 - 2 + 3 - 2 = 4$$

So, we say that the value of p (x) at x = 1 is 4. Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2 = -2$
If $p(x) = 0$ then we say x as root of the polynomial.

Example 4 :

If $x = 4/3$ is a root of the polynomial $f(x) = 6x^3 - 11x^2 + kx - 20$ then find the value of k.

Sol. $f(x) = 6x^3 - 11x^2 + kx - 20$

$$f\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + k\left(\frac{4}{3}\right) - 20 = 0 \Rightarrow 6.\frac{64}{9.3} - 11.\frac{16}{9} + \frac{4k}{3} - 20 = 0$$

$\Rightarrow 128 - 176 + 12k - 180 = 0 \Rightarrow 12k + 128 - 356 = 0 \Rightarrow 12k = 228 \Rightarrow k = 19$

Example 5 :

If $x = 2$ and $x = 0$ are roots of the polynomials $f(x) = 2x^3 - 5x^2 + ax + b$. Find the values of a and b.

Sol. If $f(2) = 2(2)^3 - 5(2)^2 + a(2) + b = 0 \Rightarrow 16 - 20 + 2a + b = 0 \Rightarrow 2a + b = 4$

$\Rightarrow f(0) = 2(0)^3 - 5(0)^2 + a(0) + b = 0 \Rightarrow b = 0 \Rightarrow 2a = 4 \Rightarrow a = 2, b = 0$

Basic operations with polynomials : The sum of two polynomials can be found by grouping like power terms, retaining their signs and adding the coefficients of like powers.

Also, the negative of a polynomial P (x) is a polynomial, to be denoted by P (x) and it is obtained by replacing each coefficient by its additive inverse.

i.e., Change the sign of the term to be subtracted and add this new term with the first term from which subtraction is to be made.

Example 6 :

Add : $(4a^3 - 5a^2 + 6a - 3), (2 + 8a^2 - 3a^3), (9a - 3a^2 + 2a^3 + a^4), (1 - 2a - 3a^3)$

Sol. Arranging columnwise with like terms in same column and adding, we get

$$\begin{array}{rrrrr} & 4a^3 & -5a^2 & +6a & -3 \\ & -3a^3 & +8a^2 & & +2 \\ a^4 & +2a^3 & -3a^2 & +9a & \\ & -3a^3 & & -2a & +1 \\ \hline a^4 & & & +13a & \\ \hline \end{array}$$

$\therefore$ Required term = $a^4 + 13a$

Example 7 :

If $p(y) = y^6 - 3y^4 + 2y^2 + 6$ and $q(y) = y^5 - y^3 + 2y^2 + y - 6$, find p (y) + q (y) and p (y) – q (y).

Sol.

$$\begin{array}{l} p(y) = y^6 + 0y^5 - 3y^4 + 0y^3 + 2y^2 + 0y + 6 \\ q(y) = \quad y^5 + 0y^4 - y^3 + 2y^2 + y - 6 \\ \hline \end{array}$$

$\therefore$ $p(y) + q(y) = y^6 + y^5 - 3y^4 - y^3 + 4y^2 + y + 0$

Also, $-q(y) = -y^5 - 0y^4 + y^3 - 2y^2 - y + 6$

Now, $p(y) = y^6 + 0y^5 - 3y^4 + 0y^3 + 2y^2 + 0y + 6$

And, $-q(y) = -y^5 - 0y^4 + y^3 - 2y^2 - y + 6$

$\therefore$ $p(y) - q(y) = y^6 - y^5 - 3y^4 + y^3 + 0y^2 - y + 12$

Thus, $p(y) - q(y) = y^6 - y^5 - 3y^4 + y^3 - y + 12$

Multiplication of Monomials :

Product of monomials = (Product of their numerical coefficients) × (Product of their variable parts)

Multiplication of Two monomials :

Multiply each term of the multiplicand by each term of the multiplier and take the algebraic sum of these products.

Example 8 :

Find the product of $(x + 3)$ and $(x^2 + 4x + 5)$.

Sol. $(x + 3)(x^2 + 4x + 5) = x(x^2 + 4x + 5) + 3(x^2 + 4x + 5) = (x^3 + 4x^2 + 5x) + (3x^2 + 12x + 15)$

$= x^3 + (4 + 3)x^2 + [5 + 12]x + 15 = x^3 + 7x^2 + 17x + 15$

Example 9 :

Multiply : $p(t) = t^4 - 6t^3 + 5t - 8$ and $q(t) = t^3 + 2t^2 + 7$.

Also, find the degree of p (t). q (t).

Sol.

$$
\begin{array}{r}
t^4 - 6t^3 + 5t - 8 \\
t^3 + 2t^2 + 0t + 7 \\
\hline
t^7 - 6t^6 + 0t^5 + 5t^4 - 8t^3 \\
+ 2t^6 - 12t^5 + 0t^4 + 10t^3 - 16t^2 \\
+ 7t^4 - 42t^3 + 0t^2 + 35t - 56 \\
\hline
\end{array}
$$

$\therefore p(t).q(t) = t^7 - 4t^6 - 12t^5 + 12t^4 - 40t^3 - 16t^2 + 35t - 56$

The highest power term is t^7, and its exponent is 7. $\therefore$ degree of p (t). q (t) is 7.

Division of polynomial by a Monomial : Let us divide $= 4x^4 + 2x^3 + 2x^2$ by $2x^2$

$(4x^4 + 2x^3 + 2x^2) \div 2x^2 = 2x^2 + x + 1$

We can write $4x^4 + 2x^3 + 2x^2 = (2x^2)(2x^2 + x + 1)$

We say $2x^2$ and $2x^2 + x + 1$ are factors of $4x^4 + 2x^3 + 2x^2$

Let us consider another example : Divide $(6x^2 + 2x + 1)$ by x ; $(6x^2 + 2x + 1) \div x = 3x + 2 + (1 \div x)$

We cannot divide 1 by x to get a polynomial term.

Division Alogrithm for Polynomial : On dividing a polynomial p (x) by a polynomial d (x), let the quotient be q (x) and the remainder be r(x), then $p(x) = d(x) . q(x) + r(x)$, where either $r(x) = 0$ or deg. $r(x) < \text{deg}.d(x)$

Here , Dividend = p (x), Divisor = d (x), Quotient = q (x) and Remainder = r (x).

Division of a polynomial by a polynomial :

Step 1 : Arrange the terms of the dividend and the divisor in descending order of their degrees.

Step 2 : Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

Step 3 : Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4 : Consider the remainder as new dividend and proceed as before.

Step 5 : Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Example 10 :

Divide $x^3 + x^2 - 2x - 30$ by $x - 3$

Sol. $(x - 3)$ $x^3 + x^2 - 2x - 30$ $x^2 + 4x + 10$

$$
\begin{array}{l}
x^3 - 3x^2 \\
\hline
\quad 4x^2 - 2x - 30 \\
\quad 4x^2 - 12x \\
\hline
\qquad 10x - 30 \\
\qquad 10x - 30 \\
\hline
\qquad \times
\end{array}
$$

Example 11 :

Divide : $(15x^2 - 32y^2 + 38xy)$ by $(3x - 2y)$

Sol. Arranging the terms of the dividend and the divisor in descending order of powers of x and then dividing, we get

$$
\begin{array}{l}
3x - 2y \;)\; 15x^2 + 38xy - 32y^2 \;(\; 5x + 16y \\
\qquad\quad 15x^2 - 10xy \\
\hline
\qquad\qquad 48xy - 32y^2 \\
\qquad\qquad 48xy - 32y^2 \\
\hline
\qquad\qquad\quad \times \\
\hline
\end{array}
$$

$\therefore\ (15x^2 - 32y^2 + 38xy) \div (3x - 2y) = (5x + 16y)$

Example 12 :

Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.

Sol. By long division

$$
\begin{array}{l}
\qquad x^2 - x + 1 \\
x + 1 \;)\; x^3 + 1 \\
\qquad x^3 + x^2 \\
\qquad - \;\; - \\
\hline
\qquad\; - x^2 \qquad + 1 \\
\qquad\; - x^2 - x \\
\qquad\; + \quad\; + \\
\hline
\qquad\qquad x + 1 \\
\qquad\qquad x + 1 \\
\qquad\qquad - \;\; - \\
\hline
\qquad\qquad 0
\end{array}
$$

Factor of a polynomial : Let p (x) be any polynomial and let a, b, c be any real numbers such that $p(x) = (x - a)(x - b)(x - c)$.

Then, clearly each one of $(x - a)$, $(x - b)$, $(x - c)$ is a linear factor of p (x).

To express a given polynomial as the product of linear factors or factors of degree less than that of the given polynomial such that no such a factor has a linear factor, is known as factorization.

Formulas useful for factorization :

* $(x + a)^2 = x^2 + 2ax + a^2$
* $(x - a)^2 = x^2 - 2ax + a^2$
* $x^2 - a^2 = (x + a)(x - a)$
* $x^3 + a^3 = (x + a)(x^2 - ax + a^2) = (x + a)^3 - 3xa(x + a)$
* $x^3 - a^3 = (x - a)(x^2 + ax + a^2) = (x - a)^3 + 3xa(x - a)$
* $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
* $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
* $(a - b)^3 = a^3 + b^3 - 3ab(a - b)$
* $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
* If $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

FACTORIZATION

Type I : Factorization of an expression when all the terms have a common factor.

When each term of an expression has a common factor, we divide each term of the expression by this factor and take it out as a multiple, as shown below.

Example 13 :

$8a^2b^3c^4 - 14a^2b^3c^2$

Sol. $8a^2b^3c^4 - 14a^2b^3c^2 = 2a^2b^3c^2 [4c^2 - 7]$

Example 14 :

$x(x - 3) - y(3 - a)$

Sol. $x(a - 3) - y(3 - a) = x(a - 3) + y(a - 3) = (a - 3)(x + y)$

Example 15 :

$ax + by - cx - ay + cy - bx$

Sol. $ax + by - cx - ay + cy - bx = ax - bx - cx - ay + by + cy$

$= x(a - b - c) - y(a - b - c) = (a - b - c)(x - y)$

Type II : Factorization by grouping terms :

The terms of the expression may be grouped so as to have a common factor.

Example 16 :

$3ax - 30cz - 5az - 175by + 42cy + 75bx - 18cx + 7ay + 125bz$

Sol. $-3ax + 75bx - 18cx + 7ay - 175by + 42cy - 5az + 125bz - 30cz$

$= -3x[a - 25b + 6c] + 7y[a - 25b + 6c] - 5z[a - 25b + 6c]$

$= (a - 25b + 6c)(-3x + 7y - 5z)$

Type III : Factorization by making a trinomial a perfect square

We know that sometimes a given expression consists of a trinomial which can be written as square of a binomial expression. In such case the following two formulas can be used :

$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$

It is clear from the formulas that $(a + b)$ and $(a + b)$ i.e. $(a + b)^2$ are the factors of $a^2 + 2ab + b^2$ and $(a - b)$ and $(a - b)$ i.e. $(a - b)^2$ are the factors of $a^2 - 2ab + b^2$.

Hence in expression if three terms are such that two of them are perfect square (and are of same sign) and the third term is twice the product of the square root of the two terms or can be expressed as above then factors can be obtained by these formulas.

Example 17 :

Factorize : $x(x-2)(x-4)+4x-8$

Sol. $x(x-2)(x-4)+4x-8 = x(x-2)(x-4)+4(x-2)$

$$= (x-2)[x(x-4)+4] = (x-2)(x^2-4x+4) = (x-2)(x-2)^2 = (x-2)^3$$

Example 18 :

Factorize : $9x^4 - 2 + \frac{1}{9x^4}$

Sol. $9x^4 - 2 + \frac{1}{9x^4} = (3x^2)^2 - 2(3x^2)\left(\frac{1}{3x^2}\right) + \left(\frac{1}{3x^2}\right)^2 = \left(3x^2 - \frac{1}{3x^2}\right)^2$

Type IV : Factorization by using the formula for the difference of two square by formula

Sometimes an expression is neither a perfect square nor it can be expressed as $a^2 - b^2$, then we add an appropriate expression to it and subtract the same so as the value of the original expression does not change. By doing so we get a perfect square term which with the remaining term, takes the form $a^2 - b^2$ and we can factorize it by the above method.

Example 19 :

Factorize : $x^2 - 1 - 2a - a^2$

Sol. $x^2 - 1 - 2a - a^2 = x^2 - (1 + 2a + a^2) = x^2 - (1+a)^2 = [x + (1+a)][x - (1+a)] = (x+1+a)(x-1-a)$

Example 20 :

Factorize : $81x^4 - 256y^4$

Sol. $81x^4 - 256y^4 = (9x^2)^2 - (16y^2)^2 = (9x^2 + 16y^2)(9x^2 - 16y^2)$

$$= (9x^2 + 16y^2)[(3x)^2 - (4y)^2] = (9x^2 + 16y^2)(3x+4y)(3x-4y)$$

Type V : By splitting the middle term :

When a trinomial is of the form $ax^2 + bx + c$ or $ax^2 + bxy + cy^2$, where $(a \neq 0)$, then we factorize the product $a \times c$ into two factors whose sum is equal to b.

If the sign of $a \times c$ is positive then either both the factors are positive or both are negative. If the sign of b is positive then both the factors are positive and if the sign of b is negative then both the factors are negative.

Again if the sign $a \times c$ is negative then one of the factors is positive and the other is negative. If the sign of b is positive then the greater factor will be positive and the smaller factor will be negative and if the sign of b is negative then the greater factor will be negative and the smaller factor will be positive.

Method :

(i) Let the given trinomial expression be $ax^2 - bx + c$, $a \neq 0$

(ii) We choose two terms p and q in such a way that

$p + q = b$, (b is the coefficient of x) and $pq = a \times c$ (a is the coefficient of x^2) and c is constant term)

(iii) Write the term bx of the given expression as $px + qx$

(iv) Now get the factors by grouping the terms obtained in step (iii).

Example 21 :

Factorize : $x^2 - 14x + 24$

Sol. Product $ac = 24$ and $b = -14$

$\therefore$ split the middle term as -12 and -2

$\Rightarrow x^2 - 14x + 24 = x^2 - 12x - 2x + 24 = x(x - 12) - 2(x - 12) = (x - 12)(x - 2)$

Example 22 :

Factorize : $6x^2 - 5x - 6$

Sol. $6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6 = 3x(2x - 3) + 2(2x - 3) = (2x - 3)(3x + 2)$

REMAINDER THEOREM

Let p (x) be a polynomial of degree greater than or equal to 1 and let a be any real number.

If p (x) is divided by (x – a), then the remainder is p (a).

Proof : When a polynomial p (x) is divided by (x – a), then by division alogrithm, we obtain quotient q (x) and a constant remainder c such that .

$$p(x) = (x - a).\ q(x) + c \qquad \text{........... (1)}$$

On putting x = a in (1), we get

$$p(a) = (a - a)\ .\ q(a) + c = 0\ .\ q(a) + c = 0 + c = c. \text{ Hence, remainder} = p(a)$$

Deduction : If a polynomial p(x) is divided by (x + a), (ax – b), (ax + b), (b – ax) then the remainder is the value of p (x) at $x = -a,\ \frac{b}{a}, -\frac{b}{a}, \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right), p\left(-\frac{b}{a}\right), p\left(\frac{b}{a}\right)$ respectively.

Example 23 :

If $f(x) = (2x^3 + x^2 + bx - 6)$ leaves a remainder 36 when divided by (x – 3), find the value of b.

Sol. Given $f(x) = (2x^3 + x^2 + bx - 6)$

It is given that f (x) when divided by (x – 3) leaves a remainder 36. So, f (3) = 36.

Now, $f(3) = (2 \times 3^3 + 3^2 + 3b - 6) = (3b + 57)$

$\therefore\ f(3) = 36 \Rightarrow 3b + 57 = 36 \Rightarrow 3b = (36 - 57) = -21 \Rightarrow b = -7$. Hence $b = -7$

Example 24 :

Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Sol. Here, $p(x) = x^4 + x^3 - 2x^2 + x + 1$, and the zero of $x - 1$ is 1.

So, $p(1) = (1)^4 - (1)^3 - 2(1)^2 + 1 + 1 = 2$

So, by the Remainder theorem, 2 is the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divide by $x - 1$.

FACTORIZATION OF POLYNOMIAL

A polynomial g (x) is called a factor of polynomial p (x), if g (x) divides p (x) exactly, i.e., on dividing p (x) by g (x), we get 0 as remainder.

Factor theorem : Let p (x) be a polynomial of degree greater than or equal to 1 and a be a real number such that p (a) = 0, then (x – a) is a factor of p (x). Conversely, if (x – a) is a factor of p (x), then p (a) = 0. If (x + a), (ax + b), (ax – b) is a factor of p (x) if p (–a) = 0, $p\left(-\frac{b}{a}\right)$ or $p\left(\frac{b}{a}\right) = 0$ respectively.

Proof : By the remainder theorem, p (x) = (x – a) q (x) + p (a)
(i) If p (a) = 0, then p (x) = (x – a) q (x), which shows that x – a is a factor of p (x)
(ii) Since x – a is a factor of p (x), q (x) = (x – a) g (x) for same polynomial g (x).
In this case, p (a) = (a – a) g (a) = 0

Use of factor theorem in factorization :
Hit and Trial method for Finding A linear factor of f (x) : Let f (x) be the given polynomial
Case I. Let the constant term in f (x) be 3 or – 3
Then, its factors are 1, –1, 3 and –3. See which one is zero.
If f (1) = 0, then (x – 1) is a factor of f (x)
If f (–1) = 0, then (x + 1) is a factor of f (x) and soon.
Case II. Let the constant term in f (x) be 4 or – 4
Then, its factors are 1, –1, 2, –2, 4 and – 4
Find f (1), f (–1), f (2), f (–2), f (4) and f (– 4)
See which one is zero and find the factor of f (x) as mentioned above.
Similarly, by taking the factors of the constant term in the given polynomial, we can find a linear factor of f (x) by hit and trial method.

Example 25 :
Show that x + 1 and 2x – 3 are factors of $2x^3 - 9x^2 + x + 12$

Sol. To prove that (x + 1) and (2x – 3) are factors of $2x^3 - 9x^2 + x + 12$ it is sufficient to show that p (–1) and p $\left(\frac{3}{2}\right)$ both are equal to zero.

$$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12 = -2 - 9 - 1 + 12 = -12 + 12 = 0$$

and $p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12 = \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{-81 + 81}{4} = 0$

Example 26 :
For what value of k is the polynomial, $2x^4 + 3x^3 + 2kx^2 + 3x + 6$ exactly divisible by (x + 2) ?
Sol. Let p (x) = $2x^4 + 3x^3 + 2kx^2 + 3x + 6$.
We may write, (x + 2) = [x – (–2)]
Now, by factor theorem, p (x) will be exactly divisible by (x + 2) only when p (–2) = 0
Now, p (–2) = $[2 \times (-2)^4 + 3 \times (-2)^3 + 2k(-2)^2 + 3 \times (-2) + 6]$ = (32 – 24 + 8k – 6 + 6) = (8k + 8)
$\therefore$ p (–2) = 0 $\Rightarrow$ 8k + 8 = 0 $\Rightarrow$ 8k = – 8 i.e., k = – 1

Example 27 :
For what value of 'a' is (x – 2) a factor of f (x) = $x^2 - ax + 6$
Sol. If (x – 2) is factor of f (x), then f (x) = 0 [By factor theorem]
$\therefore$ f (2) = $(2)^2$ – a (2) + 6 = 0 $\therefore$ 4 – 2a + 6 = 0 $\therefore$ a = 5

ADDITIONAL EXAMPLES

Example 1 :

What must be subtracted from $x^3 - 6x^2 - 15x + 80$ so that the result is exactly divisible by $x^2 + x - 12$.

Sol. In $p(x) = x^3 - 6x^2 - 15x + 80$ we subtracted $ax + b$ so that it is exactly divisible by $x^2 + x - 12$.

$\therefore\ s(x) = x^3 - 6x^2 - 15x + 80 - (ax + b) = x^3 - 6x^2 - (15 + a)x + (80 - b)$

Dividend = Divisior × quotient + remainder

But remainder will be zero.

$\therefore$ Dividend = Divisor × quotient $\Rightarrow s(x) = (x^2 + x - 12) \times$ quotient

$\Rightarrow s(x) = x^3 - 6x^2 - (15 + a)x + (80 - b) = x(x^2 + x - 12) - 7(x^2 + x - 12)$

$= x^3 + x^2 - 7x^2 - 12x - 7x + 84 = x^3 - 6x^2 - 19x + 84$

Hence, $x^3 - 6x^2 - 19x + 84 = x^3 - 6x^2 - (15 + a)x + 80 - b$

$\Rightarrow -15 - a = -19 \Rightarrow a = +4$ and $80 - b = 84 \Rightarrow b = -4$

Hence if in p (x) we added $4x - 4 = ax + b$ then it is exactly divisible $x^2 + x - 12$

Example 2 :

Find the value of $(99)^2$

Sol. $(99)^2 = (100 - 1)^2 = 100^2 - 2(100)(1) + (1)^2 = 10000 - 200 + 1 = 9801$

Example 3 :

Factorize $6x^2 - 5xy - 4y^2 + x + 17y - 15$

Sol. $6x^2 - 5xy - 4y^2 + x + 17y - 15 = 6x^2 + x[1 - 5y] - [4y^2 - 12y - 5y + 15]$

$= 6x^2 + x[1 - 5y] - [4y(y - 3) - 5(y - 3)] = 6x^2 + x[1 - 5y] - (4y - 5)(y - 3)$

$= 6x^2 + 3(y - 3)x - 2(4y - 5)x - (4y - 5)(y - 3)$

$= 3x[2x + y - 3] - (4y - 5)(2x + y - 3) = (2x + y - 3)(3x - 4y + 5)$

Example 4 :

Find α and β if $x + 1$ and $x + 2$ are factors of $p(x) = x^3 + 3x^2 - 2\alpha x + \beta$.

Sol. Put $x + 1 = 0$ or $x = -1$ and $x + 2 = 0$ or $x = -2$ in p (x)

Then, $p(-1) = (-1)^3 + 3(-1)^2 - 2\alpha(-1) + \beta = 0$

$\Rightarrow -1 + 3 + 2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha - 2$ .... (1)

$p(-2) = (-2)^3 + 3(-2)^2 - 2\alpha(-2) + \beta = 0$

$\Rightarrow -8 + 12 + 4\alpha + \beta = 0 \Rightarrow \beta = -4\alpha - 4$ .... (2)

By equalising both of the above equation

$-2\alpha - 2 = -4\alpha - 4 \Rightarrow 2\alpha = -2 \Rightarrow \alpha = -1$

$\alpha = -1$ put in eq. (1) $\Rightarrow \beta = -2(-1) - 2 = 2 - 2 = 0$. Hence $\alpha = -1, \beta = 0$

Example 5 :

Solve : $0.645 \times 0.645 + 2 \times 0.645 \times 0.355 + 0.355 \times 0.355$

Sol. $0.645 \times 0.645 + 2 \times 0.645 \times 0.355 + 0.355 \times 0.355$

$= (0.645)^2 + 2 \times (0.645)(0.355) + (0.355)^2 = (0.645 + 0.355)^2 = (1)^2 = 1$

Example 6:

Factorize : $x^2 - x\left(\frac{a^2-1}{a}\right) - 1$

Sol. $x^2 - x\left(\frac{a^2-1}{a}\right) - 1 = x^2 - x\left(a - \frac{1}{a}\right) - 1 = x^2 - ax + \frac{x}{a} - 1 = x(x-a) + \frac{1}{a}(x-a) = (x-a) + \left(x + \frac{1}{a}\right)$

Example 7 :

Solve : $\frac{0.73 \times 0.73 - 0.27 \times 0.27}{0.73 - 0.27}$

Sol. $\frac{0.73 \times 0.73 - 0.27 \times 0.27}{0.73 - 0.27} = \frac{(0.73)^2 - (0.27)^2}{0.73 - 0.27} = \frac{(0.73+0.27)(0.73-0.27)}{(0.73-0.27)} = 0.73 + 0.27 = 1$

Example 8 :

Using remainder theorem, find the remainder when $x^3 + x^2 - 2x + 1$ is divided by $x - 3$.

Sol. Let $p(x) = x^3 + x^2 - 2x + 1$

By remainder theorem, we know that p (x) when divided by (x – 3) gives a remainder equal to p (3)

Now, $p(3) = [3^3 + 3^2 - 2 \times 3 + 1] = (27 + 9 - 6 + 1) = 31$

Hence, the remainder is 31.

Example 9 :

Divide : $3x^4 + 2x^3 - 5x^2 + 3x + 1$ by $2x$

Sol. $2x$) $3x^4 + 2x^3 - 5x^2 + 3x + 1$ ($(3/2)x^3 + x^2 + (5/2)x + (3/2)$

$3x^4$

$2x^3 - 5x^2 + 3x + 1$

$2x^3$

$-5x^2 + 3x + 1$

$-5x^2$

$3x + 1$

$3x$

1

$\therefore$ Quotient $= \left(\frac{3}{2}x^3 + x^2 - \frac{5}{2}x + \frac{3}{2}\right)$ and remainder = 1

Example 10:

Factorize $8x^3 + 27y^3 + 36x^2y + 54xy^2$

Sol The given expression can be written as

$(2x)^3 + (3y)^3 + 3(4x^2)(3y) + 3(2x)(9y)^2 = (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2$

$= (2x + 3y)^3 = (2x + 3y)(2x + 3y)(2x + 3y)$