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INTRODUCTION TO EUCLID’S GEOMETRY

INTRODUCTION

The term 'Geometry' is composed of two Greek words ‘'Geo’ and ‘Metron’. The word 'Geo' means earth and ‘Metron] means ‘to measure’. In this way the evolution of Geometry can be linked with that period of the development of human civilization when the man felt the need of measuring land areas for the first time.

In most of the modern books it has been established that the people of Egypt, for the first time, found out the solutions of the problems of mensuration such as finding the areas of triangles, rectangles and linear shapes. After this the people of Babylon also formulated formulae for finding out the area of various linear shapes. which are available in – Rhind Papyrus (1650 BC) the ancient mathematics of the people of Babylon. But the relics found at Harappa and Mohan-jo-dro (now in Pakistan). Kalibanga (Rajasthan) and Lothal (Gujarat) testify clearly that a rich civlization flourished in a large track of land during the period extending from 2500 B.C. to 1750 B.C. in ancient India. The relics of this civilization prove that the people of this period had special knowledge of Geometry and Geometrical formations. On the basis of this knowledge they constructed buildings, roads, circles, arcs where mensuration is of great importance.

BOUNDHAYAN THEOREM (800 B.C.)

“Deergh chatur srisya khasnya Rajjuh Parshavmani
Thyakmani yatprithambhute kurutastadubhayam karoti"

The sum of the areas of squares formed on the perpendicular line and the base line of a rectangle is equal to the area of the square formed on the diagonal. It is worth noting that Pythagoras (580 B.C.) established this theorem about 300 years after ‘Boudhayan’. Hence it is proper to call this theorem as Boudhayan theorem.

Among Indian Geometricians are Bhraham Gupta (598 A.D.) who found out the area of cyclic quadrilateral in terms of its perimeter and arms, Arya Bhatt (476 A.D.) who found out the area of equilateral triangle, volume of pyramid and value of π and Bhaskar-II (1114 A.D.) who proved the Boudhayan theorem by split method.

Later on the Greek mathematicians (300 B.C.) systematized his knowledge by providing its facts through inductive reasoning and published it in the book titled ‘Elements’. These days we study Geometry in this way.

EUCLID’S DEFINITIONS, AXIOMS AND POSTULATES

Euclid was an ancient Greek mathematician from Alexandria who is best known for his major work, Elements. In Euclid's method, deductions are made from premises or axioms. This deductive method, as modified by Aristotle, was the sole procedure used for demonstrating scientific certitude ("truth") until the seventeenth century.

At the time of its introduction, Elements was the most comprehensive and logically rigorous examination of the basic principles of geometry.

Euclid's Elements form one of the most beautiful and influential works of science in the history of humankind.

Its beauty lies in its logical development of geometry and other branches of mathematics. It has influenced all branches of science but none so much as mathematics and the exact sciences.

Euclid divided his axioms into two categories, calling the first five postulates and the next five "common notions." The distinction between postulates and common notions is that the postulates are geometric in character, whereas common notions were considered by Euclid to be true in general.

A system of axioms is called consistent, if it is impossible to deduce from these axioms a statement that contradicts any axiom or previously proved statement. So, when any system of axioms is given, it needs to be ensured that the system is consistent.

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called propositions or theorems. Euclid deduced 465 propositions in a logical chain using his axioms, postulates, definitions and theorems proved earlier in the chain.

23 DEFINITIONS FROM BOOK 1

Definition 1 : A point is that which has no part.

Definition 2 : A line is breadthless length.

Definition 3 : The ends of a line are points.

Definition 4 : A straight line is a line which lies evenly with the points on itself.

Definition 5 : A surface is that which has length and breadth only.

Definition 6 : The edges of a surface are lines.

Definition 7 : A plane surface is a surface which lies evenly with the straight lines on itself.

Definition 8 : A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9 : And when the lines containing the angle are straight, the angle is called rectilinear.

Definition 10 : When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Definition 11 : An obtuse angle is an angle greater than a right angle.

Definition 12 : An acute angle is an angle less than a right angle.

Definition 13 : A boundary is that which is an extremity of anything.

Definition 14 : A figure is that which is contained by any boundary or boundaries.

Definition 15 : A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16 : And the point is called the centre of the circle.

Definition 17 : A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Definition 18 : A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Definition 19 : Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Definition 20 : Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definition 21 : Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22 : Of quadrilateral figures, a square is that which is both equilateral and right-angled; an along that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Definition 23 : Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

COMMON NOTIONS

Common notion 1 : Things which equal the same thing also equal one another.

Common notion 2 : If equals are added to equals, then the wholes are equal.

Common notion 3 : If equals are subtracted from equals, then the remainders are equal.

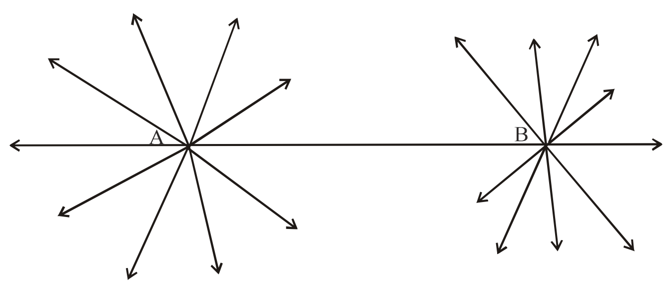
Common notion 4 : Things which coincide with one another equal one another.

Common notion 5 : The whole is greater than the part.

POSTULATES

Let the following be postulated:

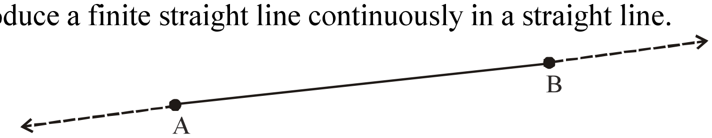
Postulate 1 : To draw a straight line from any point to any point.



This first postulate says that given any two points such as A and B, there is a line AB which has them as end points. This is one of the constructions that may be done with a straight edge (the other being described in the next postulate).

Although it doesn't explicitly say so, there is a unique line between the two points. Since Euclid uses this postulate as if it includes the uniqueness as part of it, he really ought to have stated the uniqueness explicitly.

Postulate 2 : To produce a finite straight line continuously in a straight line.



Here we have the second ability of a straightedge, namely, to extend a given line AB to CD. This postulate does not say how far a line can be extended. Sometimes it is used so that the extension equals some other line. Other times it is extended arbitrarily far.

Postulate 3 : To describe a circle with any centre and radius.

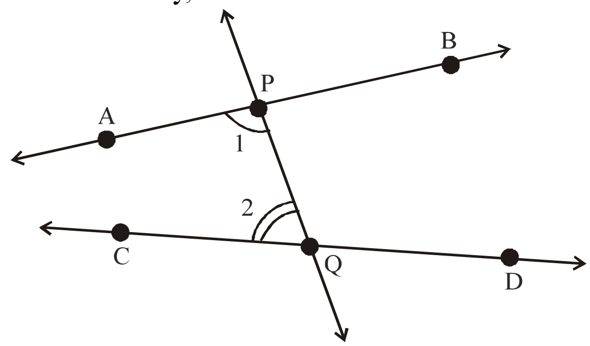
Circles were defined as plane figures with the property that there is a certain point, called the center of the circle, such that all straight lines from the centre to the boundary are equal. That is, all the radii are equal. The given data are (1) a point A to be the centre of the circle, (2) another point B to be on the circumference of the circle, and (3) a plane in which the two points lie. In the first few books of the Elements, there is but one plane under consideration and needn't be mentioned, but in the last three books which develop solid geometry, the plane has to be specified.

Note that this postulate does not allow for the compass to be moved. The usual way that a compass is used is that is opened to a given width, then the pivot is placed on the drawing surface, then a circle is drawn as the compass is rotated around the pivot. But this postulate does not allow for transferring distances. It is as if the compass collapses as soon as it's removed from the plane

Postulate 4 : That all right angles equal one another.

Postulate 5 : That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

For example, the line PQ in Fig. falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.

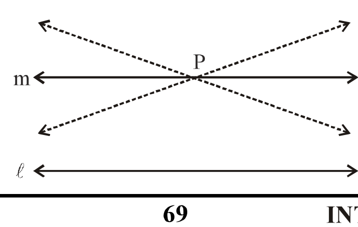


This postulate is usually called the "parallel postulate" since it can be used to prove properties of parallel lines.

The parallel postulate is historically the most interesting postulate. Geometers throughout the ages have tried to show that it could be proved from the remaining postulates so that it wasn't necessary to assume it. The process tried was to assume its falsehood, then derive a contradiction. Many strange conclusions follow from denying the parallel postulate, and several geometers found such great absurdities that they concluded that the parallel postulate did follow from the rest.

Equivalent versions of Euclid's fifth Postulate : Starting almost immediately after the publication of the Elements and continuing into the nineteenth century, mathematicians tried to demonstrate that Euclid's fifth postulate was unnecessary. That is, they attempted to upgrade the fifth postulate to a theorem by deducing it logically from the other nine. Many thought they had succeeded; invariably, however, some later mathematician would discover that in the course of his "proofs" he had unknowingly made some extra assumption, beyond the allowable set of postulates, that was in fact logically equivalent to the fifth postulate. In the early nineteenth century, after more than 2,000 years of trying to prove Euclid's fifth postulate, mathematicians began to entertain the idea that perhaps it was not provable after all and that Euclid had been correct to make it an axiom. Not long after that, several mathematicians, working independently, realized that if the fifth postulate did not follow from the others, it should be possible to construct a logically consistent geometric system without it.

One of the many statements that were discovered to be equivalent to the fifth postulate (in the course of the many failed attempts to prove it) is "Given a straight line, and a point P not on that line, there exists at most one straight line passing through P that is parallel to the given line. Given by John Dlayfar "The first "non-Euclidean" geometers took as axioms all the other nine postulates of Euclidean geometry but replaced the fifth postulate with the statement "There exists a straight line, and a point P not on that line, such that there are two straight lines passing through P that are parallel to the given line." That is, they replaced the fifth postulate with its negation and started exploring the geometric system that resulted.



Non-Euclidean Geometry : Non-Euclidean geometry is any geometry that is different from Euclidean geometry. Each Non-Euclidean geometry is a consistent system of definitions, assumptions, and proofs that describe such objects as points, lines and planes. The two most common non-Euclidean geometries are spherical geometry and hyperbolic geometry. The essential difference between Euclidean geometry and these two non-Euclidean geometries is the nature of parallel lines: In Euclidean geometry, given a point and a line, there is exactly one line through the point that is in the same plane as the given line and never intersects it. In spherical geometry there are no such lines. In hyperbolic geometry there are at least two distinct lines that pass through the point and are parallel to (in the same plane as and do not intersect) the given line.

Spherical Geometry : Spherical geometry is a plane geometry on the surface of a sphere. In a plane geometry, the basic concepts are points and lines. In spherical geometry, points are defined in the usual way, but lines are defined such that the shortest distance between two points lies along them. Therefore, lines in spherical geometry are great circles. A great circle is the largest circle that can be drawn on a sphere. The longitude lines and the equator are great circles of the Earth. Latitude lines, except for the equator, are not great circles. Great circles are lines that divide a sphere into two equal hemispheres.
Spherical geometry is used by pilots and ship captains as they navigate around the globe. Working in spherical geometry has some non-intuitive results. For example, did you know that the shortest flying distance from Florida to the Philippine Islands is a path across Alaska? The Philippines are south of Florida - why is flying north to Alaska a short-cut? The answer is that Florida, Alaska, and the Philippines are collinear locations in spherical geometry (they lie on a great circle). Another odd property of spherical geometry is that the sum of the angles of a triangle is always greater then 180°. Small triangles, like those drawn on a football field, have very, very close to 180°. Big triangles, however, (like the triangle with veracities: New York, New Delhi and Tokyo) have significantly more than 180°.
Hyperbolic Geometry : Hyperbolic geometry is the geometry of which the NonEuclid software is a model. Hyperbolic geometry is a "curved" space, and plays an important role in Einstein's General theory of Relativity.

ADDITIONAL EXAMPLES

Example 1 :

If A, B and C are three points on a line, and B lies between A and C (Fig.), then prove that AB + BC = AC.

Sol. In the figure given above, AC coincides with AB + BC.
Also, Euclid’s Axiom (4) says that things which coincide with one another are equal to one another.
So, it can be deduced that AB + BC = AC.
Note that in this solution, it has been assumed that there is a unique line passing through two points.

Example 2 :

Consider the following statement : There exists a pair of straight lines that are everywhere equidistant from one another. Is this statement a direct consequence of Euclid’s fifth postulate? Explain.

Sol. Take any line ℓ and a point P not on ℓ. Then, by Playfair’s axiom, which is equivalent to the fifth postulate, we know that there is a unique line m through P which is parallel to ℓ.
Now, the distance of a point from a line is the length of the perpendicular from the point to the line. This distance will be the same for any point on m from ℓ and any point on ℓ from m. So, these two lines are everywhere equidistant from one another.

Example 3 :