4

LINEAR EQUATION IN TWO VARIABLES

INTRODUCTION

Many times we come across a situation when we have to find the value of unknown quantity which is not directly related to any other known quantity. In this condition if we know that the unknown quantity is indirectly related with some known quantities then we obtain two algebraic expressions containing these known and unknown quantities which are equal. By applying mathematical operations on the equations obtained above, we find the value of the unknown quantity.

Let us review some basic terms you have already learned in your earlier classes.

Equality : The statement which contains the '=' sign is called equality.

Equation : The statement which expresses the equality of two algebraic expressions having one or more unknown and known quantities, is called an equation.

(i) $4x - 7 = 2x + 3$ (ii) $y + \frac{1}{y} = 5$ (iii) $(x + 1)(x - 2) = 3$ are some examples of equations.

Variable : The unknown quantities used in any equation are known as variables. Generally, they are denoted by the last English alphabet x, y, z etc.

An equation is a statement of equality of two algebraic expressions, which involve one or more unknown quantities, called the variables. If in an equation the degree of the variable is are then its equation is called a linear equation.

Equation of higher degree : If in an equation the degree of the variable is more than then this equation is called an equation of higher degree.

According to the degree of the variable in an equation they are pronounced as follows.

2^{nd} degree = Quadratic ; 3^{rd} degree = Cubic l 4^{th} degree = Biquadratic

Linear equation in two variables : An equation of the form $ax + by + c = 0$ where a, b, c are real numbers and x, y are variables, is called a linear equation in two variables.

Here a is called coefficient of x, b is called coefficients of y and c is called constant term.

eg. $6x + 2y + 5 = 0$, $5x - 2y + 3 = 0$ etc.

SOLUTIONS OR ROOTS OF LINEAR EQUATION

The value of the variable which when substituted for the variable in the equation satisfies the equation i.e. L.H.S. and R.H.S. of the equation becomes equal, is called the solution or root of the equation.

Following rules can be used to solve the simple equations :

Rules for Solving an equation :

(i) Same quantity can be added to both sides of an equation without changing the equality.

(ii) Same quantity can be subtracted from both sides of an equation without changing the equality.

(iii) Both sides of an equation may be multiplied by a same non-zero number without changing the equality.

(iv) Both sides of an equation may be divided by a same non-zero number without changing the equality.

Example :

(1) 3 is root or solution of $5x - 8 = 7$.

$5x - 8 = 7$; L.H.S. $= 5x - 8 = 5 \times 3 - 8 = 15 - 8 = 7$, (on putting $x = 3$]

and R.H.S. $= 7$ $\therefore$ On putting $x = 3$, we find that L.H.S. = R.H.S.

Hence, the solution of the given equation is $x = 3$.

(2) $x = 2$ is not a solution of $5x + 7 = 3$ as we substituting $x = 2$ in the equation.
L.H.S. = $5 \times 2 + 7 = 17 \neq$ R.H.S.
$x = 3, y = 2$ is a solution of $3x - 2y = 5$,
because L.H.S. = $3x - 2y = 3 \times 3 - 2 \times 2 = 9 - 4 = 5$ R.H.S.
i.e. $x = 3, \ y = 2$ satisfies the equation $3x - 2y = 5$ $\therefore$ it is a solution of the given equation.

Example 1 :
Solve the equation $0.3x + 0.4 = 0.28x + 1.16$.

Sol. $0.3x + 0.4 = 0.28x + 1.16$

$\Rightarrow \frac{3}{10}x + \frac{4}{10} = \frac{28}{100}x + \frac{116}{100}$ [converting decimal fraction into simple fraction]

$\Rightarrow \frac{3x+4}{10} = \frac{28x+116}{100}$

$\Rightarrow$ $100(3x + 4) = 10(28x + 116)$, [By cross multiplication]
$\Rightarrow 10(3x + 4) = 28x + 116$ [Dividing both sides by 10]
$\Rightarrow 30x + 40 = 28x + 116$ [Opening the bracket]
$\Rightarrow 30x - 28x = 116 - 40$ [By transposition]
$\Rightarrow 2x = 76 \Rightarrow x = 76/2 \ \therefore x = 38$

Example 2 :
Solve : $\frac{6x-7}{2x+1} = \frac{3x+1}{x+5}$

Sol. $\frac{6x-7}{2x+1} = \frac{3x+1}{x+5} \Rightarrow (6x - 7)(x + 5) = (3x + 1)(2x + 1)$ (by cross multiplication)
$\Rightarrow 6x^2 + 23x - 35 = 6x^2 + 5x + 1 \Rightarrow 6x^2 + 23x - 6x^2 - 5x = 1 + 35$ (on transposing)
$\Rightarrow x = 36/18 = 2$

Example 3 :
Divide 100 into two parts such that the first part is four times the second.

Sol. Let the first part be = x $\therefore$ Second part = $100 - x$
As per the question $x = 4(100 - x)$ or $x = 400 - 4x$ or $x + 4x = 400$ or $5x = 400$
or $x = 400/5$ $\therefore$ $x = 80$
$\therefore$ First part = 80 and second part is $100 - 80 = 20$. Hence, the required two parts of 100 are 80 and 20.

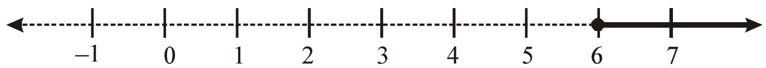
Example 4 :
The sum of two numbers is 58 and their difference is 12. Find the numbers.

Sol. Let one number = $(58 - x)$ $\therefore$ $x - (58 - x) = 12$
$\Rightarrow$ $x - 58 + x = 12 \Rightarrow 2x = 12 + 58 = 70 \Rightarrow x = 70/2 = 35$ $\therefore$ one number = 35

Example 5 :
Solve the inequation $12 + 1\frac{5}{6}x \le 5 + 3x, \ x \in R$. Represent the solution set on a number line.

Sol. We have $12 + 1\frac{5}{6}x \le 5 + 3x \Rightarrow 12 + \frac{11}{6}x \le 5 + 3x$

$\Rightarrow$ $72 + 11x \le 30 + 18$ [Multiplying both sides by 6]
$\Rightarrow$ $11x \le 18x - 42$ [Adding -72 on both sides]
$\Rightarrow$ $-7x \le -42$ [Adding $-18x$ on both sides]
$\Rightarrow$ $x \ge 6$ [Dividing both sides by -7]
$\therefore$ Solution set = $\{x : x \ge 6, x \in R\}$
This set can be represented on the number line, as shown below



Example 6 :
Determine the whether the $x = 2, y = -1$ is a solution of equation $3x + 5y - 2 = 0$.
Sol. Given eq. is $3x + 5y - 2 = 0$ .... (1)
Taking L.H.S. = $3x + 5y - 2 = 3 \times 2 + 5 \times (-1) - 2 = 6 - 5 - 2 = -1 \ne 0$
Here L.H.S. $\ne$ R.H.S. therefore $x = 2, y = -1$ is not a solution of given equations.

SIMULTANEOUS LINEAR EQUATION
When two linear equations in two unknowns are satisfied by the same pair of values of the unknown then these equations are called simultaneous linear equation.
Methods of solving Simultaneous linear equation :
(1) Substitution method :
Suppose we are given two linear equations x and y.
Step 1 : Express y in terms of x from one of the given equations
Step 2 : Substitute this value of y in the second equation to obtain an equation in x. Solve for x.
Step 3 : Substitute the value of x in the relation taken in step 1, to get the value of y.
Note : We may interchange the roles of x and y in the above method.

Example 7 :
Solve : $4x + 3y = 14$, $3x + 2y = 11$
Sol. The given equations are :
$4x + 3y = 14$ .... (1) $3x + 2y = 11$ .... (2)
From (2), we get $3x + 2y = 11 \Rightarrow 2y = (11 - 3x) \Rightarrow y = \frac{1}{2}(11 - 3x)$ .... (3)
Substituting $y = \frac{1}{2}(11 - 3x)$ in (1), we get $4x + \frac{3}{2}(11 - 3x) = 14 \Rightarrow 8x + 3(11 - 3x) = 28$
$\Rightarrow$ $8x + 33 - 9x = 28 \Rightarrow -x = 28 - 33 \Rightarrow -x = -5 \Rightarrow x = 5$
Substituting $x = 5$ in eq. (3), we get
$y = \frac{1}{2}(11 - 3 \times 5) = \frac{1}{2}(11 - 15) = \frac{1}{2} \times (-4) = -2$ $\therefore$ $x = 5$, $y = -2$ is the solution of the given equations.

(2) Elimination Method :
Step 1 : Multiply the given equations by suitable constants so as to make the coefficients of one of the unknowns, numerically equal.
Step 2 : Add the new equations, if the numerically equal coefficient are opposite in sign, otherwise subtract them.
Step 3 : Solve the equation so obtained.
Step 4 : Substitute the value of this unknown in any of the given equations. Solve it to get the value of the other unknown.

Solution of linear equations in two variables by elimination of making equal coefficient :

Let given equations are $a_1x + b_1y + c_1 = 0$ .... (1)

$a_2x + b_2y + c_2 = 0$ .... (2)

For making equal coefficient of y we multiply equations (1) by b_2 and equation (2) by b_1 we get

$$a_1b_2x + b_1b_2y + c_1b_2 = 0$$
$$a_2b_1x + b_2b_1y + c_2b_1 = 0$$
$$- \quad - \quad -$$

$$(a_1b_2 - a_2b_1)\, x = c_2b_1 - c_1b_2$$

$\Rightarrow \quad x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$. Putting this value of x in eq. (1) we get $y = \dfrac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2} = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

Example 8 :

Solve : $3x + 5y = 3$, $4x + 3y = 15$

Sol. The given equations are : $3x + 5y = 3$ .... (1) $4x + 3y = 15$ .... (2)

Multiplying (1) by 4 and (2) by 3, we get $12x + 20y = 12$ .... (3) $12x + 9y = 45$ .... (4)

Subtracting (4) from (3), we get, $11y = -33 \Rightarrow y = -3$

Substituting $y = -3$ in (1), we get $3x + 5 \times (-3) = 3 \Rightarrow 3x - 15 = 3 \Rightarrow 3x = 18 \Leftrightarrow x = 6$

$\therefore$ $x = 6$, $y = -3$ is the solution of the given equations.

Example 9 :

Solve the following equations : $2x - 3y = 5$; $3x + 2y = 1$

Sol. Given eq. are $2x - 3y = 5$ ..... (1) $3x + 2y = 1$..... (2)

Multiplying (1) eq. by 3 and eq. (2) by 2 we get

On subtraction $6x - 9y = 15$

$$6x + 4y = 2$$
$$- \quad - \quad -$$

$$-9y - 4y = 15 - 2 \Rightarrow -13y = 13 \Rightarrow y = 13/-13 \Rightarrow y = -1$$

Substituting in eq. (1), x = 1.

Example 10 :

The weekly incomes of A and B are in the ratio 5 : 6 and their weekly expenditures are in the ratio 8 : 11. If each saves Rs. 1050 per week, find their weekly incomes.

Sol. Let the weekly incomes of A and B be 5x and 6x rupees and their weekly expenditures be 8y and 11y rupees respectively.

Then, A's weekly saving = Rs. $(5x - 8y)$; B's weekly saving = Rs. $(6x - 11y)$

$\therefore$ $5x - 8y = 1050$ .... (1) $6x - 11y = 1050$ .... (2)

Multiplying (1) by 6, (2) by 5 and subtracting we get

$$30x - 48y = 6300$$
$$30x - 55y = 5250$$
$$- \quad + \quad -$$

$$7y = 1050 \quad \therefore \quad 7y = 1050 \Rightarrow y = 150$$

Substituting y = 150 in (1), we get $5x - 8 \times 150 = 1050 \Rightarrow 5x = 1050 + 1200 \Rightarrow 5x = 2250 \Rightarrow x = 450$

$\therefore$ A's weekly income = 5x = Rs. (5×450) = Rs. 2250

B's weekly income = 6x = Rs. (6×450) = Rs. 2700

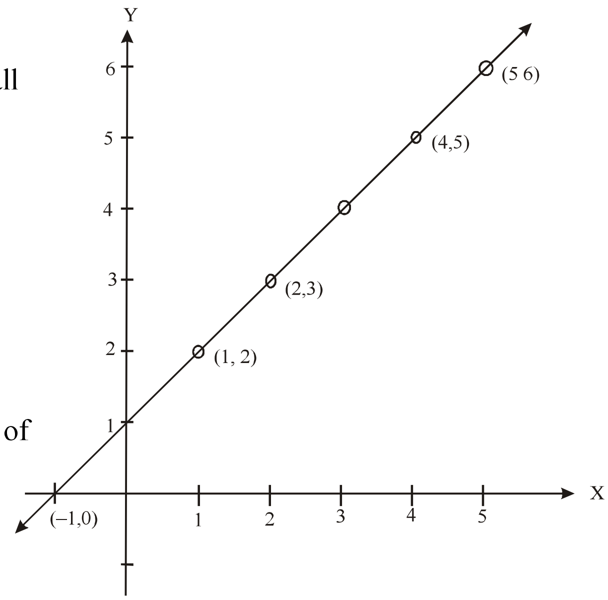
Example 11 :

Find two solutions for $3y + 4 = 0$

Sol. Writing the equation $3y + 4 = 0$ as $0.x + 3y + 4 = 0$,
we will find that $y = -4/3$ for any value of x.
Thus, two solutions can be given as $(0, -4/3)$ and $(1, -4/3)$

GRAPH OF LINEAR EQUATIONS

The graph of an equation in x and y is the set of all points whose coordinates satisfy the equation :



Example 12 :

The linear equation $y = x + 1$ is satisfied by the values of x and y, given in the table $x, y \in I$

x	-1	1	2	4	5
y	0	2	3	5	6

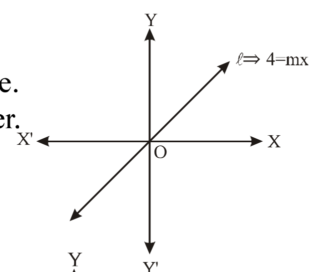
Linear function : $y = mx + c$ where m = slope of the line, c = intercept on the y-axis

How to draw the graph of a linear equation ?

$ax + by + c = 0$

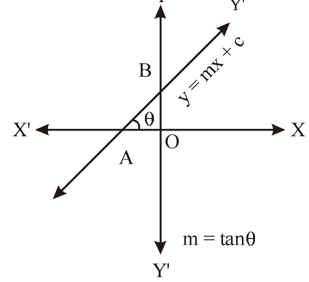
Steps

1. Make y as the subject of the formula (i.e. $y = mx + c$)
2. Select at least three values of x, such that $x, y \in I$
3. Make a table for the ordered pair : (x, y)
4. Plot these ordered pairs (points) on a graph paper, selecting suitable scale.
5. Draw a straight line passing through the points plotted on the graph paper.



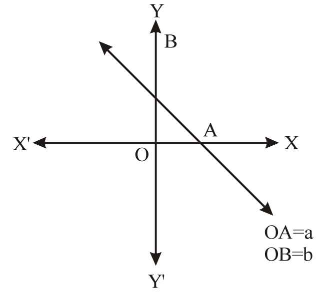
Different forms of a line :

(i) The equation of a line passing through origin is $y = mx$.
Slope of line : If a line makes an angle θ with positive direction of x-axis then tangent of this angle is called the slope of a line, it is denoted by m, i.e., $m = \tan\theta$.



(ii) Slope-intercept form is $y = mx + c$ where m is the slope of line and c is intercept made by line with y-axis.

(iii) Intercept form of a line is $\frac{x}{a} + \frac{y}{b} = 1$
where a and b are intercepts on both axis respectively made by line.



Note : A point which lies on the line is a solution of that equation. A point not lying on the line is not a solution of the equation.

(B) In order to draw the graph of a linear equation ax + by + c = 0 may follow the following algorithm.

Step 1 : Obtain the linear equation ax + by + c = 0

Step 2 : Express y in terms of x i.e. $y = -\left(\frac{ax + b}{c}\right)$

Step 3 : Put any two or three values for x and calculate the corresponding values of y from the expression values of y from the expression obtained in step 2. Let we get points as $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$

Step 4 : Plot points $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ on graph paper.

Step 5 : Join the points marked in step 4 to obtain. The line obtained is the graph of the equation ax + by + c = 0.

Example 13 :

Draw the graph of the line x – 2y = 3

Sol. Here given equation is x – 2y = 3

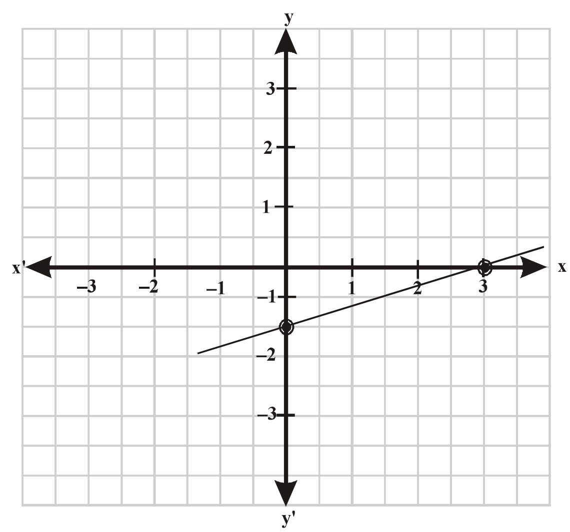
Solving it for y we get, 2y = x – 3

$\Rightarrow \quad y = \frac{x-3}{2}$

Let x = 0, then $y = \frac{0-3}{2} = \frac{-3}{2}$

x = 3, then $y = \frac{3-3}{2} = 0$

x = – 2, then $y = \frac{-2-3}{2} = \frac{-5}{2}$



Hence we get,

x	0	3	-2
y	-3/2	0	-5/2

Solving a pair of simultaneous equations graphically :

To solve a pair of simultaneous equations graphically, we first draw the graphs of the two given equations on the same graph paper. The co-ordinates of the point of intersection of the two lines give the solution.

Example 14 :

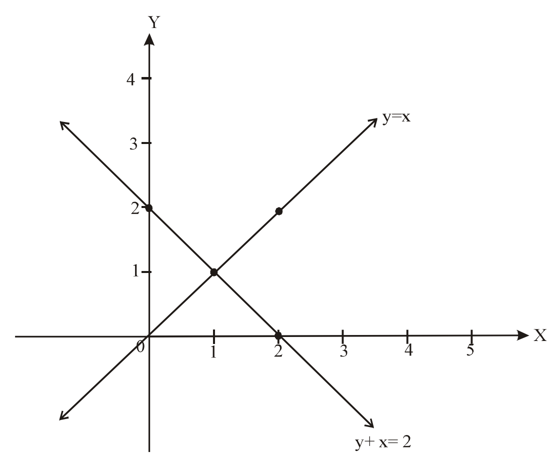
Solve graphically y = x and y = – x + 2

Sol. Table (i) y = x

x	0	1	2
y	0	1	2

Table (ii) y = – x + 2

x	0	1	2
y	2	1	0



From the graph, we find that the lines meet at (1, 1). Hence solution set = [1, 1]

ADDITIONAL EXAMPLES

Example 1 :

Express y in terms of x, given that 2y – 4x = 7. Check whether (–1, –1) is a solution of the line.

Sol. Given equation is : $2y - 4x = 7 \Rightarrow 2y = 7 + 4x \Rightarrow y = \frac{7+4x}{2}$

Now substituting x = – 1, y = – 1 in the equation, we get

$-1 = \frac{7+4(-1)}{2} \Rightarrow -1 = \frac{7-4}{2} \Rightarrow -1 \neq \frac{3}{2}$ $\therefore$ Hence LHS $\neq$ RHS

Therefore point (–1, –1) does not lie on the line 2y – 4x = 7

Example 2 :

Find the solution of the form x = a, y = 0 and x = 0, y = b for equations 3x + 2y = 12 and 5x – 2y = 10

Sol. The given equation is $3x + 2y = 6$

Since the solution of the form (a, 0) is required, therefore substituting y = 0, we get

$3x + 2 \times 0 = 6 \Rightarrow 3x = 6 \Rightarrow x = 2 \Rightarrow$ (2, 0) is a solution.

When the solution is of the form (0, b), substitute x = 0

$3 \times 0 + 2y = 6 \Rightarrow y = 3$, so solution is (0, 3)

Thus (2, 0) and (0, 3) are two solutions. Now given equation is $5x - 2y = 10$

Since the solution (a, 0) is required, put y = 0 in the equation

$5x - 2 \times 0 = 10 \Rightarrow x = 2$ $\therefore$ Solution is (2, 0)

Now solution of the form (0, b)

Substitute x = 0, $5 \times 0 - 2y = 10 \Rightarrow y = -5$. So solution is (0, –5)

$\therefore$ The two solutions for the given equation are (2, 0) and (0, –5)

Hence common solution in both the equation is (2, 0)

Example 3 :

Draw the graph of the equation x – y + 3 = 0. Use it to find some solution of the equations and check from the graph that x = 0 and y = 3 is a solution.

Sol. The given equation is $x - y + 3 = 0$

To draw the graph we use the table of corresponding values of x and y.

x	–2	–6	2
y	1	–3	+5

$x - y + 3 = 0 \Rightarrow x = -3 + y$

We have drawn the graph of x – y + 3 = 0 by plotting the points given in the table in figure.

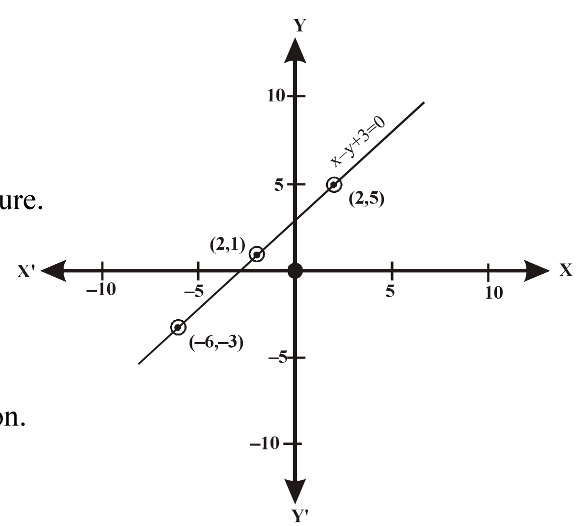
As shown in the figure some of the other solutions of x – y + 3 = 0 are :

x = 6, y = 9 ; x = – 11, y = – 8 ;

x = 1, y = 4

The point x = 0 and y = 3 is on the graph.

Hence x = 0, y = 3 is a solution of the equation.



Example 4 :

Solve the following systems of simultaneously linear equations graphically :

x + y = 3 ; 2x + 5y = 12

Sol. x + y = 3 .... (1) 2x + 5y = 12 .... (2)

(i) x + y = 3 $\Rightarrow$ y = 3 – x

Table for values of x and corresponding values of y.

x	–4	0	10
y	7	3	–7

Plot these points on graph paper. Join these points to get a line.

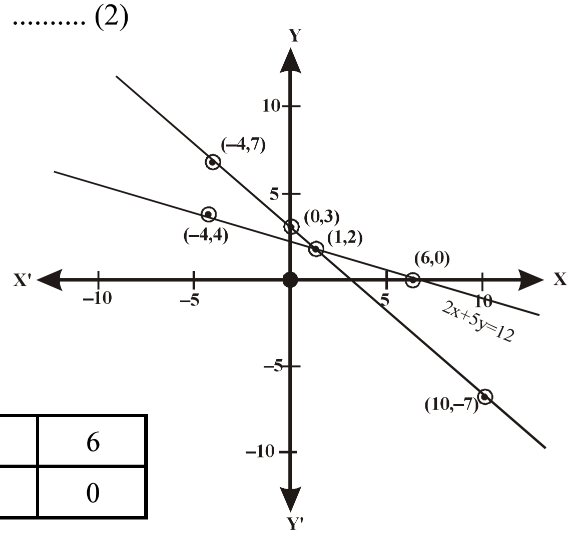
(ii) 2x + 5y = 12

$\Rightarrow$ 5y = 12 – 2x

$\Rightarrow$ $y = \frac{12-2x}{5}$

Table for values of x and corresponding values of y.

x	–4	1	6
y	4	2	0



Plot these points on graph paper.

Join these points to get a line. The graphs of the two lines are drawn in figure.

They are seen to intersect in the point (1, 2), so that the solution is x = 1, y = 2.

Example 5 :

Solve the following system of equations : 5x + 3y = 70 ; 3x – 7y = 60

Sol. 5x + 3y = 70 .... (1) 3x – 7y = 60 .... (2)

Multiplying (1) by 7 and (2) by 3, we get

35x + 21y = 490 .... (3) 9x – 21y = 180 .... (4)

Adding (3) and (4), we get 44x = 670 $\Rightarrow x = \frac{670}{40} = \frac{335}{2}$

Substituting the value of x in 5x + 3x = 70 ; $5\left(\frac{335}{22}\right) + 3y = 70 \Rightarrow \frac{1675}{22} + 3y = 70$

$\Rightarrow 3y = 70 - \frac{1675}{22} = \frac{1540-1675}{22} \Rightarrow y = \frac{-135}{22} \times \frac{1}{3} = \frac{-45}{22}$. So, the solution is $x = \frac{355}{22},\ y = \frac{-45}{22}$

Example 6 :

There are two examination halls A and B. If 10 candidates are sent from A to H, the number of students in each room becomes equal. If 20 students are sent from B to A, the number of students in A becomes double the number of students in B. Find the number of students in each room.

Sol. Let the number of students in room A be x and the number of students in room B be y.

When 10 candidates are sent from A to B, the number of students in A is x – 10

and the number of students in B is y + 10. Since the number of students becomes equal in each room,

$\therefore$ x – 10 = y + 10 or x – y = 20 .... (1)

When 20 candidates are sent from B to A the number of students in A = x + 20 and the number of students in B = y – 20. Now, the number of students in A is double the number in B

$\therefore$ x + 20 = 2 (y – 20) or x + 20 = 2y –40 or x – 2y = – 60 .... (2)

Solving (1) and (2) we get x = 100 and y = 80

$\therefore$ the number of students in A = 100 and the number of students in B = 80

Example 7 :

3 chairs and 2 tables cost Rs. 700 and 5 chairs and 3 tables cost Rs. 1100. What is the cost of 2 chairs and 2 tables.

Sol. Let the cost of one chair be Rs. x and that of one table be Rs. y

According to the question, $3x + 2y = 700$ .... (1) and $5x + 3y = 1100$ .... (2)

Multiplying (1) by (3) and (2) by 2, we have

$9x + 6y = 2100$ .... (3) and $10x + 6y = 2200$ .... (4)

Subtracting (3) from (4), we get, $x = 100$; Substituting the value of x in $3x + 2y = 700 \Rightarrow 300 + 2y = 700$

$$2y = 700 - 300 \Rightarrow 2y = 400 \Rightarrow y = \frac{400}{2} = 200$$

$\therefore$ Cost of one chair = Rs. 100 and cost of one table = Rs. 200

Cost of 2 chairs + 2 tables = Rs. 100 × 2 + Rs. 200 × 2 = Rs. 200 + Rs. 400 = Rs. 600

Example 8 :

Solve by method of substitution : $5x - 2y = 19$ and $3x + 7 = 18$

Sol. $5x - 2y = 19$ .... (1) ; $3x + 7 = 18$ .... (2). From eq. (2), $y = 18 - 3x$ .... (3)

Putting the value of y from (3) in (1), we get $5x - 2(18 - 3x) = 19$

or $5x - 36 + 6x = 19$ or $11x = 19 + 36$ or $11x = 5$ or $x = 5$

Putting the value of x in (3), we get $y = 18 - (3 \times 5) = 18 - 15 = 3$; $x = 5, y = 3$

Example 9 :

The cost of a book is twice the cost of a fountain pen. Write a linear equation in two variables to represent this statement.

Sol. Let the cost of the book = Rs. x and cost of the fountain pen = Rs. y.

Since cost of a book = Twice the cost of pen $\therefore$ $x = 2y \Rightarrow x - 2y = 0$

Example 10 :

Given the point (2, 11), find the equation of a line on which it lies. How many such equations are there ?

Sol. Given point (2, 11) i.e. x = 2 and y = 11

By addition, we get $x + y = 2 + 11 = 13$; By subtraction, we get $x - y = 2 - 11 = -9$

Hence $x + y = 13$ and $x - y = -9$ are two lines passing through (2, 11). Infinite number of lines may be drawn through the point i.e. (2, 11).

Example 11 :

The cost of petrol is Rs. 45/litre. Construct a linear equation with x representing the number of litres and y representing the total cost in rupees. Draw the graph also.

Sol. The cost of petrol = Rs. 45/litre.

Let the number of litres of petrol used = x litres.

And total cost of Petrol = Rs. y

$\therefore y = 45x$

Required table :

x	0	1
y	0	45

Plot the points (0, 0), (1, 45) and join any two points to get the line.