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INVERSE TRIGONOMETRIC FUNCTIONS

INTRODUCTION

Inverse functions relating trigonometrical ratios are called inverse trigonometric functions.

Let $\sin\theta = x$ then $\theta = \text{Arc} \sin x$

This means Arc sinx represent the angle whose sine is equal to x. $\because -1 \le \sin\theta \le 1$ and $\sin\theta = x \therefore -1 \le x \le 1$

Thus, Arc sinx is defined only when $-1 \le x \le 1$

Clearly, for every $x \in [-1, 1]$, infinite number of values of Arc sinx will be obtained. i.e., Arc sinx denotes the general value of θ satisfying $\sin\theta = x$ when we consider the principal value of the angle θ satisfying $\sin\theta = x$ then we symbolically write $\theta = \text{arc} \sin x$ or $\sin^{-1} x$ (read as sine inverse x)

Thus, $\sin^{-1}x$ or arc sinx is the principal value of the angle whose sine is equal to x. Similar definition for $\cos^{-1}x$ or arc cosx, $\tan^{-1}x$ or arc tanx etc. can be given. i.e.

(i) $\sin\theta = x \Leftrightarrow \sin^{-1}x = \theta$

(ii) $\cos\theta = x \Leftrightarrow \cos^{-1}x = \theta$

(iii) $\tan\theta = x \Leftrightarrow \tan^{-1}x = \theta$

(iv) $\cot\theta = x \Leftrightarrow \cot^{-1}x = \theta$

(v) $\sec\theta = x \Leftrightarrow \sec^{-1}x = \theta$

(vi) $\text{cosec}\,\theta = x \Leftrightarrow \text{cosec}^{-1}x = \theta$

DOMAIN AND RANGE OF ITF

Function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

INTERVALS FOR INVERSE FUNCTION

(a) $\sin^{-1}x$, $\text{cosec}^{-1}x$, $\tan^{-1}x$: belongs to I and IV quadrant.

(b) $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$: belongs to I and II quadrant.

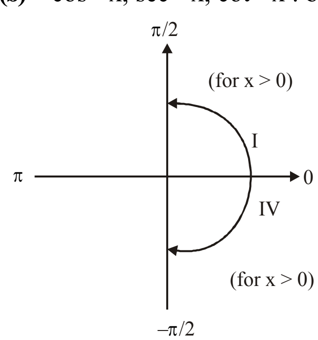


Figure (a)

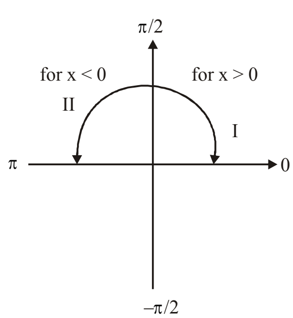


Figure (b)

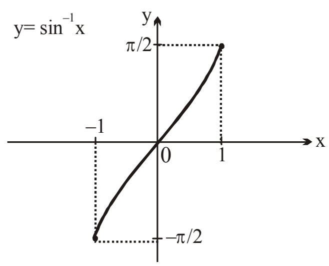
NOTE

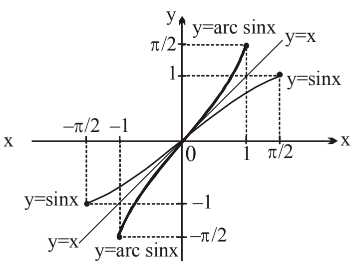
1. I quadrant is common to all the inverse functions.
2. III quadrant is not used in inverse function.
3. IV quadrant is used in the clockwise direction

i.e. $-\frac{\pi}{2} \le y \le 0$.

GRAPHS OF BASIC ITF

(1) $y = \sin^{-1} x, |x| \le 1, \; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





Note : Graph of $y = \sin^{-1}x$ and $y = \sin x$ are mirror image of each other about the line y = x.

NOTE

(i) $\sin^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

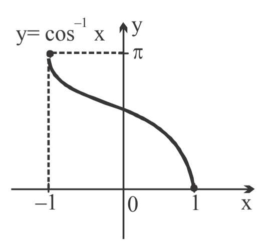
(ii) $\sin^{-1}x$ is an odd function. (symmetric about origin)

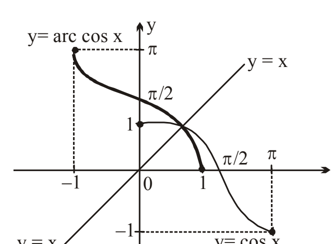
(iii) $\sin^{-1}x$ is an increasing function in its domain.

(iv) Maximum value of $\sin^{-1}x = \pi/2$, occurs at $x = 1$ and minimum value of $\sin^{-1} x = -\pi/2$, occurs at $x = -1$.

(v) $\sin^{-1}x$ is an aperiodic function.

(2) $y = \cos^{-1} x, |x| \le 1, \; y \in [0, \pi]$





Note : Graph of $y = \cos^{-1}x$ and $y = \cos x$ are mirror image of each other about the line y = x.

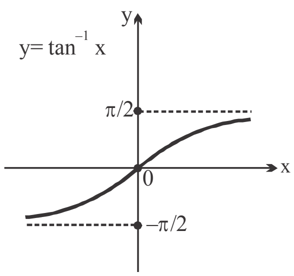
NOTE

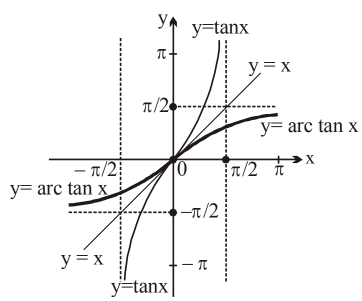
(i) $\cos^{-1}x$ is bounded in $[0, \pi]$.

(ii) $\cos^{-1}x$ is a neither odd nor even function.

(iii) $\cos^{-1}x$ is a decreasing function in its domain.
(iv) Maximum value of $\cos^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\cos^{-1} x = 0$, occurs at $x = 1$.
(v) $\cos^{-1}x$ is an aperiodic function.

(3) $y = \tan^{-1} x, x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



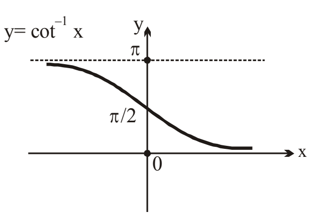


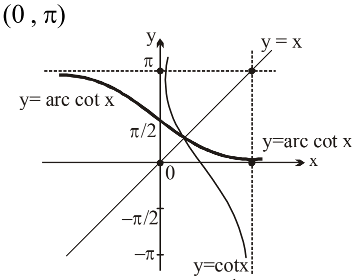
Note : Graph of $y = \tan^{-1}x$ and $y = \tan x$ are mirror image of each other about the line $y = x$.

NOTE

(i) $\tan^{-1}x$ is bounded in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
(ii) $\tan^{-1}x$ is an odd function. (symmetric about origin).
(iii) $\tan^{-1}x$ is an increasing function in its domain.
(iv) $\tan^{-1}x$ is an aperiodic function.

(4) $y = \cot^{-1} x, x \in R,\ y \in (0, \pi)$

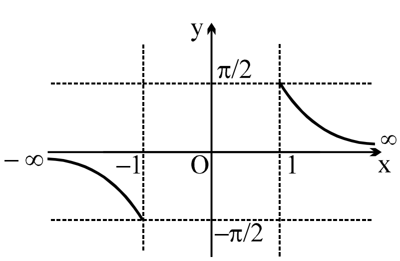




Note : Graph of $y = \cot^{-1}x$ and $y = \cot x$ are mirror image of each other about the line $y = x$.

NOTE
(i) $\cot^{-1}x$ is bounded in $(0, \pi)$.
(ii) $\cot^{-1}x$ is a neither odd nor even function.
(iii) $\cot^{-1}x$ is a decreasing function in its domain.
(iv) $\cot^{-1}x$ is an aperiodic function.

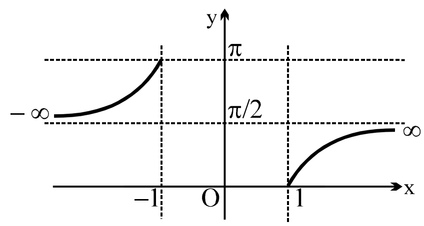
(5) $y = \operatorname{cosec}^{-1}x, |x| \ge 1,\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



NOTE

(i) $\operatorname{cosec}^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
(ii) $\operatorname{cosec}^{-1}x$ is an odd function. (symmetric about origin)
(iii) Maximum value of $\operatorname{cosec}^{-1}x = \pi/2$, occurs at $x = 1$ and minimum value of $\operatorname{cosec}^{-1}x = \pi/2$, occurs at $x = -1$.
(iv) $\operatorname{cosec}^{-1}x$ is a decreasing function.
(v) $\operatorname{cosec}^{-1}x$ is an aperiodic function.

(6) $y = \sec^{-1} x, |x| \ge 1,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



NOTE
(i) $\sec^{-1}x$ is bounded in $[0, \pi]$.
(ii) $\sec^{-1}x$ is a neither odd nor even function.
(iii) Maximum value of $\sec^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\sec^{1} x = 0$, occurs at $x = 1$.
(iv) $\sec^{-1}x$ is an increasing function.
(v) $\sec^{-1}x$ is an aperiodic function.
(vi) $\tan^{-1}(x)$ and $\cot^{-1}(x)$ are continuous and monotonic on $R \not\Rightarrow$ that their range is R
(vii) If $f(x)$ is continuous and has a range $R \not\Rightarrow$ it is monotonic.
e.g. $y = x^3 - 3x$.

Example 1 :
Find the principal value of $\sin^{-1}(1/2)$.

Sol. We know that $\sin^{-1}x$ denotes an angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x for $x \in [-1, 1]$

$\because \sin^{-1}\left(\frac{1}{2}\right)$ = An angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example 2 :
Find domain and range of the following
(a) $\sin^{-1}[x]$ (b) $\cos^{-1}\{x\}$
(c) $\sin^{-1}(e^x)$ (d) $f(x) = \tan^{-1}(\log_{4/5}(5x^2 - 8x + 4))$
(where [x] denotes the greatest integer function and {x} denotes the fractional part function.)

Sol. (a) $\sin^{-1}[x]$ defined when $-1 \le [x] \le 1 \Rightarrow -1 \le x < 2$
domain : $x \in [-1, 2)$
In this domain [x] takes the values –1, 0, 1
$\Rightarrow$ Range of $\sin^{-1}[x] = \{\sin^{-1} -1, \sin^{-1} 0, \sin^{-1} 1\}$

Range $= \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

(b) $\cos^{-1}\{x\}$ defined when $-1 \le \{x\} \le 1$

$\Rightarrow$ domain : $x \in R$ $\quad (\because \{x\} \in [0, 1))$

Range $= \cos^{-1}[0, 1) = (\cos^{-1} 1, \cos^{-1} 0]$

Range $= (0, \pi/2]$

(c) $\sin^{-1} e^x$ defined when

$-1 \le e^x \le 1 \Rightarrow e^x \ge -1$ holds always true

So $e^x \le 1 \Rightarrow x \le 0$

domain $x \in (-\infty, 0]$

In this domain $e^x \in (0, 1]$

$\Rightarrow$ Range of $\sin^{-1} e^x = \sin^{-1}(0, 1]$

$= \sin^{-1}(0, 1] = (\sin^{-1} 0, \sin^{-1} 1]$

Range $= (0, \pi/2]$

(d) $f(x)$ is defined when $5x^2 - 8x + 4 > 0$

$\because$ $a > 0, D < 0 \Rightarrow 5x^2 - 8x + 4 > 0$ is true for all $x \in R$

$\Rightarrow$ domain : $x \in R$

Now, $5x^2 - 8x + 4 = 5\left[\left(x - \frac{4}{5}\right)^2 + \frac{4}{25}\right]$

$\Rightarrow$ $5x^2 - 8x + 4 \in \left[\frac{4}{5}, \infty\right)$ for $x \in R$

$\Rightarrow$ Range of $f(x) = \tan^{-1}\left(\log_{4/5}\left[\frac{4}{5}, \infty\right)\right)$

Range $= \tan^{-1}(-\infty, 1] = \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS:

Property-1

(i) $\sin^{-1}(\sin\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta) = \theta$, Provided that $0 \le \theta \le \pi$

(iii) $\tan^{-1}(\tan\theta) = \theta$, Provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot\theta) = \theta$, Provided that $0 < \theta < \pi$

(v) $\sec^{-1}(\sec\theta) = \theta$, Provided that $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ Provided that

$-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

Example 3:

Evaluate $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

Sol. $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$ (by the property I $\sin^{-1}(\sin\theta) = \theta$)

Example 4 :

Evaluate : $\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$

Sol. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \ne \frac{7\pi}{6}$

[because $\frac{7\pi}{6}$ does not lie between 0 and π]

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$

$\left(\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right)$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ $\quad [\because \cos(2\pi - \theta) = \cos\theta]$

Property-2

(i) $\sin(\sin^{-1}x) = x$, Provided that $-1 \le x \le 1$

(ii) $\cos(\cos^{-1}x) = x$, Provided that $-1 \le x \le 1$

(iii) $\tan(\tan^{-1}x) = x$, Provided that $-\infty < x < \infty$

(iv) $\cot(\cot^{-1}x) = x$, Provided that $-\infty < x < \infty$

(v) $\sec(\sec^{-1})x = x$, Provided that $-\infty < x \le 1$ or $1 \le x < \infty$

(vi) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$,

Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

Example 5 :

Find the value of $\tan\left(\cot^{-1}\frac{1}{a}\right)$.

Sol. $\tan\left(\cot^{-1}\frac{1}{a}\right) = \tan(\tan^{-1} a) = a$

Property-3

(i) $\sin^{-1}(-x) = -\sin^{-1}x$ Provided that $-1 \le x \le 1$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$ Provided that $-1 \le x \le 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$ Provided that $-\infty < x < \infty$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$ Provided that $-\infty < x < \infty$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$ Provided that $-\infty < x \le 1$ or $1 \le x < \infty$

(vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

Example 6 :

Find the value of $\cos^{-1}(-1)$.

Sol. $\cos^{-1}(-1) = \pi - \cos^{-1}(1) = \pi - 0 = \pi$

Property-4

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $\quad x \in [-1,1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $\quad x \in R$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ $\quad x \in (-\infty, -1] \cup [1, \infty]$

Example 7 :

Find the value of $\sin\left\{\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right\}$.

Sol. $\sin\left\{\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right\} = \sin\left(\frac{\pi}{2}\right)$
$=1$

$\left(\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$

Example 8 :

If $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$, then find the value of x.

Sol. $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6} \Rightarrow \frac{\pi}{2}-\cos^{-1}x-\cos^{-1}x=\frac{\pi}{6}$

$\Rightarrow 2\cos^{-1}x=\frac{\pi}{2}-\frac{\pi}{6}=\frac{2\pi}{6}=\frac{\pi}{3}$

$\Rightarrow \cos^{-1}x=\frac{\pi}{6}$ $\quad \therefore x=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

Property-5 - Conversion Property :

(i) $\sin^{-1}x=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$; $|x|\le 1, x\ne 0$

and $\operatorname{cosec}^{-1}x=\sin^{-1}\left(\frac{1}{x}\right)$, $|x|\ge 1$

(ii) $\cos^{-1}x=\sec^{-1}\left(\frac{1}{x}\right)$; $|x|\le 1, x\ne 0$

and $\sec^{-1}x=\cos^{-1}\left(\frac{1}{x}\right)$, $|x|\ge 1$

(iii) $\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$, $x>0$

$=\pi+\tan^{-1}\left(\frac{1}{x}\right)$; $x<0$

NOTE

(i) $\operatorname{cosec}^{-1}x$ and $\sin^{-1}(1/x)$ are identical function

(ii) $\sin^{-1}x$ and $\operatorname{cosec}^{-1}(1/x)$ are not identical because domain of $\sin^{-1}x$ and $\operatorname{cosec}^{-1}(1/x)$ is not equal.

(iii) $\sec^{-1}x$ and $\cos^{-1}(1/x)$ are identical function

(iv) $\cos^{-1}x$ and $\sec^{-1}(1/x)$ are not identical because domain of $\cos^{-1}x$ and $\sec^{-1}(1/x)$ is not equal.

Example 9 :

Evaluate : $\sin^{-1}\left(\frac{2}{5}\right)$

Sol. $\sin^{-1}x=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) \Rightarrow \sin^{-1}\left(\frac{2}{5}\right)=\operatorname{cosec}^{-1}\left(\frac{5}{2}\right)$

Property-6

(i) $\tan^{-1}x+\tan^{-1}y$

$$=\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy<1\\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x>0, y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x<0, y<0 \text{ and } xy>1\end{cases}$$

(ii) $\tan^{-1}x-\tan^{-1}y$

$$=\begin{cases}\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy>-1\\ \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x>0, y<0 \text{ and } xy<-1\\ -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x<0, y>0 \text{ and } xy<-1\end{cases}$$

(iii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

If $x>0, y>0, z>0$; $xy+yz+zx<1$; $xy<1$; $yz<1$; $zx<1$

Note : If $x_1, x_2, x_3, ...x_n \in R$, then
$\tan^{-1}x_1+\tan^{-1}x_2+....+\tan^{-1}x_n$

$=\tan^{-1}\left(\frac{S_1-S_3+S_5-S_7+...}{1-S_2+S_4-S_6+...}\right)$

where S_k denotes the sum of the products of $x_1, x_2, ... x_n$ taken k at a time.

Example 10 :

Evaluate $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

Sol. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}\frac{\frac{5}{6}}{\frac{5}{6}}=\tan^{-1}1=\frac{\pi}{4}$

Example 11 :

Prove that :

(i) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

(ii) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

(iii) $\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3}=2$

Sol.(i) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\tan^{-1}1+\left(\pi+\tan^{-1}\frac{2+3}{1-2.3}\right)$

$=\tan^{-1}1+(\pi+\tan^{-1}(-1))=\frac{\pi}{4}+\pi-\frac{\pi}{4}=\pi$

(ii) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}1 + \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$

$= \tan^{-1}1 + \tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}1 + \tan^{-1}1 = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

(iii) $\frac{\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3}{\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3} = 2$

$= \frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}} = \frac{\pi}{\left(\frac{\pi}{2}\right)} = 2$

Example 12 :

If $\tan^{-1}2$ and $\tan^{-1}3$ be two angles of a triangle, then find the third angle.

Sol. $A + B = \tan^{-1}2 + \tan^{-1}3 = \tan^{-1}\frac{2+3}{1-2.3}+\pi \quad (\because ab > 11)$

$= \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4}$

Hence the third angle $= \pi - (A+B) = \pi - \left(\pi - \frac{\pi}{4}\right) = \frac{\pi}{4}$

Property-7

(i) $\sin^{-1}x + \sin^{-1}y =$

$$\begin{cases} \sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } -1\le x,\ y<0 \text{ and } x^2+y^2>1 \end{cases}$$

(ii) $\sin^{-1}x - \sin^{-1}y =$

$$\begin{cases} \sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, & \text{if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, & \text{if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1 \end{cases}$$

Example 13 :

Solve the equation $\sin^{-1}x + \sin^{-1}2x = \pi/3$.

Sol. $\sin^{-1}x + \sin^{-1}2x = \pi/3$

$\sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}x = \sin^{-1}\left[\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right]$

$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$

$\Rightarrow \left(\frac{5x}{2}\right)^2 = \frac{3}{4}(1-x^2) \Rightarrow 28x^2 = 3$

$\Rightarrow x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}}$

$\left(\because x = -\frac{1}{2}\sqrt{\frac{3}{7}} \text{ makes L.H.S. of (1) negative}\right)$

Example 14 :

If $\sin^{-1}6x + \sin^{-1}6\sqrt{3}\ x = -\frac{\pi}{2}$, then find the value of x.

Sol. $\sin^{-1}6\sqrt{3}\ x = -\frac{\pi}{2} - \sin^{-1}6x = -(\sin^{-1}1 + \sin^{-1}6x)$

$= -\sin^{-1}(1.\sqrt{1-(6x)^2} - 6x.\sqrt{1-1})$

$\Rightarrow -\sin^{-1}\sqrt{1-36x^2} = \sin^{-1}(-\sqrt{1-36x^2})$

$\Rightarrow 6\sqrt{3}\,x = -\sqrt{1-36x^2} \Rightarrow 108x^2 = 1-36x^2 \Rightarrow 144x^2 = 1$

$\Rightarrow x^2 = \frac{1}{144} \Rightarrow x \pm \frac{1}{12}$

But only $x = -1/12$ satisfies the equation.

Property-8

(i) $\cos^{-1}x + \cos^{-1}y =$

$$\begin{cases} \cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\ge 0 \\ 2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\le 0 \end{cases}$$

(ii) $\cos^{-1}x - \cos^{-1}y =$

$$\begin{cases} \cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x,y\le 1 \text{ and } x\le y \\ -\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y \end{cases}$$

Example 15 :

$\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a$, then find the value of x.

Sol. $\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a$

$\frac{a}{x}.\frac{1}{a} - \sqrt{\left(1-\frac{a^2}{x^2}\right)}\sqrt{\left(1-\frac{1}{a^2}\right)} = \frac{1}{b}.\frac{b}{x} - \sqrt{\left(1-\frac{b^2}{x^2}\right)}\sqrt{\left(1-\frac{1}{b^2}\right)}$

or $b^2(x^2-a^2)(a^2-1) = a^2(x^2-b^2)(b^2-1)$

or $x^2(a^2-b^2) = a^2b^2(a^2-b^2)$

$\therefore x = ab$

Property-9

(i) $$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) \quad , & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) \ , & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}) \ , & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), & \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$$

Property-10

(i) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1) \quad , & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), & \text{if } -1 \le x \le 0 \end{cases}$$

(ii) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x) \quad , & \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x) \ , & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x) \ , & \text{if } -1 \le x - \frac{1}{2} \end{cases}$$

Property-11

(i) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad , & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \ , & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \quad , & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \ , & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

Property-12

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad , & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) \ , & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x \le 0 \end{cases}$$

Example 16 :

Find the solution of

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

Sol. Put $a = \tan\theta$

$$\therefore \frac{2a}{1+a^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$$

$$\therefore \sin^{-1}\left(\frac{2a}{1+a^2}\right) = \sin^{-1}(\sin 2\theta) \Rightarrow 2\theta = 2\tan^{-1}a$$

Put $b = \tan\phi$

$$\therefore \frac{1-b^2}{1+b^2} = \frac{1-\tan^2\phi}{1+\tan^2\phi} = \cos 2\phi$$

$$\therefore \cos^{-1}\frac{1-b^2}{1+b^2} = 2\phi = 2\tan^{-1}b$$

$$\therefore \text{L.H.S.} = 2(\tan^{-1}a - \tan^{-1}b) = 2\tan^{-1}\frac{a-b}{1+ab}$$

Put $x = \tan\alpha$

$$\therefore \frac{2x}{1-x^2} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$$

$$\therefore \text{R.H.S.} = \tan^{-1}\frac{2x}{1-x^2} = 2\alpha = 2\tan^{-1}x$$

L.H.S. = R.H.S.

$$\Rightarrow 2\tan^{-1}\frac{a-b}{1+ab} = 2\tan^{-1}x \quad \Rightarrow x = \frac{a-b}{1+ab}$$

Example 17 :

Find the value of $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]$

Sol. $2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}$

$$=\tan^{-1}\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}.\frac{1}{5}}=\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}=\tan^{-1}\frac{5}{12}$$

$$\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)=\tan\left(\tan^{-1}\frac{5}{12}-\frac{\pi}{4}\right)$$

$$=\frac{\tan\left(\tan^{-1}\frac{5}{12}\right)-\tan\frac{\pi}{4}}{1+\tan\left(\tan^{-1}\frac{5}{12}\right)\tan\frac{\pi}{4}}=\frac{\frac{5}{12}-1}{1+\frac{5}{12}.1}=\frac{-\frac{7}{12}}{\frac{17}{12}}=-\frac{7}{17}$$

Property-13

(i) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$$=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

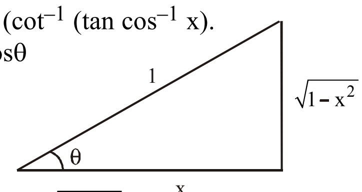
(iii) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\sqrt{1+x^2}=\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Example 18 :

Find the value of $\sin^{-1}(\cot^{-1}(\tan\cos^{-1}x))$.

Sol. Put $\cos^{-1}x=\theta\ \therefore x=\cos\theta$

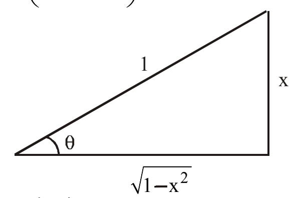


$\therefore\ \tan\theta=\frac{\sqrt{1-x^2}}{x}$

$\therefore\ \tan(\cos^{-1}x)=\tan\theta=\frac{\sqrt{1-x^2}}{x}$

$\therefore\ \cot^{-1}(\tan(\cos^{-1}x))=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\phi$ (say)

$\therefore\ \cot\phi=\frac{\sqrt{1-x^2}}{x}$



$\therefore\ \sin\phi=\frac{x}{1}$

$\therefore\ \sin(\cot^{-1}(\tan(\cos^{-1}x))=\sin\phi=x$

Example 19 :

Find the value of $\sin(\cot^{-1}x)$.

Sol. Put $\cot^{-1}x=A\ \therefore\ x=\cot A$

$$\therefore\ \sin A=\frac{1}{\sqrt{x^2+1}}=(1+x^2)^{-1/2}$$

TRY IT YOURSELF

Q.1 Find the value of following
(a) $\sin^{-1}(\sin 1)$ (b) $\sin^{-1}(\sin 10)$

Q.2 Find the value of following
(a) $\cos^{-1}(\cos 1)$ (b) $\cos^{-1}(\cos 5)$

Q.3 Find the value of following
(a) $\tan^{-1}(\tan 2)$ (b) $\tan^{-1}(\tan 10)$

Q.4 Find the integral solution of inequality
$6x^2-5x<\cos^{-1}(\cos 5)-2\sin^{-1}(\sin 3)$

Q.5 Find the value of following
(a) $\cos^{-1}\sin\left(-\frac{\pi}{4}\right)$ (b) $\sin^{-1}\cos\left(\frac{33\pi}{10}\right)$

Q.6 Find the value of of x if $4\sin^{-1}x+\cos^{-1}x=\frac{3\pi}{4}$

Q.7 If $\tan^{-1}2+\tan^{-1}4=\cot^{-1}(\lambda)$ then find λ.

Q.8 Find the value of $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$.

Q.9 If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ then value of $x^2+y^2+z^2+2xyz$ is equal to –
(A) 1 (B) –1
(C) 0 (D) 3

Q.10 Find the x satisfying the equation

$$2\cot^{-1}2-\cos^{-1}\frac{4}{5}=\operatorname{cosec}^{-1}x$$

ANSWERS

(1) (a) 1, (b) $3\pi-10$ **(2)** (a) 1, (b) $2\pi-5$
(3) (a) $\pi-2$ (b) $3\pi-10$ **(4)** 0
(5) (a) $3\pi/4$ (b) $-\pi/5$ **(6)** $\frac{\sqrt{3}-1}{2\sqrt{2}}$
(7) $-7/6$ **(8)** $\pi/6$
(9) (A) **(10)** 25/7

IDENTITIES INVOLVING ITF

1. $2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$

2. $\tan^{-1}x = 2\tan^{-1}[\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)]$ $(x \neq 0)$

INFINITE SERIES OF ITF

1. $\sin^{-1}(x) = x+\left(\frac{1}{2}\right)\frac{x^3}{3}+\left(\frac{1}{2}.\frac{3}{4}\right)\frac{x^5}{5}+\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\right)\frac{x^7}{7}+....$

$$=\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{(2n+1)};|x|\le 1$$

2. $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$

$$=\frac{\pi}{2}-\left[x+\left(\frac{1}{2}\right)\frac{x^3}{3}+\left(\frac{1}{2}.\frac{3}{4}\right)\frac{x^5}{5}+\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\right)\frac{x^7}{7}+....\right]$$

$$=\frac{\pi}{2}-\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{(2n+1)};|x|\le 1$$

3. $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+....$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)};|x|\le 1 \quad x\neq i,-i$$

4. $\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x=\frac{\pi}{2}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...\right)$

$$=\frac{\pi}{2}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1},\ |x|\le 1 \quad x\neq i,-i$$

5. $\sec^{-1}x=\cos^{-1}(x^{-1})$

$$=\frac{\pi}{2}-\left[x^{-1}+\left(\frac{1}{2}\right)\frac{x^{-3}}{3}+\left(\frac{1}{2}.\frac{3}{4}\right)\frac{x^{-5}}{5}+\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\right)\frac{x^{-7}}{7}+....\right]$$

$$=\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{-(2n+1)}}{2x+1},\ |x|\ge 1$$

6. $\operatorname{cosec}^{-1}(x)=\sin^{-1}(x^{-1})$

$$=x^{-1}+\left(\frac{1}{2}\right)\frac{x^{-3}}{3}+\left(\frac{1}{2}.\frac{3}{4}\right)\frac{x^{-5}}{5}+\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\right)\frac{x^{-7}}{7}+....$$

$$=\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{-(2n+1)}}{(2n+1)};|x|\ge 1$$

Example 20 :

The value of

$\operatorname{cosec}^{-1}\sqrt{5}+\operatorname{cosec}^{-1}\sqrt{65}+\operatorname{cosec}^{-1}\sqrt{325}+....\infty$

is equal to

(A) 0 (B) $\pi/4$
(C) $\pi/2$ (D) π

Sol. $\operatorname{cosec}^{-1}\sqrt{5}+\operatorname{cosec}^{-1}\sqrt{65}+\operatorname{cosec}^{-1}\sqrt{325}+....\infty$

$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}+....\infty$

$T_r=\tan^{-1}\frac{1}{2r^2}=\tan^{-1}\frac{2}{4r^2}=\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r-1)(2r+1)}\right)$

$T_r=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$

$S_n=\sum_{r=1}^{n}T_r = (\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}5-\tan^{-1}3)$

$+..+(\tan^{-1}(2n+1)-\tan^{-1}(2n-1))$

$=\tan^{-1}(2n+1)-\tan^{-1}1$, when $n\to\infty$

$S_\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

INEQUATIONS INVOLVING ITF

Example 21 :

Find the x satisfying the inequality $\cos^{-1}x>\cos^{-1}x^2$.

Sol. $\cos^{-1}x>\cos^{-1}x^2$

$\Rightarrow$ $x^2-x>0\Rightarrow x(x-1)>0\Rightarrow x\in(-\infty,0)\cup(1,\infty)$

$\because$ $\cos^{-1}x$ defined for $x\in[-1,1]$

$\Rightarrow$ $x\in[-1,0)$

SOME IMPORTANT RESULTS

* $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{2}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$
* $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45°$
* $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$
* $\cos^{-1}\frac{1}{2}+\cos^{-1}\left(-\frac{1}{7}\right)+\cos^{-1}\left(\frac{13}{14}\right)=\pi$
* $\sin^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}\left(\frac{16}{65}\right)$
* $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$
* $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$
* $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$
* $\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3}=2$

* If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

* If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$ then $x^2 + y^2 + z^2 + 2xyz = 1$

* $f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = \pi$ if $x \geq 1$

* $f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = -\pi$ if $x \leq -1$

ADDITIONAL EXAMPLES

Example 1 :

Find the value of : (a) $\sin\left(2\sin^{-1}\frac{3}{5}\right)$

(b) $\cos(2\tan^{-1}2) + \sin(2\tan^{-1}3)$

(c) $\cos\left(\text{arc}\sin\frac{4}{5} - \text{arc}\cos\frac{4}{5}\right)$

(d) $\tan\left(2\cot^{-1}5 - \frac{\pi}{4}\right)$

Sol. (a) $\sin^{-1}\frac{3}{5} = \theta \Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

$\sin(2\theta) = 2\sin\theta.\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

(b) Let $\tan^{-1}2 = \theta \Rightarrow \tan\theta = 2$

$\tan^{-1}3 = \phi \Rightarrow \tan\phi = 3$

$$\cos(2\theta) + \sin(2\phi) = \frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{2\tan\phi}{1+\tan^2\phi}$$

$$= \frac{1-(2)^2}{1+(2)^2} + \frac{2(3)}{1+(3)^2} = \frac{-3}{5} + \frac{3}{5} = 0$$

(c) Let $\sin^{-1}\frac{4}{5} = \theta \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$

$\cos^{-1}\frac{4}{5} = \phi \Rightarrow \cos\phi = \frac{4}{5}, \sin\phi = \frac{3}{5}$

$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= \frac{4}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$$

(d) Let $\cot^{-1}5 = \theta \Rightarrow \cot\theta = 5, \tan\theta = 1/5$

$$\tan\left(2\theta - \frac{\pi}{4}\right) = \frac{\tan 2\theta - 1}{1 + \tan 2\theta}$$

$$= \frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) - 1}{1 + \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)} = \frac{\left(\frac{2(1/5)}{1-(1/5)^2}\right) - 1}{1 + \left(\frac{2(1/5)}{1-(1/5)^2}\right)} = \frac{\frac{5}{12} - 1}{1 + \frac{5}{12}} = \frac{-7}{17}$$

Example 2 :

Find the domain of definition of following functions.

(a) $f(x) = \text{arc}\cos\frac{2x}{1+x}$

(b) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

Sol. (a) $f(x) = \cos^{-1}\frac{2x}{1+x}$

f(x) defined when $-1 \leq \frac{2x}{1+x} \leq 1$

$\frac{2x}{1+x} \geq -1 \Rightarrow \frac{2x}{1+x} + 1 \geq 0 \Rightarrow \frac{3x+1}{1+x} \geq 0$

$\Rightarrow x \in (-\infty, -1) \cup [-1/3, \infty)$ ... (1)

and $\frac{2x}{1+x} \leq 1 \Rightarrow \frac{2x}{1+x} - 1 \leq 0 \Rightarrow \frac{x-1}{1+x} \leq 0$

$\Rightarrow x \in (-1, 1]$ ... (ii)

From (i) and (ii), $x \in [-1/3, 1]$

(b) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

f(x) defined when $-1 \leq \frac{x-3}{2} \leq 1$ and $4 - x > 0$

Now $-1 \leq \frac{x-3}{2} \leq 1$

$\Rightarrow -2 \leq x - 3 \leq 2 \Rightarrow 1 \leq x \leq 5 \Rightarrow x \in [1, 5]$ ... (i)

and $4 - x > 0 \Rightarrow x < 4 \Rightarrow x \in (-\infty, 4)$ ... (ii)

From (i) and (ii), $x \in [1, 4)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

Now, $3 - x \geq 0 \Rightarrow x \leq 3 \Rightarrow x \in (-\infty, 3]$ ... (i)

$-1 \leq \frac{3-2x}{5} \leq 1 \Rightarrow -5 \leq 3 - 2x \leq 5$

$\Rightarrow -2 \le 2x \le 8 \Rightarrow -1 \le x \le 4 \Rightarrow x \in [-1, 4]$ ... (ii)

$2|x| - 3 > 0 \Rightarrow |x| > \frac{3}{2} \Rightarrow x \in \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ ... (iii)

$-1 \le \log_2 x \le 1 \Rightarrow \frac{1}{2} \le x \le 2 \Rightarrow x \in \left[\frac{1}{2}, 2\right]$ .. (iv)

From (i), (ii), (iii) and (iv), $x \in (3/2, 2]$

Example 3 :

Find the value of following

(a) $\sin^{-1}(\sin 2)$ (b) $\sin^{-1}(\sin 3)$
(c) $\sin^{-1}(\sin 4)$ (d) $\sin^{-1}(\sin 5)$

Sol. (a) $\sin^{-1}(\sin 2) \ne 2$ $\left(\because 2 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\Rightarrow \sin^{-1}(\sin 2) = \sin^{-1}(\sin(\pi - 2)) = \pi - 2$

$\left(\because \pi - 2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(b) $\sin^{-1}(\sin 3) = \sin^{-1}(\sin(\pi - 3)) = \pi - 3$

$\left(\because \pi - 3 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(c) $\sin^{-1}(\sin 4) = \sin^{-1}(\sin(\pi - 4)) = \pi - 4$

$\left(\because \pi - 4 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(d) $\sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi$

$\left(\because 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

Example 4 :

Find the value of following

(a) $\cos^{-1}(\cos 2)$ (b) $\cos^{-1}(\cos 3)$
(c) $\cos^{-1}(\cos 4)$ (d) $\cos^{-1}(\cos 10)$

Sol. (a) $\cos^{-1}(\cos 2) = 2$ $(\because 2 \in [0, \pi])$

(b) $\cos^{-1}(\cos 3) = 3$ $(\because 3 \in [0, \pi])$

(c) $\cos^{-1}(\cos 4) = \cos^{-1}(\cos(2\pi - 4)) = 2\pi - 4$
$(\because 2\pi - 4 \in [0, \pi])$

(d) $\cos^{-1}(\cos 10) = \cos^{-1}(\cos(10 - 3\pi)) = 10 - 3\pi.$
$(\because 10 - 3\pi \in [0, \pi])$

Example 5 :

Find the value of following

(a) $\tan^{-1}(\tan 1)$ (b) $\tan^{-1}(\tan 3)$
(c) $\tan^{-1}(\tan 4)$ (d) $\tan^{-1}(\tan 5)$

Sol. (a) $\tan^{-1}(\tan 1) = 1$ $\left(\because 1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(b) $\tan^{-1}(\tan 3) = -\tan^{-1}(\tan(\pi - 3)) = 3 - \pi$

$\left(\because \pi - 3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(c) $\tan^{-1}(\tan 4) = -\tan^{-1}(\tan(\pi - 4)) = 4 - \pi$

$\left(\because \pi - 4 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(d) $\tan^{-1}(\tan 5) = \tan^{-1}(\tan(5 - 2\pi)) = 5 - 2\pi$

$\left(\because 5 - 2\pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

Example 6 :

Find the value of following

(a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right)$

(b) $\cos^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right)$

(c) $\sin^{-1}\left(\cos\frac{33\pi}{10}\right)$

Sol. (a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right) = \sin^{-1}\sin\left(\pi + \frac{2\pi}{11}\right)$

$= \sin^{-1}\left(-\sin\left(\frac{2\pi}{11}\right)\right) = \sin^{-1}\sin\left(-\frac{2\pi}{11}\right) = -\frac{2\pi}{11}$

(b) $\cos^{-1}\sin\left(-\frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \cos^{-1}\cos\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

(c) $\sin^{-1}\cos\left(\frac{33\pi}{10}\right) = \sin^{-1}\cos\frac{13\pi}{10} = \sin^{-1}\left(-\cos\frac{3\pi}{10}\right)$

$= \sin^{-1}\left(-\sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)\right) = \sin^{-1}\left(-\sin\frac{\pi}{5}\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right) = -\frac{\pi}{5}$

Example 7 :

Are $\tan(\cot^{-1}x)$ and $\cot(\tan^{-1}x)$ are identical ?

Sol. [**True**], as both functions have same graph.

Example 8 :

Find the greater of the two angles

$A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

Sol. $A = 2\tan^{-1}(2\sqrt{2} - 1) = 2\tan^{-1}(1.828)$

$\therefore A > 2\tan^{-1}\sqrt{3}$ $[\sqrt{3} = 1.732 < 1.828]$

$\Rightarrow A > \frac{2\pi}{3}$ ....(1)

We have $\sin^{-1}\frac{1}{3} < \sin^{-1}\frac{1}{2} = \frac{\pi}{6} \Rightarrow 3\sin^{-1}\frac{1}{3} < \frac{\pi}{2}$

Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

We have $\sin^{-1}\frac{1}{3} = \sin^{-1}\left(3\times\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right)$

$$= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$$

$\therefore \quad 3\sin^{-1}\frac{1}{3} < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \left(\because \frac{\sqrt{3}}{2} = 0.868 > 0.852\right)$

i.e. $3\sin^{-1}\frac{1}{3} < \frac{\pi}{3}$ ...(2)

Also $\sin^{-1}\frac{3}{5} = \sin^{-1}(0.6) < \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\therefore \ \sin^{-1}\frac{3}{5} < \frac{\pi}{3} \quad \therefore \ B = 3\sin^{-1}\frac{1}{3} + \sin\frac{3}{5} < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore \ B < \frac{2\pi}{3}$ ....(3), By eq. (1) and eq. (3), $A > B$

Example 9 :

If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then find the value of $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$.

Sol. Let $\sin^{-1} a = x \ \therefore a = \sin x$; $\sin^{-1} b = y \ \therefore b = \sin y$

$\sin^{-1} c = z \ \therefore c = \sin z$

$\therefore \ a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$

$= \sin x \cos x + \sin y \cos y + \sin z \cos z$

$= (1/2)(\sin 2x + \sin 2y + \sin 2z) = (1/2)(4 \sin x \sin y \sin z)$

$= 2 \sin x \sin y \sin z = = 2abc$

Example 10 :

If x_1, x_2, x_3, x_4 are roots of equation

$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x\cos\beta - \sin\beta = 0$,

then find $\sum_{i=1}^{4} \tan^{-1} x_i$

Sol. $p_1 = \Sigma x_1 = \sin 2\beta$; $p_2 = \Sigma x_1 x_2 = \cos 2\beta$

$p_3 = \Sigma x_1 x_2 x_3 = \cos\beta$; $p_4 = \Sigma x_1 x_2 x_3 x_4 = -\sin\beta$

$\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4$

$$= \tan^{-1}\frac{p_1 - p_3}{1 - p_2 + p_4}$$

$p_1 - p_3 = \sin 2\beta - \cos\beta = \cos\beta(2\sin\beta - 1)$

$1 - p_2 + p_4 = 1 - \cos 2\beta - \sin\beta$

$= 2\sin^2\beta - \sin\beta = \sin\beta(2\sin\beta - 1)$

$\therefore \quad \tan^{-1}\frac{p_1 - p_3}{1 - p_2 + p_4} = \tan^{-1}\left(\frac{\cos\beta}{\sin\beta}\right) = \tan^{-1}(\cot\beta)$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \beta\right)\right] = \frac{\pi}{2} - \beta$$

Example 11 :

Evaluate

$$\tan^{-1}\left(\frac{c_1 x - y}{c_1 y + x}\right) + \tan^{-1}\left(\frac{c_2 - c_1}{1 + c_2 c_1}\right) + \tan^{-1}\left(\frac{c_3 - c_2}{1 + c_3 c_2}\right) + ... + \tan^{-1}\left(\frac{1}{c_n}\right)$$

Sol. $\tan^{-1}\left(\frac{c_1 x - y}{c_1 y + x}\right) + \tan^{-1}\left(\frac{c_2 - c_1}{1 + c_2 c_1}\right) + \tan^{-1}\left(\frac{c_3 - c_2}{1 + c_3 c_2}\right) + ... + \tan^{-1}\left(\frac{1}{c_n}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{1}{c_1}}{1 + \frac{x}{y}\frac{1}{c_1}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_1} - \frac{1}{c_2}}{1 + \frac{1}{c_1 c_2}}\right)$$

$$+ \tan^{-1}\left(\frac{\frac{1}{c_2} - \frac{1}{c_3}}{1 + \frac{1}{c_2 c_3}}\right) + ... + \tan^{-1}\left(\frac{1}{c_n}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{1}{c_1}\right) + \tan^{-1}\left(\frac{1}{c_1}\right)$$

$$- \tan^{-1}\left(\frac{1}{c_2}\right) + \tan^{-1}\left(\frac{1}{c_2}\right) - \ldots - \tan^{-1}\left(\frac{1}{c_n}\right) + \tan^{-1}\left(\frac{1}{c_n}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right)$$

Example 12 :

Find the principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Sol. Let $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = y$.

Then $\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\frac{2\pi}{3}\right) = \frac{-1}{\sqrt{3}}$.

Hence, principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$.

Example 13 :

Prove that the equation $\sin[\cot^{-1}(\cos(\tan^{-1} x))] = \frac{a}{b}$ has solutions provided $\frac{1}{\sqrt{2}} \le \left|\frac{a}{b}\right| < 1$ and find the solutions.

Sol. $\sin[\cot^{-1}(\cos(\tan^{-1} x))]$

$$= \sin\left[\cot^{-1}\left(\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right]$$

$$= \sin\left[\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \sin\left[\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right] = \sqrt{\frac{1+x^2}{2+x^2}}$$

The given equation reduces to $x^2 = \frac{2a^2 - b^2}{b^2 - a^2}$

As $|\sin\theta| < 1, |a| < |b|$ and real values of x exist provided

$2a^2 \ge b^2$ i.e., $\frac{1}{\sqrt{2}} \le \left|\frac{a}{b}\right| < 1$ and then $x = \pm\sqrt{\frac{2a^2 - b^2}{b^2 - a^2}}$

Example 14 :

If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - (\sin 2\beta)x^3 + (\cos 2\beta)x^2 - (\cos\beta)x - \sin\beta = 0$$

prove that

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 = n\pi + \frac{\pi}{2} - \beta.$$

Sol. The sums of the products of the roots taken one, two, three and four at a time are given by

$\Sigma x_1 = -(-\sin 2\beta)$, $\Sigma x_1x_2 = (-1)^2 \cos 2\beta$,

$\Sigma x_1x_2x_3 = (-1)^3(-\cos\beta)$ & $\Sigma x_1x_2x_3x_4 = (-1)^4(-\sin\beta)$

$\tan(\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4)$

$$= \frac{\Sigma x_1 - \Sigma x_1x_2x_3x_4}{1 - \Sigma x_1x_2 + x_1x_2x_3x_4} = \frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \cot\beta$$

$$\sum_1^4 \tan^{-1}x_1 = n\pi + \frac{\pi}{2} - \beta, \ n = 0, 1, 2, \ldots\ldots\ldots\ldots$$

Example 15 :

Prove that

$$\tan^{-1}\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}} + \tan^{-1}\sqrt{\left\{\frac{b(a+b+c)}{ac}\right\}} + \tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}} = \pi$$

Sol. $$\text{LHS} = \tan^{-1}\frac{\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}} + \sqrt{\left\{\frac{b(a+b+c)}{ac}\right\}}}{1 - \sqrt{\left\{\frac{a(a+b+c)}{bc}\right\} \cdot \left\{\frac{b(a+b+c)}{ac}\right\}}} + \tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}$$

$$= \tan^{-1}\frac{\sqrt{\{(a+b+c)\}}}{-\sqrt{\{ab\}}(a+b)} + \tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}$$

$$= \tan^{-1}\left[\sqrt{\frac{(a+b+c)c}{ab}}\right] + \tan^{-1}\left[-\sqrt{\left[\frac{c(a+b+c)}{ab}\right]}\right]$$

$$= \pi - \tan^{-1}\left[\sqrt{\frac{c(a+b+c)}{ab}}\right] + \tan^{-1}\left[-\sqrt{\left[\frac{c(a+b+c)}{ab}\right]}\right] = \pi$$

Example 16 :

Prove that $3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} = \frac{1}{4}\pi - \tan^{-1}\frac{1}{1985}$

Sol. Since $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

$$\therefore\ 3\tan^{-1}\frac{1}{4} = \tan^{-1}\left(\frac{3\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^3}{1 - 3\left(\frac{1}{4}\right)}\right)$$

or $3\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{47}{52}$

$$\therefore\ 3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} = \tan^{-1}\frac{47}{52} + \tan^{-1}\frac{1}{20}$$

$$= \tan^{-1}\frac{\frac{47}{52} + \frac{1}{20}}{1 - \frac{47}{52}\cdot\frac{1}{20}} = \tan^{-1}\frac{992}{993}$$

and $\frac{1}{4}\pi - \tan^{-1}\frac{1}{1985} = \tan^{-1}\frac{1-\frac{1}{1985}}{1+\frac{1}{1985}}$

$= \tan^{-1}\frac{1984}{1986} = \tan^{-1}\frac{992}{993}$

Example 17 :

Solve the equation $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$

Sol. Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right] + \tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]} = \tan\{\tan^{-1}(-7)\} = -7$$

i.e., $\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}} = -7$ i.e., $\frac{2x^2-x+1}{1-x} = -7$,

so that $x = 2$

The value $x = 2$ is a solution of the equation

$$\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \pi + \tan^{-1}(-7)$$

Example 18 :

Solve for x : $\tan^{-1}(1+x) + \tan^{-1}x + \tan^{-1}(x-1) = \tan^{-1}3$

Sol. $\tan^{-1}(1+x) + \tan^{-1}(x-1) = \tan^{-1}3 - \tan^{-1}x$

$\tan^{-1}\left(\frac{1+x+x-1}{1-x^2+1}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$,

when $x^2 - 1 < 1$ and $3x < 1$

$\Rightarrow \frac{2x}{2-x^2} = \frac{3-x}{1+3x}$, when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow 2x(1+3x) = (3-x)(2-x^2)$. when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow 2x + 6x^2 = 6 - 2x - 3x^2 + x^3 \Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

$\Rightarrow (x+1)(x^2 - 10x + 6) = 0$, when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow x = -1$ and neglecting $x^2 - 10x + 6 = 0$ as its root does not

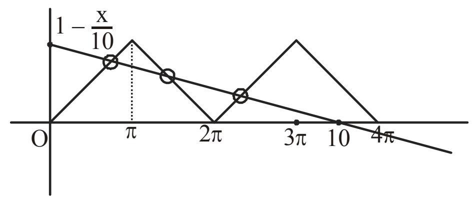
$\in\left(-\sqrt{2}, \frac{1}{3}\right)$ $\therefore\ x = -1$

Example 19 :

Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation

$f(x) = \frac{10-x}{10}$ is –

Sol. 3. $f : [0, 4\pi] \to [0, \pi]$, $f(x) = \cos^{-1}(\cos x)$.



$f(x) = 1 - \frac{x}{10}$

Example 20 :

Number of positive solutions satisfying the equation

$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is

Sol. 1. $\tan^{-1}\left(\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{(2x+1)(4x+1)}}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \frac{6x+2}{8x^2+6x} = \frac{2}{x^2} \Rightarrow \frac{3x+1}{4x+3} = \frac{2}{x}$ (where $x \neq 0$)

$\Rightarrow 3x^2 - 7x - 6 = 0$

$\Rightarrow x = 3, -2/3$

But $2x + 1 > 0$ and $4x + 1 > 0$

So, solution are $x = 3$.

Example 21 :

Solve the inequality satisfying

$\text{arc}\tan^2 x - 3\,\text{arc}\tan x + 2 > 0$

Sol. $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 > 0$

$\Rightarrow (\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$

$\because \tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ $(\tan^{-1}x - 2)$ is always negative

$\Rightarrow$ $(\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$ holds true only when $\tan^{-1}x - 1 < 0$

$\Rightarrow$ $\tan^{-1}x < 1 \Rightarrow x < \tan 1$

$\Rightarrow$ $x \in (-\infty, \tan 1)$

Example 22 :

If $\sin^{-1}(\sin 9) - \cos^{-1}(\cos 15)$ can be written in the form $a\pi - b$, then find the value of $a + b$. ($a, b \in N$).

Sol. $\sin^{-1}(\sin 9) = \sin^{-1}\sin(3\pi - 9) = 3\pi - 9$

$\left(\because 3\pi - 9 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\cos^{-1}(\cos 15) = \cos^{-1}(\cos(15 - 4\pi)) = 15 - 4\pi$

$(\because 15 - 4\pi \in [0, \pi])$

$\Rightarrow \sin^{-1}(\sin 9) - \cos^{-1}(\cos 15) = (3\pi - 9) - (15 - 4\pi)$

$= 7\pi - 24 \Rightarrow a = 7, b = 24$; $a + b = 7 + 24 = 31$.

Example 23 :

If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$

$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

can be written in the from of $\frac{a\pi}{b}$ (where a, b ∈ N) then find the minimum value of a + b.

Sol. $\sin^{-1}\left(\sin\left(\frac{33\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(5\pi-\frac{2\pi}{7}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{2\pi}{7}\right)\right) = \frac{2\pi}{7}$

$\cos^{-1}\left(\cos\left(\frac{46\pi}{7}\right)\right) = \cos^{-1}\left(\cos\left(6\pi+\frac{4\pi}{7}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right) = \frac{4\pi}{7}$

$\tan^{-1}\left(-\tan\left(\frac{13\pi}{8}\right)\right) = \tan^{-1}\left(\tan\left(2\pi-\frac{13\pi}{8}\right)\right)$

$= \tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right) = \frac{3\pi}{8}$

$\cot^{-1}\left(\cot\left(\frac{-19\pi}{8}\right)\right) = \cot^{-1}\left(\cot\left(3\pi-\frac{19\pi}{8}\right)\right)$

$= \cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right) = \frac{5\pi}{8}$

$\Rightarrow \sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$

$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

$=\frac{2\pi}{7}+\frac{4\pi}{7}+\frac{3\pi}{8}+\frac{5\pi}{8}=\frac{6\pi}{7}+\pi=\frac{13\pi}{7}$

$\Rightarrow$ a = 13, b = 7 $\Rightarrow$ a + b = 13 + 7 = 20

Example 24 :

Find the value(s) of x satisfying the equation

$$\cot^{-1}\frac{x^2-1}{2x}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2x}{3}$$

Sol. Case (i) : $\frac{x^2-1}{2x}>0$

$\cot^{-1}\frac{x^2-1}{2x}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$ ($\because \cot^{-1}x=\tan^{-1}\frac{1}{x}$; x> 0)

$\Rightarrow \tan^{-1}\frac{2x}{x^2-1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3} \Rightarrow \tan^{-1}\frac{2x}{x^2-1}=\frac{\pi}{3}$

$\Rightarrow \frac{2x}{x^2-1}=\sqrt{3} \Rightarrow \sqrt{3}x^2-2x-\sqrt{3}=0$

$\Rightarrow x=\frac{2\pm4}{2\sqrt{3}} \Rightarrow x=\frac{-1}{\sqrt{3}},\sqrt{3}$

Case (ii) : $\frac{x^2-1}{2x}<0$

$\cot^{-1}\frac{x^2-1}{2x}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$

$\Rightarrow \pi+\tan^{-1}\frac{2x}{x^2-1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$

($\because \cot^{-1}x=\pi+\tan^{-1}(1/x)$; x < 0)

$\Rightarrow \tan^{-1}\frac{2x}{x^2-1}=\frac{-\pi}{6} \Rightarrow \frac{2x}{x^2-1}=\frac{-1}{\sqrt{3}}$

$\Rightarrow x^2+2\sqrt{3}x-1=0$

$\Rightarrow x=-\sqrt{3}\pm2 \Rightarrow x=-(2+\sqrt{3}),\ 2-\sqrt{3}$

From case (i) and (ii)

$\Rightarrow x=\sqrt{3},-\frac{1}{\sqrt{3}},-(2+\sqrt{3}),(2-\sqrt{3})$

Example 25 :

Find the value of x if $5\tan^{-1}x+3\cot^{-1}x=\frac{7\pi}{4}$

Sol. $5\tan^{-1}x+3\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{7\pi}{4}$

$2\tan^{-1}x=\frac{7\pi}{4}-\frac{3\pi}{2} \Rightarrow 2\tan^{-1}x=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{8} \Rightarrow x=\tan\frac{\pi}{8} \Rightarrow x=\sqrt{2}-1$

Example 26 :

Find the maximum and minimum values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$

Sol. $(\sin^{-1}x)^3+(\cos^{-1}x)^3$

$=(\sin^{-1}x+\cos^{-1}x)((\sin^{-1}x)^2+(\cos^{-1}x)^2-\sin^{-1}x\cdot\cos^{-1}x)$

$=\frac{\pi}{2}((\sin^{-1}x)+(\cos^{-1}x))^2-3\sin^{-1}x\cdot\cos^{-1}x)$

$=\frac{\pi}{2}\left[\left(\frac{\pi}{2}\right)^2-3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)\right]$

$$= \frac{\pi}{2}\left[\frac{\pi^2}{4} - \frac{3\pi}{2}\sin^{-1}x + 3(\sin^{-1}x)^2\right]$$

$$= \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{12}\right]$$

$$= \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right]$$

Maximum value occurs when $\sin^{-1}x = -\pi/2$

Maximum value $= \frac{3\pi}{2}\left[\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right] = \frac{3\pi}{2}\frac{7\pi^2}{12} = \frac{7\pi^3}{8}$

Minimum value occurs when $\sin^{-1}x = \pi/4$

$\Rightarrow$ Minimum value $= \frac{3\pi}{2}\left[\frac{\pi^2}{48}\right] = \frac{\pi^3}{32}$

Example 27 :

Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Sol. $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

domain : $x \in [-1, 1]$

$$f(x) = \frac{\pi}{2} + [\tan^{-1} -1, \tan^{-1} 1]$$

$$= \frac{\pi}{2} + \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Example 28 :

Find the value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$, for $0 < A < \pi/4$.

Sol. For $0 < A < \pi/4$, $\cot A > 1 \Rightarrow (\cot A)(\cot^3 A) > 1$

Then $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1-\cot^4 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right) = \pi$$

Example 29 :

If $f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x$ then find

(a) $f(100)$ (b) $\cos(f(-10))$

Sol. $f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = \pi$ if $x \geq 1$

$f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = -\pi$ if $x \leq -1$

$\Rightarrow$ (a) $f(100) = \pi$

(b) $\cos(f(-10)) = \cos(-\pi) = -1$

QUESTION BANK

CHAPTER 2 : INVERSE TRIGONOMETRIC FUNCTIONS

EXERCISE - 1 [LEVEL-1]

Q.1 The value of $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$ is
(A) -2 (B) $8\pi - 26$
(C) $4\pi + 2$ (D) None of these

Q.2 $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] =$
(A) 25/24 (B) 25/7
(C) 24/25 (D) None of these

Q.3 $\left[\sin\left(\tan^{-1}\frac{3}{4}\right)\right]^2 =$
(A) 3/5 (B) 5/3
(C) 9/25 (D) 25/9

Q.4 $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$ is equal to
(A) $\pi/4$ (B) $\pi/6$
(C) $\pi/3$ (D) $2\pi/3$

Q.5 $2\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{24}{25} =$
(A) $\pi/2$ (B) $2\pi/3$
(C) $5\pi/3$ (D) None of these

Q.6 $\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) =$
(A) $\pi/4$ (B) $\pi/2$
(C) $\cos^{-1}(4/5)$ (D) π

Q.7 $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} =$
(A) $\pi/2$ (B) $\pi/3$
(C) $\pi/4$ (D) None of these

Q.8 The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is
(A) 0 (B) $\pi/2$
(C) $2\pi/3$ (D) $10\pi/3$

Q.9 If $\theta = \tan^{-1}a, \phi = \tan^{-1}b$ and $ab = -1$, then $\theta - \phi =$
(A) 0 (B) $\pi/4$
(C) $\pi/2$ (D) None of these

Q.10 $\text{Sin}^{-1}\left(\frac{\pi}{6} - \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right) =$
(A) 1 (B) $-1/2$
(C) 1/4 (D) not possible

Q.11 If $\alpha = \sin^{-1}(\cos(\sin^{-1}x))$ and $\beta = \cos^{-1}(\sin(\cos^{-1}x))$, then–
(A) $\tan\alpha = \cot\beta$ (B) $\tan\alpha = -\cot\beta$
(C) $\tan\alpha = \tan\beta$ (D) $\tan\alpha = -\tan\beta$

Q.12 $\cos(\tan^{-1}x) =$
(A) $\sqrt{1+x^2}$ (B) $\frac{1}{\sqrt{1+x^2}}$
(C) $1+x^2$ (D) None of these

Q.13 $\tan(\cos^{-1}x)$ is equal to
(A) $\frac{\sqrt{1-x^2}}{x}$ (B) $\frac{x}{1+x^2}$
(C) $\frac{\sqrt{1+x^2}}{x}$ (D) $\sqrt{1-x^2}$

Q.14 The solution set of the equation $\sin^{-1}x = 2\tan^{-1}x$ is
(A) {1, 2} (B) {–1, 2}
(C) {–1,1, 0} (D) {1, 1/2, 0}

Q.15 $\sin^{-1}x + \sin^{-1}\frac{1}{x} + \cos^{-1}x + \cos^{-1}\frac{1}{x} =$
(A) π (B) $\pi/2$
(C) $3\pi/2$ (D) None of these

Q.16 $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right] =$
(A) 6/17 (B) 17/6
(C) 7/16 (D) 16/7

Q.17 If $\tan^{-1}\frac{x-1}{x+2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then $x =$
(A) $\frac{1}{\sqrt{2}}$ (B) $-\frac{1}{\sqrt{2}}$
(C) $\pm\sqrt{\frac{5}{2}}$ (D) $\pm\frac{1}{2}$

Q.18 If $\tan^{-1}\frac{a+x}{a} + \tan^{-1}\frac{a-x}{a} = \frac{\pi}{6}$, then $x^2 =$
(A) $2\sqrt{3}a$ (B) $\sqrt{3}a$
(C) $2\sqrt{3}a^2$ (D) None of these

Q.19 If $\cos^{-1}x + \cos^{-1}y = 2\pi$, then $\sin^{-1}x + \sin^{-1}y$ is equal to
(A) π (B) $-\pi$
(C) $\pi/2$ (D) None of these

Q.20 $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is
(A) $\pi/2$ (B) $\pi/3$
(C) $\pi/4$ (D) $\pi/4$ or $-3\pi/4$

Q.21 The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$ is

(A) $\frac{6}{17}$ (B) $\frac{6}{\sqrt{13}}$

(C) $\frac{\sqrt{13}}{5}$ (D) $\frac{17}{6}$

Q.22 $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=$

(A) 17/7 (B) –17/7

(C) 7/17 (D) –7/17

Q.23 $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=$

(A) $\cot^{-1}\sqrt{x}$ (B) $\tan^{-1}\sqrt{x}$

(C) $\tan^{-1}x$ (D) $\cot^{-1}x$

Q.24 If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$ then

(A) $x+y-xy=1$ (B) $x+y+xy=1$

(C) $x+y+xy+1=0$ (D) $x+y-xy+1=0$

Q.25 $\sin\left[3\sin^{-1}\left(\frac{1}{5}\right)\right]=$

(A) 71/125 (B) 74/125

(C) 3/5 (D) 1/2

Q.26 If $\cos^{-1}p+\cos^{-1}q+\cos^{-1}r=\pi$ then $p^2+q^2+r^2+2pqr=$

(A) 3 (B) 1

(C) 2 (D) –1

Q.27 The equation $\sin^{-1}x-\cos^{-1}x=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has

(A) No solution

(B) Unique solution

(C) Infinite number of solutions

(D) None of these

Q.28 $2\tan^{-1}(\cos x)=\tan^{-1}(\text{cosec}^2x)$, then x =

(A) π/2 (B) π

(C) π/6 (D) π/3

Q.29 If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$, then find $9x^2-12xy\cos\theta+4y^2$

(A) $16\sin^2\theta$ (B) $36\sin^2\theta$

(C) $18\sin^2\theta$ (D) $22\sin^2\theta$

Q.30 If $\text{cosec}^{-1}x=2\cot^{-1}7+\cos^{-1}\frac{3}{5}$ then x =

(A) 44/117 (B) 125/117

(C) 24/7 (D) 5/3

Q.31 If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then find the value of $\cos^{-1}x+\cos^{-1}y$

(A) π (B) π/2

(C) π/4 (D) π/3

Q.32 $\sin(\cot^{-1}\cos\tan^{-1}x)$ is equal to –

(A) $\sqrt{\frac{x^2+1}{x^2+2}}$ (B) $\sqrt{\frac{x+2}{x^2+1}}$

(C) $\sqrt{\frac{x^2-1}{x^2+2}}$ (D) $\sqrt{\frac{x^2+2}{x^2+1}}$

Q.33 There exists a positive real number x satisfying $\cos(\tan^{-1}x)=x$, the value of $\cos^{-1}(x^2/2)$ is –

(A) π/10 (B) π/5

(C) 2π/5 (D) 4π/5

Q.34 The value of $\cos(\tan^{-1}\tan 4)$ is –

(A) $\frac{1}{\sqrt{17}}$ (B) $-\frac{1}{\sqrt{17}}$

(C) – cos 4 (D) + cos 4

Q.35 If $\frac{3\pi}{2}\le x\le\frac{5\pi}{2}$ then $\sin^{-1}\sin x$ is –

(A) x (B) –x

(C) x – 2π (D) 2π – x

Q.36 $\cos\left(\cos^{-1}\cos\left(\frac{8\pi}{7}\right)+\tan^{-1}\tan\left(\frac{8\pi}{7}\right)\right)$ has the value equal to

(A) 1 (B) –1

(C) $\cos\frac{\pi}{7}$ (D) 0

Q.37 If a > b > 0, $\sec^{-1}\left(\frac{a+b}{a-b}\right)=2\sin^{-1}x$, then x =

(A) $-\sqrt{\frac{b}{a+b}}$ (B) $\sqrt{\frac{b}{a+b}}$

(C) $-\sqrt{\frac{a}{a+b}}$ (D) $\sqrt{\frac{a}{a+b}}$

Q.38 If $\tan^{-1}x=\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{3}\right)$, then x is –

(A) 1/6 (B) 1/4

(C) 1/2 (D) 1/3

Q.39 $2\cos^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$ is valid for all values of x satisfying –

(A) $0\le x\le\frac{1}{\sqrt{2}}$ (B) $-1\le x\le 1$

(C) $0\le x\le 1$ (D) $\frac{1}{\sqrt{2}}\le x\le 1$

Q.40 If $\frac{(x+1)^2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$, then $\sin^{-1} A + \tan^{-1} B + \sec^{-1} C$

(A) $\pi/2$ (B) $\pi/6$
(C) 0 (D) $5\pi/6$

Q.41 The value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$, x, y > 0 is –

(A) $\pi/4$ (B) $-\pi/4$
(C) $\pi/2$ (D) $-\pi/2$

Q.42 The value of $\sin(2\sin^{-1} 0.8)$ is equal to

(A) 0.48 (B) sin 1.2°
(C) sin 1.6° (D) 0.96

EXERCISE - 2 [LEVEL-2]

Q.1 If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then find the value of x.

(A) 1 (B) 2
(C) 4 (D) 6

Q.2 The sum $\sum_{n=1}^{\infty} \text{arc tan}\left(\frac{2}{n^2}\right)$ equals

(A) $\pi/4$ (B) $\pi/2$
(C) $3\pi/4$ (D) π

Q.3 The greatest and the least value of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ are

(A) $-\frac{\pi}{2}, \frac{\pi}{2}$ (B) $-\frac{\pi^3}{8}, \frac{\pi^3}{8}$
(C) $\frac{7\pi^3}{8}, \frac{\pi^3}{32}$ (D) None of these

Q.4 For the equation $\cos^{-1} x + \cos^{-1} 2x + \pi = 0$, the number of real solution is

(A) 1 (B) 2
(C) 0 (D) ∞

Q.5 Find the value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$

(A) 3/4 (B) 3/4
(C) 1/2 (D) 1/4

Q.6 Find the number of solution of the equation $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$.

(A) 1 (B) 2
(C) 3 (D) 4

Q.7 Find : $\tan^{-1}(1/3) + \tan^{-1}(1/7) + \tan^{-1}(1/13) +..... + \tan^{-1}\left(\frac{1}{n^2+n+1}\right) +....\infty$

(A) $\pi/4$ (B) $\pi/2$
(C) $\pi/3$ (D) $\pi/6$

Q.8 Find x for : $\tan^{-1}(1+x) + \tan^{-1} x + \tan^{-1}(x-1) = \tan^{-1} 3$

(A) 2 (B) 4
(C) –2 (D) –1

Q.9 The value of $\tan\left(\text{arc}\sin\left(-\frac{4}{5}\right) - \text{arc}\cos\left(-\frac{5}{13}\right)\right) =$

(A) 25/63 (B) – 3/7
(C) – 33/56 (D) 16/63

Q.10 If $\frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$, then $\sin^{-1}\frac{A}{B} =$

(A) $\pi/4$ (B) $\pi/2$
(C) $\pi/3$ (D) $\pi/6$

Q.11 The number of real solutions of the equation $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is –

(A) infinitely many (B) one
(C) four (D) two

Q.12 $\cos\left[2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right] =$

(A) 1/5 (B) $-2\sqrt{6}/5$
(C) –1/5 (D) $\sqrt{6}/5$

Q.13 Given $0 \le x \le 1/2$ then the value of $\tan\left[\sin^{-1}\left\{\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right\} - \sin^{-1} x\right]$ is –

(A) 1 (B) $\sqrt{3}$
(C) –1 (D) $1/\sqrt{3}$

Q.14 If the non-zero numbers x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are also in A.P. then –

(A) xy = yz (B) x = y = z
(C) $x^2 = yz$ (D) $z^2 = xy$

Q.15 The value of $\tan^{-1}\left(\frac{\sin 2 - 1}{\cos 2}\right)$ is equal to –

(A) 2 (B) $2 - \frac{\pi}{2}$
(C) $1 - \frac{\pi}{4}$ (D) $\frac{\pi}{4} - 1$

Q.16 The value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$ is

(A) $4\tan^{-1}(1)$ (B) $2\tan^{-1}(2)$
(C) 0 (D) none

Q.17 The number of solutions of the equation $\tan^{-1}\left(\frac{x}{3}\right) + \tan^{-1}\left(\frac{x}{2}\right) = \tan^{-1} x$ is

(A) 3 (B) 2
(C) 1 (D) 0

Q.18 Which of the following is the solution set of the equation

$$2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{2x\sqrt{1-x^2}}\right)$$

(A) (0, 1) (B) (–1, 1) – {0}
(C) (–1, 0) (D) [–1, 1]

Q.19 Sum of the roots of the equation,
arc cot x – arc cot (x + 2) = π/12 is

(A) $\sqrt{3}$ (B) 2
(C) –2 (D) $-\sqrt{3}$

Q.20 The range of values of p for which the equation

$\sin\cos^{-1}\left(\cos(\tan^{-1}x)\right) = p$ has a solution is:

(A) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (B) [0, 1)
(C) $\left[\frac{1}{\sqrt{2}}, 1\right)$ (D) (– 1, 1)

Q.21 For the equation
$2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$,
which of the following are not possible –
(1) $a^2x + 2a = x$ (2) $a^2 + 2ax + 1 = 0$
(3) $a \neq 0$ (4) $a \neq -1, 1$
(A) (1) & (2) (B) (2) & (3)
(C) (1), (2) & (3) (D) All of these

Q.22 If $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\left(\frac{x}{2-x^2}\right)$ $= \cos^{-1}k + \pi$, then the value of k =

(A) 1 (B) $-1/\sqrt{2}$
(C) $1/\sqrt{2}$ (D) None of these

Q.23 If α and β are the roots of the equation
$x^2 - 4x + 1 = 0$ (α > β) then the value of

$$f(\alpha,\beta) = \frac{\beta^3}{2}\text{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$$

(A) 56 (B) 66
(C) 40 (D) 18

Q.24 Which of the following is not a rational number –

(A) $\sin\left(\tan^{-1}3 + \tan^{-1}\frac{1}{3}\right)$ (B) $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right)$

(C) $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right)$ (D) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Q.25 The value of a for which

$$ax + \sec^{-1}\sqrt{2x^2 - x^4} + \text{cosec}^{-1}\sqrt{2x^2 - x^4} = 0$$

(A) π/4 (B) –π/2
(C) 2/π (D) –2/π

Q.26 If $\sin^{-1}\sin(5) > x^2 - 4x$ then the number of possible integral values of x is –
(A) 1 (B) 2
(C) 3 (D) 4

Q.27 If $\sum_{k=1}^{100}\tan^{-1}\frac{1}{2k^2} = \tan^{-1}\frac{p}{q}$, where p and q are coprime number, then 2p – q is –
(A) 101 (B) 99
(C) 201 (D) 98

Q.28 Choose the correct options –
(A) No. of solution of the equation

$\sin^{-1}x - \cos^{-1}(-x) = \frac{\pi}{2}$ is one.

(B) Solution set of the equation

$\sin^{-1}(x^2+4x+3) + \cos^{-1}(x^2+6x+8) = \frac{\pi}{2}$ is $\left\{-\frac{5}{2}\right\}$.

(C) $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$ is equal to π
(D) $2[\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3]$ is equal to π.

Q.29 Number of solution(s) of the equation

$\cos^{-1}(1-x) - 2\cos^{-1}x = \frac{\pi}{2}$ is

(A) 1 (B) 2
(C) 3 (D) 4

Q.30 The smallest positive integer x so that

$\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\frac{\pi}{4}$, is

(A) 1 (B) 2
(C) 3 (D) 8

Directions : Assertion-Reason type questions.
(A) Statement- 1 is True, Statement-2 is True, Statement2 is a correct explanation for Statement -1
(B) Statement -1 is True, Statement -2 is True; Statement2 is NOT a correct explanation for Statement - 1
(C) Statement - 1 is True, Statement- 2 is False
(D) Statement -1 is False, Statement -2 is True

Q.31 **Statement 1 :** The domain of the function $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ is [–1, 1]
Statement 2 : $\sin^{-1}x, \cos^{-1}x$ are defined for $|x| \le 1$ and $\tan^{-1}x$ is defined for all x.

Q.32 **Statement 1 :** $\text{cosec}^{-1}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)$

Statement 2 : $\text{cosec}^{-1}x > \sec^{-1}x$ if $1 \le x < \sqrt{2}$

Q.33 Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Statement 1 : $f'(2) = -\frac{2}{5}$

Statement 2 : $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, \quad \forall x > 1$

Q.34 **Statement 1 :** $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement 2 : $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall x, y \in (0, 1)$

Passage (Q.35-Q.37) :

Given that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}x & , |x| \le 1 \\ -\pi + 2\tan^{-1}x & , x > 1 \\ \pi + 2\tan^{-1}x & , x < -1 \end{cases}$

$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & , |x| \le 1 \\ \pi - 2\tan^{-1}x & , x > 1 \\ -(\pi + 2\tan^{-1}x) & , x < -1 \end{cases}$ and

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \le x \le 1$

Q.35 $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent at x then

(A) $x \in [-3, 4]$ (B) $x \in [-2, 2]$
(C) $x \in [-1, 1]$ (D) $x \in [1, \infty]$

Q.36 If $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$ then $x \in$

(A) $(1/3, \infty)$ (B) $(-1, \infty)$
(C) $(-\infty, -1)$ (D) None of these

Q.37 If $(x-1)(x^2+1) > 0$ then

$\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} - \tan^{-1}x\right) =$

(A) 1 (B) $1/\sqrt{2}$
(C) –1 (D) None of these

Passage (Q.38-Q.40)

It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$ where $x > 0$ and $B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$ where $t \in [0, 1/\sqrt{2}]$, and $\sin^{-1}x + \cos^{-1}x = \pi/2$ for $-1 \le x \le 1$ and $\tan^{-1}x + \cot^{-1}x = \pi/2$ for all $x \in R$.

Q.38 The interval in which A lies is –

(A) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right)$ (B) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$

(C) $\left(\frac{\pi^3}{40}, \frac{\pi^3}{10}\right)$ (D) None of these

Q.39 The maximum value of B is –

(A) $\pi^2/8$ (B) $\pi^2/16$
(C) $\pi^2/4$ (D) None of these

Q.40 If least value of A is λ and maximum value of B is μ then

$\cot^{-1}\cot\left(\frac{\lambda - \mu\pi}{\mu}\right) =$

(A) $\pi/8$ (B) $-\pi/8$
(C) $7\pi/8$ (D) $-7\pi/8$

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.41 $\tan\left(\text{arc tan}\left(\frac{-2}{3}\right) + \text{arc tan}(5)\right)$ equals

Q.42 Number of solution(s) of the equation

$\cos^{-1}(1-x) - 2\cos^{-1}x = \frac{\pi}{2}$ is

Q.43 Number of value of x satisfying the equation

$\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$ is

Q.44 The value of the expression,

$14\tan\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \tan^{-1}\frac{1}{31}\right)$

is an integer which is equal to

Q.45 The smallest positive integer x so that

$\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\frac{\pi}{4}$, is

Q.46 Let $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ where

$x \in (-1, 0)$ then f simplifies to

Q.47 Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$.
If y simplifies to $a\pi + b$ then find $(a - b)$.

Q.48 If $\sin^{-1}\sin(5) > x^2 - 4x$ then the number of possible integral values of x is –

Q.49 The number of solutions of the equation,

$\tan^{-1}(4\{x\}) + \cot^{-1}(x + [x]) = \frac{p}{2}$, is (where [ ] denotes greatest integer function and { } denotes fractional part function)

Q.50 The number of real solution of

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is –

Q.51 Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

Q.52 Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation

$f(x) = \frac{10-x}{10}$ is –

Q.53 Number of positive solutions satisfying the equation

$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is

Q.54 $\sin^{-1}(1-x) - 2\sin^{-1}x = \pi/2$, then x is equal to –

EXERCISE - 3 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 The value of $\cos^{-1}(-1) - \sin^{-1}(1)$ is- **[AIEEE 2002]**
(A) π (B) $\pi/2$
(C) $3\pi/2$ (D) $-3\pi/2$

Q.2 The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for – **[AIEEE 2003]**
(A) $|a| \le \frac{1}{\sqrt{2}}$ (B) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$
(C) all real values of a (D) $|a| < \frac{1}{2}$

Q.3 If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to- **[AIEEE 2005]**
(A) $2\sin 2\alpha$ (B) 4
(C) $4\sin^2\alpha$ (D) $-4\sin^2\alpha$

Q.4 If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then a value of x is- **[AIEEE 2007]**
(A) 1 (B) 3
(C) 4 (D) 5

Q.5 The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is **[AIEEE 2008]**
(A) 3/17 (B) 1/17
(C) 2/17 (D) 6/17

Q.6 Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$. where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is – **[JEE MAIN 2015]**
(A) $\frac{3x+x^3}{1-3x^2}$ (B) $\frac{3x-x^3}{1+3x^2}$
(C) $\frac{3x+x^3}{1+3x^2}$ (D) $\frac{3x-x^3}{1-3x^2}$

Q.7 Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$.
A normal to $y = f(x)$ at $x = \pi/6$ also passes through the point **[JEE MAIN 2017]**
(A) $(0, 2\pi/3)$ (B) $(\pi/6, 0)$
(C) $(\pi/4, 0)$ (D) $(0, 0)$

Q.8 If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then g (x) equals – **[JEE MAIN 2017]**
(A) $\frac{3x}{1-9x^3}$ (B) $\frac{3}{1+9x^3}$
(C) $\frac{9}{1+9x^3}$ (D) $\frac{3x\sqrt{x}}{1-9x^3}$

Q.9 If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$ then x is equal to : **[JEE MAIN 2019 (Jan)]**
(A) $\frac{\sqrt{145}}{12}$ (B) $\frac{\sqrt{145}}{10}$
(C) $\frac{\sqrt{146}}{12}$ (D) $\frac{\sqrt{145}}{11}$

Q.10 If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to: **[JEE MAIN 2019 (Jan)]**
(A) π (B) 7π
(C) 0 (D) 10

Q.11 The value of $\sin^{-1}(12/13) - \sin^{-1}(3/5)$ is equal to: **[JEE MAIN 2019 (April)]**
(A) $\pi - \sin^{-1}(63/65)$ (B) $\pi - \sin^{-1}(33/65)$
(C) $\pi/2 - \sin^{-1}(56/65)$ (D) $\pi/2 - \sin^{-1}(9/65)$

Q.12 Let $f(x) = \{(\sin(\tan^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1$ where $|x| > 1$ and $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(\sin^{-1}f(x))$.
If $y(\sqrt{3}) = \frac{\pi}{6}$ then $y(-\sqrt{3}) =$ **[JEE MAIN 2020 (Jan)]**
(A) $5\pi/6$ (B) $-\pi/6$
(C) $\pi/3$ (D) $2\pi/3$

Q.13 If $f'(x) = \tan^{-1}(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $f(0) = 0$, then $f(1)$ is equal to : **[JEE MAIN 2020 (Jan)]**
(A) $\frac{\pi-1}{4}$ (B) $\frac{\pi+2}{4}$
(C) $\frac{\pi+1}{4}$ (D) $\frac{1}{4}$

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
A	A	D	C	D	A	A	A	A	C	D	A	B	A	C	A	B	C	C	B	C	D
Q	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
A	D	B	B	A	B	B	D	B	B	D	A	C	C	C	B	B	C	D	D	A	D

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	A	C	C	C	A	C	A	D	D	D	D	B	A	B	C	A	A	A	C	B
Q	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
A	B	C	A	D	B	C	B	B	A	D	A	A	A	A	B	A	C	B	C	A
Q	41	42	43	44	45	46	47	48	49	50	51	52	53	54						
A	1	1	1	10	8	0	53	3	2	2	1	3	1	0						

EXERCISE - 3

Q	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	A	C	B	D	D	A	C	A	A	C	B	C

CHAPTER- 2: INVERSE TRIGONOMETRIC FUNCTIONS

SOLUTIONS TO TRY IT YOURSELF

(1) (a) $\sin^{-1}(\sin 1) = 1 \quad \left(\because\ 1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(b) $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = 3\pi - 10$

$\left(\because\ 3\pi - 10 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(2) (a) $\cos^{-1}(\cos 1) = 1 \ ; \ (\because\ 1 \in [0, \pi])$

(b) $\cos^{-1}(\cos 5) = \cos^{-1}(\cos(2\pi - 5)) = 2\pi - 5;$
$(\because\ 2\pi - 5 \in [0, \pi])$

(3) (a) $\tan^{-1}(\tan 2) \neq 2 \quad \left(\because\ 2 \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\Rightarrow \tan^{-1}(\tan 2) = \tan^{-1}(\tan(\pi - 2)) = \pi - 2$

$\left(\because\ \pi - 2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(b) $\tan^{-1}(\tan 10) = \tan^{-1}(\tan(3\pi - 10)) = 3\pi - 10;$

$\left(\because\ 3\pi - 10 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(4) $6x^2 - 5x < \cos^{-1}(\cos(2\pi - 5)) - 2\sin^{-1}(\sin(\pi - 3))$
$6x^2 - 5x < 2\pi - 5 - 2\pi + 6$

$6x^2 - 5x - 1 < 0 \ ; \ (6x+1)(x-1) < 0 \Rightarrow x \in \left(-\frac{1}{6}, 1\right)$

Integral solution is $x = 0$

(5) (a) $\cos^{-1}\sin\left(-\frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \cos^{-1}\cos\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

(b)

$\sin^{-1}\cos\left(\frac{33\pi}{10}\right) = \sin^{-1}\cos\frac{13\pi}{10} = \sin^{-1}\left(-\cos\frac{3\pi}{10}\right)$

$= \sin^{-1}\left(-\sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)\right) = \sin^{-1}\left(-\sin\frac{\pi}{5}\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right) = -\frac{\pi}{5}$

(6) $4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \frac{3\pi}{4}$

$\Rightarrow 3\sin^{-1}x = \frac{\pi}{4} \Rightarrow \sin^{-1}x = \frac{\pi}{12} \Rightarrow x = \sin\frac{\pi}{12}$

$\Rightarrow x = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(7) $\tan^{-1}2 + \tan^{-1}4 = \pi + \tan^{-1}\left(\frac{2+4}{1-2\times 4}\right) = \pi + \tan^{-1}\left(\frac{6}{-7}\right)$

$= \pi - \tan^{-1}\frac{6}{7} = \pi - \cot^{-1}\frac{7}{6} = \cot^{-1}\left(-\frac{7}{6}\right) \Rightarrow \lambda = -\frac{7}{6}$

(8) $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}\right)$

$= \tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{3}\cdot\sqrt{2}}\right)$

$= \tan^{-1}\frac{1}{\sqrt{2}} - (\tan^{-1}\sqrt{3} - \tan^{-1}\sqrt{2})$

$= \cot^{-1}\sqrt{2} - \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{2}$

$= \frac{\pi}{2} - \tan^{-1}\sqrt{3} = \cot^{-1}\sqrt{3} = \frac{\pi}{6}$

(9) **(A).** $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$
$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(-z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

Squaring both sides
$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$
$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

(10) $2\cot^{-1}2 - \cos^{-1}\frac{4}{5} = 2\tan^{-1}\frac{1}{2} - \cos^{-1}\frac{4}{5}$

$= \tan^{-1}\frac{2(1/2)}{1-(1/2)^2} - \tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\frac{3}{4}$

$= \tan^{-1}\left(\frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\times\frac{3}{4}}\right) = \tan^{-1}\left(\frac{7}{24}\right) = \operatorname{cosec}^{-1}\left(\frac{25}{7}\right)$

$= \operatorname{cosec}^{-1}x = \operatorname{cosec}^{-1}\left(\frac{25}{7}\right) \Rightarrow x = \frac{25}{7}$

CHAPTER- 2 : INVERSE TRIGONOMETRIC FUNCTIONS
EXERCISE-1

(1) **(A).** $\cos^{-1}(\cos 12)-\sin^{-1}(\sin 14) \Rightarrow 12-14=-2$

(2) **(D).** $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]=\cot\left[\cot^{-1}\left(\frac{7}{24}\right)\right]=\frac{7}{24}$.

(3) **(C).** $\left[\sin\left(\tan^{-1}\frac{3}{4}\right)\right]^2=\left[\sin\left(\sin^{-1}\frac{3}{5}\right)\right]^2=\left(\frac{3}{5}\right)^2=\frac{9}{25}$.

(4) **(D).** $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=\frac{\pi}{3}+\frac{2\pi}{6}=\frac{2\pi}{3}$

(5) **(A).**

$$2\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{24}{25}=\sin^{-1}2\times\frac{3}{5}\sqrt{1-\frac{9}{25}}+\cos^{-1}\frac{24}{25}$$

$$=\sin^{-1}\frac{24}{25}+\cos^{-1}\frac{24}{25}=\frac{\pi}{2}$$

(6) **(A).** $\sin^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\left(\frac{(3/4)+(1/7)}{1-(3/4)\times(1/7)}\right)=\tan^{-1}\left(\frac{25}{25}\right)=\tan^{-1}1=\frac{\pi}{4}.$$

(7) **(A).** $\sin^{-1}\frac{4}{5}=\tan^{-1}\frac{4}{3},\ 2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}=\cot^{-1}\frac{4}{3}$

and $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$.

(8) **(A).** $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

$$=\cos^{-1}\left[\cos\left(2\pi-\frac{\pi}{3}\right)\right]+\sin^{-1}\left[\sin\left(2\pi-\frac{\pi}{3}\right)\right]$$

$$=\frac{\pi}{3}-\frac{\pi}{3}=0.$$

(9) **(C).** Given that $\theta=\tan^{-1}a$ and $\phi=\tan^{-1}b$ and $ab=-1$.

$$\Rightarrow \tan\theta\tan\phi=-1\Rightarrow\tan\theta=-\cot\phi\ \Rightarrow\theta-\phi=\frac{\pi}{2}.$$

(10) **(D).** $\sin^{-1}\left(\frac{\pi}{6}+\sin^{-1}\frac{\sqrt{3}}{2}\right)$

$=\sin^{-1}\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\sin^{-1}\left(\frac{\pi}{2}\right)=\sin^{-1}(1.57)$ not possible

(11) **(A).**

$$\sin\alpha=\cos(\sin^{-1}x)=\cos\left(\frac{\pi}{2}-\cos^{-1}x\right)=\sin(\cos^{-1}x)$$

$\therefore\ \cos\beta=\sin(\cos^{-1}x)$

$\Rightarrow\ \sin\alpha=\cos\beta$

$\Rightarrow\ \tan\alpha=\cot\beta$

(12) **(B).** Let $\theta=\tan^{-1}x\Rightarrow x=\tan\theta$

$$\therefore\cos\theta=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{1}{\sqrt{1+x^2}}$$

Hence $\cos\theta=\cos(\tan^{-1}x)=\frac{1}{\sqrt{1+x^2}}$.

(13) **(A).** Let $\cos^{-1}x=\theta.$ Then $x=\cos\theta$

$$\Rightarrow\tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{x^2}-1}=\sqrt{\frac{1-x^2}{x}}$$

$$\therefore\tan(\cos^{-1}x)=\tan\theta=\frac{\sqrt{1-x^2}}{x}.$$

(14) **(C).** $\sin^{-1}x=2\tan^{-1}x\ \Rightarrow\sin^{-1}x=\sin^{-1}\frac{2x}{1+x^2}$

$\Rightarrow\frac{2x}{1+x^2}=x\ \Rightarrow x^3-x=0$

$\Rightarrow x(x+1)(x-1)=0\Rightarrow x=\{-1,1,0\}$.

(15) **(A).** $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}$

$$=\{\sin^{-1}(x)+\cos^{-1}(x)\}+\left\{\sin^{-1}\left(\frac{1}{x}\right)+\cos^{-1}\left(\frac{1}{x}\right)\right\}$$

$$=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

(16) **(B).** $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]$

$$=\tan\left[\tan^{-1}\frac{\sqrt{\left(1-\frac{16}{25}\right)}}{\frac{4}{5}}+\tan^{-1}\frac{2}{3}\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right]=\tan.\tan^{-1}\frac{17}{6}=\frac{17}{6}.$$

(17) (C). We have $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2x(x+2)}{4x+5}=\tan\frac{\pi}{4}=1$$

$$\Rightarrow 2x^2+4x=4x+5 \quad \Rightarrow x=\pm\sqrt{\frac{5}{2}}.$$

(18) (C). Given equation is $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{a+x}{a}+\frac{a-x}{a}}{1-\frac{a+x}{a}.\frac{a-x}{a}}\right)=\frac{\pi}{6}$$

$$\Rightarrow \frac{2a^2}{x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow x^2=2\sqrt{3}a^2.$$

(19) (B). $\cos^{-1}x+\cos^{-1}y=2\pi$

$$\Rightarrow \frac{\pi}{2}-\sin^{-1}x+\frac{\pi}{2}-\sin^{-1}y=2\pi$$

$$\Rightarrow \pi-(\sin^{-1}x+\sin^{-1}y)=2\pi$$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y=-\pi$$

(20) (C). $\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$=\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{1-y/x}{1+y/x}\right)=\tan^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}-\frac{\pi}{4}$$

$$=\tan^{-1}\frac{x}{y}+\cot^{-1}\frac{x}{y}-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

(21) (D). $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$

$$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)$$

$$=\tan\left[\tan^{-1}\frac{17}{12}\times\frac{12}{6}\right]=\frac{17}{6}$$

(22) (D). $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]$

$$=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]=\tan\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)=-\frac{7}{17}.$$

(23) (B). Let $x=\tan^2\theta\Rightarrow\theta=\tan^{-1}\sqrt{x}$

Now, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$=\frac{1}{2}\cos^{-1}\cos 2\theta=\frac{2\theta}{2}=\theta=\tan^{-1}\sqrt{x}.$$

(24) (B). $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4};\ \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}1$

$$\frac{x+y}{1-xy}=1;\ x+y+xy=1.$$

(25) (A). $\sin\left[3\sin^{-1}\frac{1}{5}\right]=\sin\left[\sin^{-1}\left\{3\left(\frac{1}{5}\right)-4\left(\frac{1}{5}\right)^3\right\}\right]$

$$=\sin\left[\sin^{-1}\left\{\frac{3}{5}-\frac{4}{125}\right\}\right]=\sin\left[\sin^{-1}\left(\frac{75-4}{125}\right)\right]$$

$$=\sin\left[\sin^{-1}\frac{71}{125}\right]=\frac{71}{125}.$$

(26) (B). According to given condition, we put

$p=q=r=\frac{1}{2}$. Then, $p^2+q^2+r^2+2pqr$

$$=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+2.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{2}{8}=1.$$

(27) (B). We have $\sin^{-1}x-\cos^{-1}x=\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$

But $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$\therefore\ \sin^{-1}x=\frac{\pi}{3}$ and $\cos^{-1}x=\frac{\pi}{6}$

$\Rightarrow x=\frac{\sqrt{3}}{2}$ is the unique solution.

(28) (D). $2\tan^{-1}(\cos x)=\tan^{-1}(\operatorname{cosec}^2 x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}\left(\frac{1}{\sin^2 x}\right)$$

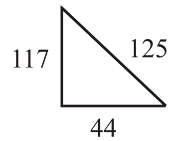
$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 2\cos x = 1 \;\Rightarrow x = \frac{\pi}{3}.$$

(29) **(B).** $\frac{x}{2}\cdot\frac{y}{3} - \sqrt{\left(1-\frac{x^2}{4}\right)}\sqrt{\left(1-\frac{y^2}{9}\right)} = \cos\theta$

$\therefore\; (xy - 6\cos\theta)^2 = (4 - x^2)(9 - y^2)$

$\therefore\; 9x^2 - 12xy\cos\theta + 4y^2 = 36(1-\cos^2\theta) = 36\sin^2\theta$

(30) **(B).** $2\cot^{-1}7 + \cos^{-1}\frac{3}{5}$



$= \cot^{-1}\frac{49-1}{2\times 7} + \cot^{-1}\frac{3}{4}$

$= \tan^{-1}\frac{7}{24} + \tan^{-1}\frac{4}{3} = \tan^{-1}\left[\frac{\frac{7}{24}+\frac{4}{3}}{1-\frac{28}{72}}\right] = \tan^{-1}\frac{117}{44}$

Hence, $\operatorname{cosec}^{-1}x = \operatorname{cosec}^{-1}\frac{125}{117}$

(31) **(D).** We have, $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$

$\Rightarrow \left(\frac{\pi}{2} - \cos^{-1}x\right) + \left(\frac{\pi}{2} - \cos^{-1}y\right) = \frac{2\pi}{3}$

$\Rightarrow \pi - (\cos^{-1}x + \cos^{-1}y) = \frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

(32) **(A).** $\sin(\cot^{-1}\cos\tan^{-1}x) = \sin(\cot^{-1}\cos\theta)$,

where $\theta = \tan^{-1}x$ or $x = \tan\theta$

$= \sin\left\{\sin^{-1}\left(\frac{1}{\sqrt{1+\cos^2\theta}}\right)\right\} = \frac{1}{\sqrt{1+\cos^2\theta}}$

$= \frac{\sec\theta}{\sqrt{1+\sec^2\theta}} = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

(33) **(C).** $\tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow x = \frac{1}{\sqrt{1+x^2}} \Rightarrow x^2 = \frac{-1+\sqrt{5}}{2}$

$\Rightarrow \frac{x^2}{2} = \frac{\sqrt{5}-1}{4}$; $\cos^{-1}\left(\frac{x^2}{2}\right) = \frac{2\pi}{5}$

(34) **(C).** Since $\tan^{-1}\tan 4 = 4 - \pi$

Hence, $\cos\tan^{-1}\tan 4 = \cos(4-\pi) = -\cos 4$

(35) **(C).** $\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$ nearest π is 2π.

Then solution is $x - 2\pi$

(36) **(B).** $\cos^{-1}\cos\frac{8\pi}{7} = \cos^{-1}\cos\left(-\frac{\pi}{7}\right)$

$= \cos^{-1}\cos\left(\pi - \frac{\pi}{7}\right) = \cos^{-1}\cos\left(\frac{6\pi}{7}\right) = \frac{6\pi}{7}$

$\tan^{-1}\left(\tan\frac{8\pi}{7}\right) = \tan^{-1}\left(\tan\frac{\pi}{7}\right) = \frac{\pi}{7}$

(37) **(B).** Given : $\sec^{-1}\left(\frac{a+b}{a-b}\right) = 2\sin^{-1}x$

Let $\sec^{-1}\left(\frac{a+b}{a-b}\right) = \theta = \cos^{-1}\left(\frac{a-b}{a+b}\right)$

Then $x = \sin\frac{\theta}{2}$, $\cos\theta = \frac{a-b}{a+b}$

$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{b}{a+b}$; $\sin\frac{\theta}{2} = \sqrt{\frac{b}{a+b}}$

(38) **(C).** $\tan^{-1}x + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$; Clearly, $x = \frac{3-1}{3+1} = \frac{1}{2}$

(39) **(D).** $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

Clearly equation does not satisfy for $x = 0$

Hence (A), (B), (C) are ruled out $\therefore\; \frac{1}{\sqrt{2}} \le x \le 1$

(40) **(D).** Multiply by $x^3 + x$

$(x+1)^2 = A(x^2+1) + (Bx + C)x$

Compare coefficient $\therefore$ $A = 1, B = 0, C = 2$

$\sin^{-1}A + \tan^{-1}0 + \sec^{-1}2 = 5\pi/6$

(41) **(A).** Take $x = 1, y = 1$; LHS $= \tan^{-1}(1/1) - \tan^{-1}(0) = \pi/4$

(42) **(D).** Let $\sin^{-1}0.8 = \theta \Rightarrow \sin\theta = 0.8$

$\therefore\; \cos\theta = \sqrt{1-\sin^2\theta} = 0.6$

Given exp $= \sin 2\theta = 2\sin\theta\cos\theta = 1.6 \times 0.6 = 0.96$

EXERCISE-2

(1) **(A).** We know that, $\sin^{-1}y + \cos^{-1}y = \frac{\pi}{2}, |y| \le 1$

$\therefore$ According to question

$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots = x^2 - \frac{x^4}{2} + \frac{x^6}{4}\ldots\ldots$

$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x}{1+\frac{x^2}{2}}$] $(\because 0 < |x| < \sqrt{2})$

$\Rightarrow \frac{x}{2+x} = \frac{x^2}{2+x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3 \Rightarrow x = x^2$

But $x \neq 0$ hence $x = 1$

(2) **(C).** $T_n = \tan^{-1}\left(\frac{2}{n^2}\right) = \tan^{-1}\left(\frac{2}{(1+(n^2-1)}\right)$

$= \tan^{-1}\left(\frac{2}{1+(n-1)(n+1)}\right) = \tan^{-1}\left(\frac{(n+1)-(n-1)}{1+(n-1)(n+1)}\right)$

$= \tan^{-1}(n+1) - \tan^{-1}(n-1)$

$T_1 = \tan^{-1}(2) - \tan^{-1}(0)$

$T_2 = \tan^{-1}(3) - \tan^{-1}(1)$

$T_3 = \tan^{-1}(4) - \tan^{-1}(2)$

$\vdots \quad \vdots \qquad \vdots$

$T_{n-1} = \tan^{-1}(n) - \tan^{-1}(n-2)$

$T_n = \tan^{-1}(n+1) - \tan^{-1}(n-1)$

$S = \pi - \tan^{-1}(1) = \frac{3\pi}{4}$

(3) **(C).** We have $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$= (\sin^{-1}x + \cos^{-1}x)^3$

$-3\sin^{-1}x\cos^{-1}x(\sin^{-1}x + \cos^{-1}x)$

$= \frac{\pi^3}{8} - 3(\sin^{-1}x\cos^{-1}x)\frac{\pi}{2}$

$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1}x + \frac{3\pi}{2}(\sin^{-1}x)^2$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x\right]$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$

$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$

$\therefore$ The least value is $\frac{\pi^3}{32}$

and since $\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \left(\frac{3\pi}{4}\right)^2$

$\therefore$ The greatest value is $\frac{\pi^3}{32} + \frac{9\pi^2}{16} \times \frac{3\pi}{2} = \frac{7\pi^3}{8}$.

(4) **(C).** $\cos^{-1}x + \cos^{-1}(2x) = -\pi$

$\Rightarrow \cos^{-1}2x = -\pi - \cos^{-1}x$

$\Rightarrow 2x = \cos(\pi + \cos^{-1}x)$

$\Rightarrow 2x = \cos\pi(\cos\cos^{-1}x) - \sin\pi\sin(\cos^{-1}x)$

$2x = -x \Rightarrow x = 0$

But x = 0 does not satisfy the given equation.

No solution will exist.

(5) **(A).** Let $\cos^{-1}\frac{1}{8} = \theta$, where $0 < \theta < \frac{\pi}{2}$. Then

$\Rightarrow \frac{1}{2}\cos^{-1}\frac{1}{8} = \frac{1}{2}\theta \Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\frac{1}{2}\theta$

Now, $\cos^{-1}\frac{1}{8} = \theta$

$\Rightarrow \cos\theta = \frac{1}{8} \Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8} \Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16}$

$\Rightarrow \cos\frac{\theta}{2} = \frac{3}{4}$ $\quad [\because 0 < \frac{\theta}{2} < \frac{\pi}{4}$, so $\cos\frac{\theta}{2} \neq -\frac{3}{4}]$

(6) **(C).** The given equation can be written as

$\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$\Rightarrow \tan^{-1}\frac{x-1+x+1}{1-(x-1)(x+1)} = \tan^{-1}\frac{3x-x}{1+3x^2}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2} \Rightarrow x + 3x^3 = 2x - x^3$

$\Rightarrow 4x^3 - x = 0 \Rightarrow x(4x^2 - 1) = 0$

$\Rightarrow x = 0, x = \pm 1/2$

(7) **(A).** $T_n = \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$

$= \tan^{-1}(n+1) - \tan^{-1}n$

$\Rightarrow T_1 = \tan^{-1}(1/3) = \tan^{-1}2 - \tan^{-1}1$

$T_2 = \tan^{-1}(1/7) = \tan^{-1}3 - \tan^{-1}2$

....

$T_n = \tan^{-1}(n+1) - \tan^{-1}n$

On adding

$T_1 + T_2 + T_3 + ... + T_n = \tan^{-1}(n+1) - \tan^{-1}1$

$= \tan^{-1}\left(\frac{n}{n+2}\right)$

$\therefore \lim_{n\to\infty}(T_1 + T_2 + T_3 + ... + T_n) = \lim_{n\to\infty}\tan^{-1}\left(\frac{1}{1+2/n}\right)$

$= \tan^{-1}1 = \frac{\pi}{4}$

(8) **(D).** $\tan^{-1}(1+x) + \tan^{-1}(x-1) = \tan^{-1}3 - \tan^{-1}x$

$\tan^{-1}\left(\frac{1+x+x-1}{1-x^2+1}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$, when $x^2 - 1 < 1$ and $3x < 1$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{3-x}{1+3x},$$

when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow 2x(1+3x) = (3-x)(2-x^2)$. when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow 2x + 6x^2 = 6 - 2x - 3x^2 + x^3 \Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

$\Rightarrow (x+1)(x^2 - 10x + 6) = 0$, when $-\sqrt{2} < x < \frac{1}{3}$

$\Rightarrow x = -1$ and neglecting $x^2 - 10x + 6 = 0$ as its root does not $\in \left(-\sqrt{2}, \frac{1}{3}\right)$ $\quad \therefore \quad x = -1$

(9) $\tan\left(-\sin^{-1}\left(\frac{4}{5}\right) - \pi + \cos^{-1}\left(\frac{5}{13}\right)\right)$

$$= -\tan\left(\pi + \sin^{-1}\frac{4}{5} - \cos^{-1}\frac{5}{13}\right) = -\tan(\alpha - \beta)$$

where $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{5}{13}$

$$= -\left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right) = \frac{\frac{4}{3} - \frac{12}{5}}{1 + \frac{4}{3}\cdot\frac{12}{5}} = -\left(\frac{20-36}{63}\right) = \frac{16}{63}$$

(10) **(D).**

$$\frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3) + B(x-1)}{(x-1)(x+3)}$$

$\therefore\ 3x - 1 = A(x+3) + B(x-1) \Rightarrow A = 1, B = 2$

$\sin^{-1}[A/B] = \sin^{-1}[1/2] = \pi/6$

(11) **(D).** $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

Comparing with

$$\cos^{-1}\sqrt{x^2+x+1} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\cos^{-1}\sqrt{x^2+x+1} = \tan^{-1}\sqrt{x(x+1)}$$

$$\Rightarrow \cos^{-1}\sqrt{x^2+x+1} = \cos^{-1}\frac{1}{\sqrt{x^2+x+1}}$$

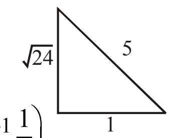
which is possible only if $\sqrt{x^2+x+1} = 1$

$\Rightarrow x^2 + x + 1 = 1 \Rightarrow x(x+1) = 0 \Rightarrow x = 0, x = -1$

$\therefore$ No. of solutions = 2

(12) **(B).** $\cos\left[\cos^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{1}{5}\right)\right]$

$$= \cos\left[\cos^{-1}\left(\frac{1}{5}\right) + \frac{\pi}{2}\right]$$



$$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\frac{1}{5}\right)$$

$$= -\sin\left(\sin^{-1}\frac{\sqrt{24}}{5}\right) = -\frac{2\sqrt{6}}{5}$$

(13) **(A).** $0 \le x \le 1/2$

$$\tan\left[\sin^{-1}\left\{\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right\} - \sin^{-1}x\right]$$

$$= \tan\left[\sin\left\{\frac{x + \sqrt{1-x^2}}{\sqrt{2}}\right\} - \sin^{-1}x\right]$$

Put $\sin^{-1}x = \theta$ or $x = \sin\theta$

Given $= \tan\left[\sin^{-1}\left\{\frac{\sin\theta + \cos\theta}{\sqrt{2}}\right\} - \theta\right]$

$$= \tan\left[\sin^{-1}\left[\sin\left(\theta + \frac{\pi}{4}\right) - \theta\right]\right]$$

$$= \tan\left[\theta + \frac{\pi}{4} - \theta\right] = \tan\frac{\pi}{4} = 1$$

(14) **(B).** Since x, y, z are in A.P.

$\Rightarrow 2y = x + z$ ..... (i)

Also, $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P.

$\Rightarrow 2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$$\tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xz} \Rightarrow y^2 = zx \Rightarrow$ x, y, z are in G.P. ..... (ii)

From (i) and (ii) $\Rightarrow x = y = z$

(15) **(C).** $\tan^{-1}\left[\frac{2\sin 1\cos 1 - \cos^2 1 - \sin^2 1}{\cos^2 1 - \sin^2 1}\right]$

$$= -\tan^{-1}\left(\frac{\cos 1 - \sin 1}{\cos 1 + \sin 1}\right) = -\tan^{-1}\left[\tan\left(\frac{\pi}{4} - 1\right)\right]$$

$$= 1 - \frac{\pi}{4}$$

(16) **(A).** $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1 - \cot^4 A}\right) + \pi$$

$(0<A<\frac{\pi}{4} \Rightarrow \cot A>1)$

$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\frac{\cot A(1+\cot^2 A)}{(1-\cot^2 A)(1+\cot^2 A)}$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right) = \pi$

$\Rightarrow 4\tan^{-1}(1)$

(17) **(A).** $\tan^{-1}\left(\frac{x}{3}\right) + \tan^{-1}\left(\frac{x}{2}\right) = \tan^{-1} x$

or $\tan^{-1}\left(\frac{x/3 + x/2}{1-x^2/6}\right) = \tan^{-1}x$

where $x>0$ & $x^2/6<1 \Rightarrow x^2<6 \Rightarrow -\sqrt{6} < x < \sqrt{6}$

now, $\left(\frac{5x}{6-x^2}\right) = x \Rightarrow x\left[\frac{5}{6-x^2} - 1\right] = 0$

$\Rightarrow x=0$ or $x^2-1=0 \Rightarrow x=\pm 1$

$\therefore x=\{-1,0,1\} \Rightarrow$ 3 solution

(18) **(A).** $2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{2x\sqrt{1-x^2}}\right)$

put $x=\cos\theta$; LHS $=2\theta$; $0\le\theta\le\pi$ and $-1\le x\le 1$ ....(1)

R.H.S. $= \cot^{-1}\left(\frac{\cos 2\theta}{2\cos\theta\,|\sin\theta|}\right) = \cot^{-1}(\cot 2\theta) = 2\theta$

if $0<2\theta<\pi$ ....(2)

or $0<\theta<\pi/2$

from (1) and (2) $0<\theta<\pi/2$ $\therefore x\in(0,1)$

(19) **(C).** $\cot^{-1}x - \cot^{-1}(x+2) = \frac{\pi}{12}$

or $\frac{\pi}{2} - \tan^{-1}(x) - \frac{\pi}{2} - \tan^{-1}(x+2) = \frac{\pi}{12}$

$\tan^{-1}(x+2) - \tan^{-1}(x) = \frac{\pi}{12}$

$\tan^{-1}\frac{(x+2)-x}{1+x(x+2)} = \frac{\pi}{12} \Rightarrow \left(2-\sqrt{3}\right) = \frac{2}{x^2+2x+1}$

$\Rightarrow x^2+2x+1 = 2\left(2+\sqrt{3}\right) \Rightarrow x^2+2x-\left(3+2\sqrt{3}\right) = 0$

(20) **(B).** $\sin\cos^{-1}\left(\cos(\tan^{-1}x)\right) = p$

for $x\in R$, $\tan^{-1}x \in (-\pi/2, \pi/2)$

$\cos(\tan^{-1}x) \in (0, 1]$

$\cos^{-1}\cos(\tan^{-1}x) \in [0, \pi/2)$

$\sin(\cos^{-1}(\cos(\tan^{-1}x))) \in [0, 1)$

(21) **(B).** $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a + \tan^{-1}a^3)$

$2x = \frac{2a}{1-a^2} + \frac{2(a+a^3)}{1-a^4}$

$\therefore a\ne\pm 1 \Rightarrow D$ (Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$)

$x = \frac{a}{1-a^2} + \frac{a}{1-a^2} = \frac{2a}{1-a^2}$

$\Rightarrow x(1-a^2) = 2a \Rightarrow a^2x + 2a = x \Rightarrow (1)$

Hence (2) & (3) are not possible.

(22) **(C).** $\sin^{-1}(x-1) \Rightarrow -1\le x-1\le 1 \Rightarrow 0\le x\le 2$

$\cos^{-1}(x-3) \Rightarrow -1\le x-3\le 1 \Rightarrow 2\le x\le 4$

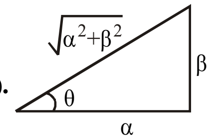
$\tan^{-1}\left(\frac{x}{2-x^2}\right) \Rightarrow x\in R,\ x\ne\sqrt{2}, -\sqrt{2}$

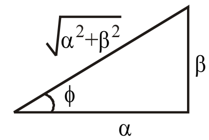
$\sin^{-1}(2-1) + \cos^{-1}(2-3) + \tan^{-1}\frac{2}{2-4} = \cos^{-1}k + \pi$

$\Rightarrow \sin^{-1}1 + \cos^{-1}(-1) + \tan^{-1}(-1) = \cos^{-1}k + \pi$

$\frac{\pi}{2} + \pi - \frac{\pi}{4} = \cos^{-1}k + \pi \Rightarrow \cos^{-1}k = \frac{\pi}{4} \Rightarrow k = \frac{1}{\sqrt{2}}$

(23) **(A).**





Let $\tan^{-1}\left(\frac{\beta}{\alpha}\right) = \theta$ and $\tan^{-1}\left(\frac{\alpha}{\beta}\right) = \phi$

$f(\alpha,\beta) = \frac{\beta^3}{2\sin^2\frac{\theta}{2}} + \frac{\alpha^3}{2\cos^2\frac{\theta}{2}} = \frac{\beta^3}{1-\cos\theta} + \frac{\alpha^3}{1+\cos\theta}$

$= \frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}} + \frac{\alpha^3}{1+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$

$= \sqrt{\alpha^2+\beta^2}\left[\beta\sqrt{\alpha^2+\beta^2} + \alpha\sqrt{\alpha^2+\beta^2}\right]$

$f(\alpha,\beta) = (\alpha^2+\beta^2)(\alpha+\beta)$

Now, $\alpha+\beta=4$ and $\alpha\beta=1$

$f(\alpha,\beta) = ((\alpha+\beta)^2 - 2\alpha\beta))(\alpha+\beta) = (16-2)(4) = 56$

(24) **(D).**

(A) $\sin\left(\tan^{-1}3 + \tan^{-1}\frac{1}{3}\right) = \sin\frac{\pi}{2} = 1$

(B) $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right) = \cos\left(\cos^{-1}\frac{3}{4}\right) = \frac{3}{4}$

(C) $\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\frac{\sqrt{63}}{8} = \theta$ So, $\sin\theta = \frac{\sqrt{63}}{8}$ if $\cos\theta = 1/8$

We have $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \frac{3}{4}$

$\sin\frac{\theta}{4} = \sqrt{\frac{1-\cos\theta/2}{2}} = \frac{1}{2\sqrt{2}}$

Now, $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right) = \log_2\frac{1}{2\sqrt{2}} = -\frac{3}{2}$

(D) $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$; $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore \tan\frac{\theta}{2} = \frac{3-\sqrt{5}}{2}$ which is irrational.

(25) **(B).** For existance of $\sec^{-1}(\sqrt{2x^2 - x^4})$

or $\operatorname{cosec}^{-1}(\sqrt{2x^2 - x^4})$

$2x^2 - x^4 \geq 1$ i.e. $(x^2-1)^2 \leq 0$. Hence $x^2 = 1 \Rightarrow x = \pm 1$

if $x = 1$, $a + \frac{\pi}{2} = 0$;

if $x = -1$, $-a + \frac{\pi}{2} = 0 \Rightarrow a = \pm\frac{\pi}{2}$

(26) **(C).** $\sin^{-1}\sin 5 = \sin^{-1}\sin(5-2\pi) = 5 - 2\pi$

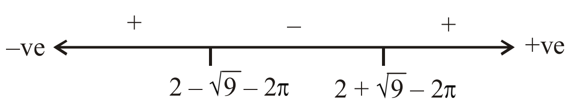
$\left(\text{As } -\frac{\pi}{2} \leq 5 - 2\pi \leq \frac{\pi}{2}\right)$

$\therefore \sin^{-1}\sin 5 > x^2 - 4x$

$5 - 2\pi > x^2 - 4x$

$x^2 - 4x + 2\pi - 5 < 0$

sign sum of $(x^2 - 4x + 2\pi - 5)$



$2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi}$

Integral values of x are 1, 2, 3

Number of integral value of x = 3

(27) **(B).**

$\sum\tan^{-1}\left(\frac{2}{1+4k^2-1}\right) = \sum\tan^{-1}\left(\frac{2}{1+(2k-1)(2k+1)}\right)$

$= \sum\tan^{-1}\left(\frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)}\right)$

$= \sum_{k=1}^{100}\tan^{-1}(2k+1) - \tan^{-1}(2k-1)$

$= (\tan^{-1}(3) - \tan^{-1}(1)) + (\tan^{-1}(5) - \tan^{-1}(3))$
$+ \ldots (\tan^{-1}(201) - \tan^{-1}(199))$

$= \tan^{-1}(201) - \tan^{-1}(1)$

$= \tan^{-1}\left(\frac{201-1}{1+201}\right) = \tan^{-1}\left(\frac{200}{202}\right) = \tan^{-1}\left(\frac{100}{101}\right)$

$2p - q = 200 - 101 = 99$

(28) **(B).**

(A) $\sin^{-1}x - \frac{\pi}{2} + \sin^{-1}(-x) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(x) + \sin^{-1}(-x) = \pi$

$0 = \pi$ which is not possible.

$\therefore$ no solution.

(B) $\sin^{-1}(x^2+4x+3) + \cos^{-1}(x^2+6x+8) = \pi/2$

$= \sin^{-1}(x^2+4x+3) + \cos^{-1}(x^2+4x+3)$

$\Rightarrow x^2+6x+8 = x^2+4x+3$

$\Rightarrow 2x = -5 \Rightarrow x = -\frac{5}{2}$

$\because x^2+4x+3 = (x+2)^2 - 1 \in [-1, 1]$ at $x = -5/2$

and $x^2+6x+8 = (x+3)^2 - 1 \in [-1, 1]$ at $x = -5/2$

$\therefore x = -5/2$

(C) $\because -1 \leq \cos(\sin^{-1}x) \leq 1$ and $-1 \leq \sin(\cos^{-1}x) \leq 1$

$\sin^{-1}\{\cos(\sin^{-1}x)\} + \cos^{-1}(\sin(\cos^{-1}x)\} = \pi/2$

(D) $2\left[\tan^{-1}\frac{1+2}{1-2} + \pi + \tan^{-1}3\right]$

$= 2[\pi - \tan^{-1}3 + \tan^{-1}3] = 2\pi$

(29) **(A)** $\frac{\pi}{2} - \cos^{-1}(1-x) + 2\cos^{-1}x = 0]$

$\sin^{-1}(1-x) + 2\cos^{-1}x = 0$, domain is $[0, 1]$

now, in $[0, 1]$, $\sin^{-1}(1-x) \in [0, \pi/2]$

and $2\cos^{-1}x \in [0, \pi]$

Hence $\sin^{-1}(1-x) = 0$ and $\cos^{-1}x = 0$ $] \Rightarrow x = 1$

(30) **(D).** $\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\pi/4$

$\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}} = 1$ or $\frac{x}{10} + \frac{1}{x+1} = 1 - \frac{x}{10}\cdot\frac{1}{x+1}$

$x(x+1) + 10 = 10(x+1) - x$

$x^2 + x + 10 = 10x + 10 - x$

$x^2 - 8x = 0$; $x = 0, 8$

(31) **(A).** Statement 1 : Domain of $\sin^{-1}x \to [-1, 1]$

$\cos^{-1}x \to [-1, 1]$

$\tan^{-1}x \to R$

Domain $= [-1, 1] \wedge [-1, 1] \wedge R = [-1, 1]$

(32) **(A).** $\text{cosec}^{-1} x > \sec^{-1} x$

$\text{cosec}^{-1} x > \pi/2 - \text{cosec}^{-1} x$; $\text{cosec}^{-1} x > \pi/4$

$1 \le x < \sqrt{2}$ and $\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right) \in [1, \sqrt{2})$

Statement 2 is true and explains statement 1.

(33) **(A).** $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x,\ x > 1$

$f'(x) = -\frac{2}{1+x^2} \Rightarrow f'(2) = -\frac{2}{5}$

Statement is true, statement-2 is true, statement-2 is correct explanation for statement-1

(34) **(A).** $\sin^{-1} x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1} x > \tan^{-1} y$

$\therefore$ statement-2 is true, $e < \pi$; $\frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

by statement-2.

$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

statement-1 is true

(35) (B), (36) (A), (37) (C).

(i) $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right)$

$= \sin^{-1}\left(\frac{2.\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1}\right) - 2\tan^{-1}\frac{x}{2} = 2\tan^{-1}\frac{x}{2} - 2\tan^{-1}\frac{x}{2} = 0$

$\left|\frac{x}{2}\right| \le 1$; $|x| \le 2 \Rightarrow -2 \le x \le 2$

(ii) $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1} 3x$

$\Rightarrow \frac{\pi}{2} - \sin^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1} 3x$

$\Rightarrow \sin^{-1}\frac{6x}{1+9x^2} = \pi - 2\tan^{-1} 3x$

$\Rightarrow \sin^{-1}\frac{2.3x}{1+(3x)^2} = \pi - 2\tan^{-1} 3x$

Above is true when $3x > 1 \Rightarrow x > \frac{1}{3}$; $x \in \left(\frac{1}{3}, \infty\right)$

(iii) $(x-1)(x^2+1) > 0 \Rightarrow x > 1$

$\therefore \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \tan^{-1} x\right]$

$= \sin\left[\frac{1}{2}(-\pi + 2\tan^{-1} x) - \tan^{-1} x\right] = \sin\left(-\frac{\pi}{2}\right) = -1$

(38) **(B).** $A = (\tan^{-1} x)^3 + (\cot^{-1} x)^3$

$A = (\tan^{-1} x + \cot^{-1} x)^3 - 3\tan^{-1}x\cot^{-1}x\,(\tan^{-1} x + \cot^{-1} x)$

$\Rightarrow A = \left(\frac{\pi}{2}\right)^3 - 3\tan^{-1} x \cot^{-1} x \frac{\pi}{2}$

$\Rightarrow A = \frac{\pi^3}{8} - \frac{3\pi}{2}\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$

$\Rightarrow A = \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\tan^{-1} x - \frac{\pi}{2}\right)^2$ as $x > 0$; $\frac{\pi^3}{32} \le A < \frac{\pi^3}{8}$

(39) **(C).** $B = (\cos^{-1} t)^2 + (\sin^{-1} t)^2$

$B = (\sin^{-1} t + \cos^{-1} t)^2 - 2\sin^{-1} t + \cos^{-1} t$

$B = \frac{\pi^3}{4} - 2\sin^{-1} t\left(\frac{\pi}{2} - \sin^{-1} t\right)$; $B = \frac{\pi^2}{8} + 2\left(\sin^{-1} t - \frac{\pi}{4}\right)^2$

$B_{max} = \frac{\pi^2}{8} + 2\frac{\pi^2}{16} = \frac{\pi^2}{4}$

(40) **(A).** $\lambda = \frac{\pi^3}{32}, \mu = \frac{\pi^2}{4}$; $\frac{\lambda}{\mu} = \frac{\pi}{8}$; $\frac{\lambda - \mu\pi}{\mu} = \frac{\pi}{8} - \pi = \frac{-7\pi}{8}$

$\cot^{-1}\cot\left(\frac{\lambda - \mu\pi}{\mu}\right) = \cot^{-1}\cot\left(-\frac{7\pi}{8}\right) = \frac{\pi}{8}$

(41) **1.** $\tan\left(\tan^{-1}(5) - \tan^{-1}\left(\frac{2}{3}\right)\right)$

$\tan^{-1}(5) - \tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left(\frac{5-(2/3)}{1+(10/3)}\right)$

$= \tan^{-1}\left(\frac{13}{13}\right) = \frac{\pi}{4}$ $\therefore \tan\frac{\pi}{4} = 1$

(42) **1.** $\frac{\pi}{2} - \cos^{-1}(1-x) + 2\cos^{-1}x = 0]$

$\sin^{-1}(1-x) + 2\cos^{-1}x = 0$

domain is [0, 1]

now, in [0, 1],

$\sin^{-1}(1-x) \in \left[0, \frac{\pi}{2}\right]$

and $2\cos^{-1}x \in [0, \pi]$

Hence $\sin^{-1}(1-x) = 0$ $\Rightarrow x = 1$

and $\cos^{-1}x = 0$

(43) **1.** $\sin^{-1}\left(\frac{5}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{12}{x}\right) = \cos^{-1}\left(\frac{12}{x}\right)$

(very obviously $x > 0$)

$\alpha = \beta$
$\tan\alpha = \tan\beta$

$$\frac{5}{\sqrt{x^2-25}} = \frac{\sqrt{x^2-144}}{12}$$

solving $x = 13$ or $x = -13$ (rejected)

(44) **10.** n^{th} term of 3, 7, 13, 21, 31 ...... is n^2+n+1

$$T_n = \tan^{-1}\frac{1}{1+n(n+1)} = \tan^{-1}\frac{(n+1)-n}{1+n(n+1)}$$

$$= \tan^{-1}(n+1) - \tan^{-1} n$$

$S = T_1 + T_2 + T_3 + T_4 + T_5$
$= (\tan^{-1}2 - \tan^{-1}1)$
$(\tan^{-1}3 - \tan^{-1}2)$
$(\tan^{-1}4 - \tan^{-1}3)$
$(\tan^{-1}5 - \tan^{-1}4)$
$(\tan^{-1}6 - \tan^{-1}5)$

$S = \tan^{-1}6 - \tan^{-1}1$

$$= \tan^{-1}\left(\frac{6-1}{1+6}\right) = \tan^{-1}\left(\frac{5}{7}\right)$$

$$\therefore \tan\left(\tan^{-1}\frac{5}{7}\right) = \frac{5}{7} \quad \therefore \text{int} = 14\cdot\frac{5}{7} = 10$$

(45) **8.** $\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\frac{\pi}{4}$

$$\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}} = 1 \quad \text{or} \quad \frac{x}{10}+\frac{1}{x+1} = 1-\frac{x}{10}\cdot\frac{1}{x+1}$$

$x(x+1)+10 = 10(x+1)-x$
$x^2+x+10 = 10x+10-x$; $x^2-8x=0$; $x = 0, 8$

(46) **0.** $f(x) = \cos^{-1}(\cos 2\theta) + \tan^{-1}(\tan 2\theta)$
where $x = \tan\theta$
and $\theta \in (-\pi/4, 0) = \cos^{-1}(\cos(-2\theta)) + \tan^{-1}(\tan 2\theta)$
$= -2\theta + 2\theta = 0$

(47) **53.** $\sin^{-1}(\sin 8) = \sin^{-1}(\sin(3\pi-8)) = 3\pi - 8$

$\tan^{-1}(\tan 10) = \tan^{-1}(\tan(10-3\pi)) = 10-3\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi-12)) = 4\pi-12$

$\sec^{-1}(\sec 9) = \sec^{-1}(\sec(9-2\pi)) = 9-2\pi$

$\cot^{-1}(\cot 6) = \cot^{-1}(\cot(6-\pi)) = 6-\pi$

$\text{cosec}^{-1}(\text{cosec } 7) = \text{cosec}^{-1}(\text{cosec}(7-2\pi)) = 7-2\pi$

$y = (3\pi-8)+(3\pi-10)+(4\pi-12)+(2\pi-9)$
$+(-\pi+6)+(2\pi-7)$

$y = 13\pi - 40$
$\Rightarrow a = 13$ and $b = -40 \Rightarrow a-b = 13-(-40) = 53$

(48) **3.** $\sin^{-1}\sin 5 = \sin^{-1}\sin(5-2\pi) = 5-2\pi$

$$\left(\text{As } -\frac{\pi}{2} \le 5-2\pi \le \frac{\pi}{2}\right)$$

$\therefore \sin^{-1}\sin 5 > x^2 - 4x$
$5-2\pi > x^2-4x$
$x^2-4x+2\pi-5 < 0$
sign sum of $(x^2-4x+2\pi-5)$

–ve ← + | $2-\sqrt{9}-2\pi$ | – | $2+\sqrt{9}-2\pi$ | + → +ve

$2-\sqrt{9-2\pi} < x < 2+\sqrt{9-2\pi}$
Integral values of x are 1, 2, 3
Number of integral value of $x = 3$

(49) **2.** $\because 4\{x\} = x+[x]$ .... (1)

$$= x + x - \{x\} \Rightarrow 5\{x\} = 2x \Rightarrow \{x\} = \frac{2x}{5}$$

$$\Rightarrow 0 \le \frac{2x}{5} < 1 \Rightarrow 0 \le x < \frac{5}{2}. \text{ Hence, } [x] = 0, 1, 2$$

Again from (1) $4x - 4[x] = x + [x]$
$3x = 5[x]$
Case-I: If $x \in [0, 1) \Rightarrow [x] = 0$
$\therefore 3x = 0 \Rightarrow x = 0$
Case-II: If $x \in [1, 2) \Rightarrow [x] = 1$
$\therefore 3x = 5 \Rightarrow x = 5/3$
Case-III : If $x \in (2, 5/2] \in [x] = 2$
$\therefore 3x = 10 \Rightarrow x = 10/3$ (reject)
$\therefore$ Number of solutions = 2.

(50) **2.** $\tan^{-1}\sqrt{[x(x+1)} = \frac{\pi}{2} - \sin^{-1}\sqrt{(x^2+x+1)}$

$$= \cos^{-1}\sqrt{x^2+x+1} = \tan^{-1}\frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}}$$

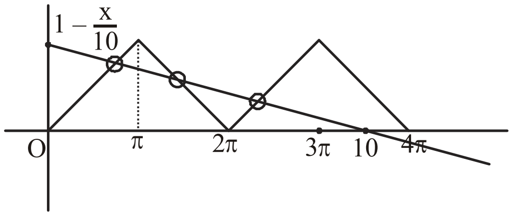
$$\Rightarrow \sqrt{x(x+1)} = \frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}}$$

$\Rightarrow x = 0, -1$ are the only real solutions.

(51) **1.** $\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \therefore f(\theta) = \tan\theta$

$$\therefore \frac{df}{d\tan\theta} = 1$$

(52) **3.** $f:[0,4\pi]\to[0,\pi],\ f(x)=\cos^{-1}(\cos x)$.



$f(x)=1-\dfrac{x}{10}$

(53) **1.** $\tan^{-1}\left(\dfrac{\dfrac{1}{2x+1}+\dfrac{1}{4x+1}}{1-\dfrac{1}{(2x+1)(4x+1)}}\right)=\tan^{-1}\left(\dfrac{2}{x^2}\right)$

$\Rightarrow \dfrac{6x+2}{8x^2+6x}=\dfrac{2}{x^2}\ \Rightarrow \dfrac{3x+1}{4x+3}=\dfrac{2}{x}$ (where $x\neq 0$)

$\Rightarrow 3x^2-7x-6=0$

$\Rightarrow x=3,-2/3$

But $2x+1>0$ and $4x+1>0$

So, solution are $x=3$.

(54) **0.** Here, $\sin^{-1}(1-x)-2\sin^{-1}x=\pi/2$

Let $x=\sin\theta$, then $\sin^{-1}(1-\sin\theta)-2\theta=\pi/2 \Rightarrow -2\theta=\dfrac{\pi}{2}$ $-\sin^{-1}(1-\sin\theta)$ $\quad(\sin^{-1}x+\cos^{-1}x=\pi/2)$

$\Rightarrow -2\theta=\cos^{-1}(1-\sin\theta)\Rightarrow \cos(-2\theta)=1-\sin\theta\Rightarrow \cos 2\theta=1-\sin\theta$

$\Rightarrow 1-2\sin^2\theta=1-\sin\theta \Rightarrow 2\sin^2\theta=\sin\theta\Rightarrow 2\sin^2\theta-\sin\theta=0\Rightarrow \sin\theta(2\sin\theta-1)=0$

Either $\sin\theta=0$ or $2\sin\theta-1=0\ \therefore\ x=0$ or $2x-1=0$

$\Rightarrow x=1/2$. But $x=1/2$ does not satisfy the equation.

EXERCISE-3

(1) **(B).** $\cos^{-1}(-1)-\sin^{-1}(1)=\pi-\dfrac{\pi}{2}$

$\{\because\ \sin^{-1}x\in[-\pi/2,\ \pi/2]$ and $\cos^{-1}x\in[0,\pi]\}$

(2) **(A).** $\sin^{-1}x=2\sin^{-1}a$

$\because -\pi/2\le\sin^{-1}x\le\pi/2,$

$\therefore -\pi/2\le 2\sin^{-1}a\le\pi/2$

$\Rightarrow -\pi/4\le\sin^{-1}a\le\pi/4\Rightarrow \sin(-\pi/4)\le a\le\sin\pi/4$

$\Rightarrow \dfrac{-1}{\sqrt2}\le a\le\dfrac{1}{\sqrt2}\ \Rightarrow |a|\le\dfrac{1}{\sqrt2}$

(3) **(C).** $\cos^{-1}x-\cos^{-1}(y/2)=\alpha$

$\Rightarrow \cos^{-1}\left(\dfrac{xy}{2}-\sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}}\right)=\alpha$

$\Rightarrow \dfrac{xy}{2}-\sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}}=\cos\alpha$

$\Rightarrow \dfrac{xy}{2}-\cos\alpha=\sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}}$

Squaring both side

$\Rightarrow \left(\dfrac{xy}{2}-\cos\alpha\right)^2=\left(\sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}}\right)^2$

$\Rightarrow \dfrac{x^2y^2}{4}+\cos^2\alpha-\dfrac{2xy}{2}\cos\alpha=(1-x^2)\left(1-\dfrac{y^2}{4}\right)$

$\Rightarrow \dfrac{x^2y^2}{4}+\cos^2\alpha-xy\cos\alpha=1-\dfrac{y^2}{4}-x^2+\dfrac{x^2y^2}{4}$

$\Rightarrow x^2+\dfrac{y^2}{4}-xy\cos\alpha+\cos^2\alpha-1=0$

$\Rightarrow 4x^2+y^2-4xy\cos\alpha+4\cos^2\alpha-4=0$

$\Rightarrow 4x^2+y^2-4xy\cos\alpha=4-4\cos^2\alpha=[1-\cos^2\alpha]=4\sin^2\alpha$

(4) **(B).** $\sin^{-1}(x/5)+\operatorname{cosec}^{-1}(5/4)=\pi/2$

$\Rightarrow \sin^{-1}(x/5)+\sin^{-1}(4/5)=\pi/2$

$\Rightarrow \sin^{-1}(x/5)+\cos^{-1}(3/5)=\pi/2$

$\Rightarrow x/5=3/5\Rightarrow x=3$

$\{\because\ \operatorname{cosec}^{-1}x=\sin^{-1}(1/x)=\cos^{-1}\sqrt{1-\dfrac{1}{x^2}}$

and $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}\}$

(5) **(D).** $\cot(\operatorname{cosec}^{-1}(5/3)+\tan^{-1}2/3)$

$=\cot(\cot^{-1}(4/3)+\cot^{-1}3/2)$

$=\dfrac{\cot\cot^{-1}(4/3)\times\cot\cot^{-1}(3/2)-1}{\cot\cot^{-1}(4/3)+\cot\cot^{-1}(3/2)}$

$=\dfrac{(4/3)\times(3/2)-1}{(4/3)+(3/2)}=\dfrac{2-1}{\dfrac{8+9}{6}}=\dfrac{6}{17}$

$\{\because \operatorname{cosec}^{-1}x=\sin^{-1}(1/x)$

$=\cos^{-1}\sqrt{1-\dfrac{1}{x^2}}=\cot^{-1}\left(\dfrac{\sqrt{1-\dfrac{1}{x^2}}}{\dfrac{1}{x}}\right)=\cot^{-1}\sqrt{x^2-1}\ \}$

and $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

(6) **(D).** $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)\ ;\ \ y=\dfrac{3x-x^3}{1-3x^2}$

(7) **(A).** $f(x) = \tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \dfrac{\pi}{2}\right)$

$$f'(x) = \frac{1}{1+\dfrac{1+\sin x}{1-\sin x}} \times \frac{1}{2\sqrt{\dfrac{1+\sin x}{1-\sin x}}}$$

$$\times \left\{\frac{(1-\sin x)(\cos x)-(1-\sin x)(-\cos x)}{(1-\sin x)^2}\right\}$$

At $x = \dfrac{\pi}{6}$

$$f'\left(\frac{\pi}{6}\right) = \frac{1}{1+\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}} \times \frac{1}{2\sqrt{\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}}} \times \left\{\frac{2\times\sqrt{3}/2}{\left(\frac{1}{2}\right)^2}\right\}$$

$$= \frac{1}{1+3} \times \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{1/4} = \frac{1}{4} \times \frac{1}{2\sqrt{3}} \times 4 \times \sqrt{3} = \frac{1}{2}$$

Slope of normal $= -2$

Point at $x = \dfrac{\pi}{6}$,

$$f\left(\frac{\pi}{6}\right) = \tan^{-1}\sqrt{\frac{1+\frac{1}{2}}{1-\frac{1}{2}}} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$\therefore$ Equation $y - \dfrac{\pi}{3} = (-2)\left(x - \dfrac{\pi}{6}\right)$

$\Rightarrow$ $y - \dfrac{\pi}{3} = -2x + \dfrac{\pi}{3} \Rightarrow y + 2x = \dfrac{2\pi}{3}$

(8) **(C).** $x \in \left(0, \dfrac{1}{4}\right)$; $3x^{3/2} \in \left(0, \dfrac{3}{8}\right)$; $\theta \in \left(0, \tan^{-1}\left(\dfrac{3}{8}\right)\right)$

$$y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left(\frac{2x^{3/2}}{1-(3x^{3/2})^2}\right)$$

Let $\tan^{-1}(3x^{3/2}) = \theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}(3x^{3/2})$$

$$y' = \frac{2}{1+9x^3} \times 3 \cdot \frac{3}{2} x^{1/2} = \frac{9}{1+9x^3}\sqrt{x}$$

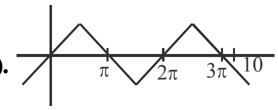
(9) **(A).** $\cos^{-1}\left(\dfrac{2}{3x}\right) + \cos^{-1}\left(\dfrac{3}{4x}\right) = \dfrac{\pi}{2}\left(x > \dfrac{3}{4}\right)$

$\cos^{-1}\left(\dfrac{3}{4x}\right) = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{2}{3x}\right)$; $\cos^{-1}\left(\dfrac{3}{4x}\right) = \sin^{-1}\left(\dfrac{2}{3x}\right)$

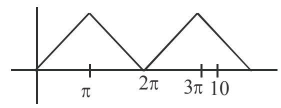
$$\cos\left(\cos^{-1}\left(\frac{3}{4x}\right)\right) = \cos\left(\sin^{-1}\left(\frac{2}{3x}\right)\right)$$

$\dfrac{3}{4x} = \dfrac{\sqrt{9x^2-4}}{3x}$; $\dfrac{81}{16} + 4 = 9x^2$

$$x^2 = \frac{145}{16\times 9} \Rightarrow x = \frac{\sqrt{145}}{12}$$

(10) **(A).** 

$x = \sin^{-1}(\sin 10) = 3\pi - 10$;



$y = \cos^{-1}(\cos 10) = 4\pi - 10$

$y - x = \pi$

(11) **(C).** $\sin^{-1}\left(\dfrac{12}{13}\right) - \sin^{-1}\left(\dfrac{3}{5}\right)$

$$\sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$$

$$= \sin^{-1}\left(\frac{33}{65}\right) = \cos^{-1}\left(\frac{56}{65}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$$

(12) **(B).** $2y = \sin^{-1} f(x) + C = \sin^{-1}(\sin(2\tan^{-1}x)) + C$

$\Rightarrow$ $2\left(\dfrac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(\dfrac{2\pi}{3}\right)\right) + C$

$\dfrac{\pi}{3} = \dfrac{\pi}{3} + C$ $\quad\therefore\ C = 0$

For $x = -\sqrt{3}$, $2y = \sin^{-1}\left(\sin\left(\dfrac{-2\pi}{6}\right)\right) + 0$

$\Rightarrow$ $2y = \dfrac{-\pi}{3} \Rightarrow y = \dfrac{-\pi}{6}$

(13) **(C).** $f'(x) = \tan^{-1}(\sec x + \tan x)$

$$f'(x) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right)$$

$\because$ $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \Rightarrow 0 < \dfrac{\pi}{4} + \dfrac{x}{2} < \dfrac{\pi}{2}$

$\Rightarrow$ $f'(x) = \dfrac{\pi}{4} + \dfrac{x}{2}$

$\therefore$ $f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4} + c$ $\quad\because f(0) = 0 \Rightarrow c = 0$

$\Rightarrow$ $f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4}$ $\quad\therefore\ f(1) = \dfrac{\pi+1}{4}$