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MATRICES AND DETERMINANTS

INTRODUCTION

Elementary matrix already has now becomes as integral part of the mathematical background necessary in field of electrical / computer engineering / chemistry.

A matrix is any rectangular array of numbers written within brackets. A matrix is usually represented by a capital letter and classified by its dimensions. The dimension of the matrices are the number of rows and columns.

A $m \times n$ matrix is usually written as

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$$

(where a_{ij} represents any number which lies i^{th} row (from top) & j^{th} column form left)

(i) The matrix is not a number. It has got no numerical value.

(ii) The determinant of matrix

$$A_{m\times m} = |A_{m\times m}| = \begin{vmatrix} a_{11}\ldots\ldots\ldots\ldots a_{1m} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ldots\ldots\ldots\ldots a_{mm} \end{vmatrix}$$

Abbreviated as :

$A = [a_{ij}]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order $m \times n$. The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, the matrix is called real matrix.

SPECIAL TYPE OF MATRICES :

(A) Row Matrix :

$A = [\,a_{11}, a_{12}, \ldots\ldots a_{1n}\,]$ having one row . $(1 \times n)$ matrix. (or row vectors)

(B) Column Matrix : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix}$

having one column. $(m \times 1)$ matrix (or column vectors)

(C) Zero or Null Matrix : $(A = O_{m \times n})$

An $m \times n$ matrix all whose entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix &

$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(D) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(E) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

Note: Every row matrix is also a Horizontal but not the converse.

|||ly every column matrix is also a vertical matrix but not the converse.

(F) Square Matrix : (Order n)

If number of rows = number of columns $\Rightarrow$ a square matrix. A real square matrix all whose elements are positive is called a positive matrices. Such matrices have application in mechanics and economics.

NOTE :

(i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements**.

e.g. in the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, a_{21} and a_{12} are conjugate elements.

(ii) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn}$ are called **Diagonal Elements** . The line along which the diagonal elements lie is called **" Principal or Leading "** diagonal.

The quantity $\Sigma\, a_{ii}$ = trace of the matrix written as, $(t_r)\,A = t_r\,(A)$

Square Matrix

- Triangular Matrix
  - Upper triangular matrix, if $a_{ij} = 0 \ \forall i > j$: $A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$;
  - Lower triangular matrix, if $a_{ij} = 0 \ \forall i < j$: $A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$
- Diagonal Matrix: atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$

  $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

  abbreviated as dia $(d_1, d_2, d_3 \,.....d_n)$
  - Scalar matrix: if $d_1 = d_2 = d_3 ..... = a \neq 0$: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
  - Unit matrix: if $d_1 = d_2 = d_3 ..... = 1$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

NOTE

(i) Minimum number of zeros in an upper or lower triangular matrix of order n

$$= 1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2}$$

(ii) Minimum number of cyphers in a diagonal/scalar/unit matrix of order n = n (n – 1)
and maximum number of cyphers = $n^2 - 1$.
"It is to be noted that with every square matrix there is a corresponding determinant formed by the elements of A in the same order." If |A| = 0 then A is called a **singular matrix** and if $|A| \neq 0$ then A is called a **non singular matrix.**

Note: If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then det. A = 0 but not conversely.

ALGEBRA OF MATRICES :

ADDITION :

$A + B = [a_{ij} + b_{ij}]$ where A & B are of the same type . (same order)
If A and B are square matrices of the same type then,
$t_r(A + B) = t_r(A) + t_r(B)$

(a) Addition of matrices is commutative :
i.e. A + B = B + A where A and B must have the same order

(b) Addition of matrices is associative :
$(A + B) + C = A + (B + C)$
Provided A , B & C have the same order.

(c) Additive inverse :
If $A + B = \mathbf{O} = B + A \quad [A = m \times n]$
and both A and B have the same order then A and B are said to be the to be the additive inverse of each other where **O** is the null matrix of the same order as that of A and B. '**O**' is the additive identity element.

$$\left.\begin{array}{l} \text{If } A + B = A + C \Rightarrow B = C \\ \text{and If } B + A = C + A \Rightarrow B = C \end{array}\right\}$$

cancellation laws hold good.

MULTIPLICATION OF A MATRIX BY A SCALAR :

$$\text{If } A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} ;\ kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$$

i.e. $k(A + B) = kA + kB$

Note:

(i) If A is a square matrix then $t_r(kA) = k[t_r(A)]$

(ii) $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ then A + A + A

$$= \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix} = 3A$$

Example 1 :

A matrix $A = [a_{ij}]$ of order 2 × 3 whose elements are such that $a_{ij} = i + j$ is –

(1) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 5 & 4 \end{bmatrix}$ (4) None of these

Sol. **(1).** a_{ij} is the element of i^{th} row and j^{th} column of matrix A

$\therefore a_{11} = 1 + 1 = 2, a_{12} = 1 + 2 = 3, a_{13} = 1 + 3 = 4$
$a_{21} = 2 + 1 = 3. a_{22} = 2 + 2 = 4, a_{23} = 2 + 3 = 5$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Example 2 :

If $A=\begin{bmatrix}1 & -3 & 2\\ 2 & k & 5\\ 4 & 2 & 1\end{bmatrix}$ is a singular matrix, then find the value of k.

Sol. A is singular $\Rightarrow |A|=0$

$$\Rightarrow \begin{vmatrix}1 & -3 & 2\\ 2 & k & 5\\ 4 & 2 & 1\end{vmatrix}=0$$

$\Rightarrow 1(k-10)+3(2-20)+2(4-4k)=0$

$\Rightarrow 7k+56=0 \Rightarrow k=-8$

MULTIPLICATION OF MATRICES

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A=[a_{ij}]_{mxn}$ and $B=[b_{ij}]_{nxp}$ then their product $AB=C=[c_{ij}]$, will be matrix of order m x p, where, $(AB)_{ij}=c_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}$

Ex. If $A=\begin{bmatrix}1 & 4 & 2\\ 2 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\ 2 & 2\\ 1 & 3\end{bmatrix}$

then $AB=\begin{bmatrix}1.1+4.2+2.1 & 1.2+4.2+2.3\\ 2.1+3.2+1.1 & 2.2+3.2+1.3\end{bmatrix}$

$AB=\begin{bmatrix}11 & 16\\ 9 & 13\end{bmatrix}$

Properties of Matrix Multiplication : If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C=A(BC)$ (Associative Law)

(iii) $IA=A=AI$

(I is identity matrix for matrix multiplication

(iv) $A(B+C)=AB+AC$ (Distributive Law)

(v) If $AB=AC$ this not implies that $B=C$

(Cancellation Law is not applicable)

(vi) If $AB=0$

It does not mean that $A=0$ or $B=0$, again product of two non-zero matrix may be zero matrix.

(vii) $tr(AB)=tr(BA)$

NOTE

(i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A+B)^2=A^2+B^2+AB+BA$

(b) $(A-B)^2=A^2+B^2-AB-BA$

(c) $(A-B)(A+B)=A^2-B^2+AB-BA$

(d) $(A+B)(A-B)=A^2-B^2-AB+BA$

(e) $A(-B)=(-A)B=-(AB)$

Positive Integral Powers of a Matrix : The positive integral powers of a matrix A are defined only when A is a square matrix. Also then

$A^2=A.A \quad A^3=A.A.A=A^2A$

Also for any positive integers m, n

(i) $A^mA^n=A^{m+n}$

(ii) $(A^m)^n=A^{mn}=(A^n)^m$

(iii) $I^n=I, I^m=I$

(iv) $A^{\circ}=I_n$ where A is a square matrices of order n.

Example 3 :

If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and $A^2-4A-nI=0$, then find the value of n.

Sol. $A^2=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$, $4A=\begin{bmatrix}8 & -4\\ -4 & 8\end{bmatrix}$, $nI=\begin{bmatrix}n & 0\\ 0 & n\end{bmatrix}$

$\Rightarrow A^2-4A-nI$

$=\begin{bmatrix}5-8-n & -4+4-0\\ -4+4-0 & 5-8-n\end{bmatrix}=\begin{bmatrix}-3-n & 0\\ 0 & -3-n\end{bmatrix}$

$\therefore A^2-4A-nI=0$

$\Rightarrow \begin{bmatrix}-3-n & 0\\ 0 & -3-n\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow -3-n=0 \Rightarrow n=-3$

Example 4 :

If $A=\begin{bmatrix}-1 & 2\\ 3 & -4\end{bmatrix}$ then find element a_{21} of A^2.

Sol. The element a_{21} is product of second row of A to the first column of A

$\therefore a_{21}=[3 \quad -4]\begin{bmatrix}-1\\ 3\end{bmatrix}=-3-12=-15$

TRANSPOSE OF A MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix A and is denoted by A^T or A'.

From the definition it is obvious that

If order of A is m x n, then order of A^T is n x m.

Ex. Transpose of Matrix

$\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}_{2\times3}$ is $\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\\ a_3 & b_3\end{bmatrix}_{3\times2}$

Properties of Transpose

(i) $(A^T)^T = A$ (ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(AB)^T = B^T A^T$ (iv) $(kA)^T = k(A)^T$

(v) $I^T = I$ (vi) $\text{tr}(A) = \text{tr}(A)^T$

(vii) $(A_1A_2A_3.....A_{n-1}A_n)^T = A_n^T A_{n-1}^T.....A_3^T A_2^T A_1^T$

Example 5 :

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ then find $(AB)^T$.

Sol. $AB = \begin{bmatrix} 3+2 & 4+12 \\ 9+0 & 12+0 \end{bmatrix} = \begin{bmatrix} 5 & 16 \\ 9 & 12 \end{bmatrix} \quad \therefore (AB)^T = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$

Example 6 :

If $A = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ then find B^TA^T

Sol. $B^TA^T = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 8-7 & -28+28 \\ 2-2 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

SYMMETRIC AND SKEW-SYMMETRIC MATRIX

Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Ex. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

NOTE

(i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$

Skew-Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix. if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Ex. $\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & 0 \end{bmatrix}$

NOTE

(i) All Principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

Properties of Symmetric and skew-symmetric matrices

(i) If A is a square matrix, then $A + A^T, AA^T, A^TA$ are symmetric matrices while $A - A^T$ is Skew-Symmetric Matrices.

(ii) If A is a Symmetric Matrix, then $-A, KA, A^T, A^n, A^{-1}, B^T AB$ are also symmetric matrices where $n \in N$, $K \in R$ and B is a square matrix of order that of A

(iii) If A is a skew symmetric matrix, then –
(a) A^{2n} is a symmetric matrix for $n \in N$
(b) A^{2n+1} is a skew-symmetric matrices for $n \in N$
(c) kA is also skew-symmetric matrix where $k \in R$
(d) $B^T AB$ is also skew - symmetric matrix where B is a square matrix of order that of A.

(iv) If A, B are two symmetric matrices, then –
(a) $A \pm B$, $AB + BA$ are also symmetric matrices.
(b) $AB - BA$ is a skew-symmetric matrix
(c) AB is a symmetric matrix when $AB = BA$.

(v) If A, B are two skew-symmetric matrices, then –
(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices
(b) $AB + BA$ is a symmetric matrix

(vi) If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of a symmetric and skew symmetric matrix i.e.

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

Example 7 :

If $A = \begin{bmatrix} -1 & 7 \\ 2 & 3 \end{bmatrix}$, then find skew-symmetric part of A.

Sol. Let $A = B + C$, where $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$ are respectively symmetric and skew-symmetric parts of A.

Now $C = \frac{1}{2}\left\{\begin{bmatrix} -1 & 7 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 7 & 3 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 \\ -5/2 & 0 \end{bmatrix}$

DETERMINANT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by $|A|$ or det (A) is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Properties of the Determinant of a matrix :

(i) $|A|$ exists $\Leftrightarrow$ A is a square matrix

(ii) $|AB| = |A||B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is a skew symmetric matrix of odd order then $|A| = 0$

(vii) If $A = \text{diag}(a_1, a_2.....a_n)$ then $|A| = a_1a_2...a_n$

(viii) $|A|^n = |A^n|$, $n \in N$.

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of matrix A and it is denoted by adj A. Thus if $A = [a_{ij}]$ be a square matrix and F_{ij} be the cofactor of a_{ij} in |A|, then Adj. $A = [F^{ij}]^T$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ a_{n1} & a_{n2} & ...a_{nn} \end{bmatrix}, \text{ then adj. } A = \begin{bmatrix} F^{11} & F^{12} & ...F^{1n} \\ F^{21} & F^{22} & ...F^{2n} \\ F^{n1} & F^{n2} & ...F^{nn} \end{bmatrix}^T$$

$$\text{Ex. if } A = \begin{bmatrix} 9 & -4 \\ -2 & 3 \end{bmatrix} \text{ then adj } A = \begin{bmatrix} 3 & 2 \\ 4 & 9 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 2 & 9 \end{bmatrix}$$

Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A (\text{adj. } A) = |A| I_n = (\text{adj } A) A$

(ii) $|\text{adj } A| = |A|^{n-1}$
(Thus A (adj A) is always a scalar matrix)

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj}(A^T) = (\text{adj } A)^T$

(vi) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(vii) $\text{adj}(A^m) = (\text{adj } A)^m, m \in N$

(viii) $\text{adj}(kA) = k^{n-1}(\text{adj. } A), k \in R$

(ix) $\text{adj}(I_n) = I_n$

(x) adj 0 = 0

(xi) A is symmetric ⇒ adj A is also symmetric

(xii) A is diagonal ⇒ adj A is also diagonal

(xiii) A is triangular ⇒ adj A is also triangular

(xiv) A is singular ⇒ |adj A| = 0

Example 8 :

$$\text{If } A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}, \text{ then find adj (adj A).}$$

Sol. $|A| = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = (2)(2)(2) = 8$

Now $\text{adj}(\text{adj } A) = |A|^{3-2}A$

$$= 8\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example 9 :

$$\text{If } A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \text{ then find } |\text{adj}(\text{adj } A)|.$$

Sol. $|A| = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2$

$\therefore\ |\text{adj}(\text{adj } A)| = |A|^{(n-1)^2} = |A|^{2^2}$ [∴ Here n = 3]

$= 2^4 = 16$

INVERSE OF A MATRIX

If A and B are two matrices such that AB = I = BA then B is called the inverse of A and it is denoted by A^{-1}, thus $A^{-1} = B \Leftrightarrow AB = I = BA$

To find inverse matrix of a given matrix A we use following formula $A^{-1} = \dfrac{\text{adj. } A}{|A|}$. Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$

Note : (i) Matrix A is called invertible if A^{-1} exists.
(ii) Inverse of a matrix is unique.

Properties of Inverse Matrix :

Let A & B are two invertible matrices of the same order, then

(i) $(A^T)^{-1} = (A^{-1})^T$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A^k)^{-1} = (A^{-1})^k, k \in N$

(iv) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

(v) $(A^{-1})^{-1} = A$

(vi) $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

(vii) If $A = \text{diag}(a_1, a_2, ..., a_n)$, then A^{-1}
$= \text{diag}(a_1^{-1}, a_2^{-1}, ...a_n^{-1})$

(viii) A is symmetric matrix ⇒ A^{-1} is symmetric matrix.

(ix) A is triangular matrix and $|A| \neq 0$ ⇒ A^{-1} is a triangular matrix.

(x) A is scalar matrix ⇒ A^{-1} is scalar matrix

(xi) A is diagonal matrix ⇒ A^{-1} is diagonal matrix

(xii) AB = AC ⇒ B = C, iff $|A| \neq 0$.

Example 10 :

Find the inverse matrix of $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$

Sol. Let the given matrix is A, then |A| = – 8

and $\text{adj } A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

Example 11 :

If $A=\begin{bmatrix}0 & -1 & 2\\ 2 & -2 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 1\\ 1 & 0\\ 1 & 1\end{bmatrix}$ and M=AB, then find M^{-1}.

Sol. $M=\begin{bmatrix}0 & -1 & 2\\ 2 & -2 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ -2 & 2\end{bmatrix}$

$|M|=6, \text{adj } M=\begin{bmatrix}2 & -2\\ 2 & 1\end{bmatrix}$

$M^{-1}=\frac{1}{6}\begin{bmatrix}2 & -2\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1/3 & -1/3\\ 1/3 & 1/6\end{bmatrix}$

Method of finding the inverse of a matrix by Elementary transformation :

Let A be a non singular matrix of order n. Then A can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots\ldots, E_4$ such that $(E_k E_{k-1} \ldots\ldots E_2E_1)A = I_n$

$\Rightarrow (E_k E_{k-1} \ldots\ldots E_2E_1)AA^{-1} = I_nA^{-1}$

(post multiplying by A^{-1})

$\Rightarrow (E_k E_{k-1} \ldots\ldots E_2E_1) I_n = A^{-1}$

$(\because \; I_nA^{-1} = A^{-1}$ and $AA^{-1} = I_n)$

$\Rightarrow A^{-1} = (E_k E_{k-1} \ldots\ldots E_2E_1) I_n$

Algorithm for finding the inverse of a non singular matrix by elementary row transformations :

Let A be non-singular matrix of order n

Step-I : Write $A = I_n A$

Step-II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step-III : Write $A^{-1} = B$

The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step-I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step-II : After introducing unity at (1, 1) place introduce zeros at all other places in first column.

Step-III : Introduce unity at the intersection 2nd row and 2nd column with the help of 2nd and 3rd row.

Step-IV : Introduce zeros at all other places in the second column except at the intersection of 2nd and 2nd column

Step-V : Introduce unity at the intersection of 3rd row and third column.

Step-VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Example 12 :

Using elementary transformation, find the inverse of the matrix $A=\begin{bmatrix}a & b\\ c & \left(\frac{1+bc}{-a}\right)\end{bmatrix}$.

Sol. $A=\begin{bmatrix}a & b\\ c & \left(\frac{1+bc}{a}\right)\end{bmatrix}$

We write, $\begin{bmatrix}a & b\\ c & \left(\frac{1+bc}{a}\right)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$

$\Rightarrow \begin{bmatrix}1 & \frac{b}{a}\\ c & \left(\frac{1+bc}{a}\right)\end{bmatrix}=\begin{bmatrix}\frac{1}{a} & 0\\ 0 & 1\end{bmatrix}A \qquad \left(R_1 \to \frac{R_1}{a}\right)$

or $\begin{bmatrix}1 & \frac{b}{a}\\ 0 & \frac{1}{a}\end{bmatrix}=\begin{bmatrix}\frac{1}{a} & 0\\ \frac{-c}{a} & 1\end{bmatrix}A \qquad (R_2 \to R_2 - cR_1)$

or $\begin{bmatrix}1 & \frac{b}{a}\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{a} & 0\\ -c & a\end{bmatrix}A \qquad (R_2 \to aR_2)$

or $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1+bc}{a} & -b\\ -c & a\end{bmatrix}A \qquad \left(R_1 \to R_1 - \frac{b}{a}R_2\right)$

$\Rightarrow A^{-1}=\begin{bmatrix}\frac{1+bc}{a} & -b\\ -c & a\end{bmatrix}$

SOME SPECIAL CASES OF MATRIX

(i) Orthogonal Matrix : A square Matrix A is called orthogonal if $AA^T = I = A^TA$ i.e. if $A^{-1} = A^T$

Ex. $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is a orthogonal matrix because here

$A^{-1} = A^T$

(ii) Idempotent Matrix : A square matrix A is called an Idempotent Matrix if $A^2 = A$

Ex. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is a Idempotent Matrix because here $A^2 = A$

(iii) Involutory Matrix : A square matrix A is called an involutory Matrix if $A^2 = I$ or $A^{-1} = A$

Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a Involutory Matrix.

(iv) Nilpotent Matrix : A square matrix A is called a nilpotent Matrix if there exist $p \in N$ such that $A^p = 0$

Ex. $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a Nilpotent Matrix

(v) The conjugate of a Matrix : The conjugate of a matrix A is a matrix $\overline{A}$ whose each element is a conjugate complex number of corresponding element of Matrix A.

Note : Conjugate transpose Matrix of matrix A is a Transpose Matrix of conjugate of matrix A and it is denoted by A^* or A^{θ}. i.e. $A^* = (\overline{A})^T$

(vi) Hermition Matrix : A square Matrix is Hermition Matrix if $^{\theta} = A$. i.e. $a_{ij} = \overline{a}_{ji}\ \forall$ i, j

(vii) Skew Hermition Matrix : A Square Matrix A is Skew-Hermition is $A = -A^{\theta}$ e.q. $a_{ij} = -\overline{a}_{ij}\ \forall$ i, j.

(viii) Period of a Matrix : If for any Matrix A, $A^{k+1} = A$ then k is called period of Matrix (where k is a least positive integer) Ex. If $A^3 = A$, $A^5 = A$, $A^7 = A$,.....then it is a periodic matrix and $A^{2+1} = A$ so its period is = 2

(ix) Differentiation of a Matrix :

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of Matrix A

Ex. if $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

(x) Submatrix : Let A be m x n matrix, then a matrix obtained by leaving some rows or columns or both of A is called a sub matrix of A

Ex. if $A' = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 2 \\ 2 & 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$ are sub matrices of

Matrix $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 2 & 4 \\ 2 & 5 & 3 & 1 \end{bmatrix}$

(xi) Rank of a Matrix : A number r is said to be the rank of a $m \times n$ matrix A if

(a) Every square sub matrix of order (r + 1) or more is singular and

(b) There exists at least one square submatrix of order r which is non-singular. Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

Ex. The rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix}$ is

We have $|A| = 0$, therefore r(A) is less then 3, we observe that $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is a non-singular square sub matrix of order 2. Hence r(A) = 2.

Note: (i) The rank of the null matrix is not defined and the rank of every non null matrix is greater than or equal to one.

(ii) The rank of matrix is same as the rank of its transpose i.e. $r(A) = r(A^T)$.

(iii) Elementary transformation do not alter the rank of matrix.

TRY IT YOURSELF-1

Q.1 The matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ be a zero divisor of the polynomial $f(x) = x^2 - 4x - 5$. Find the trace of matrix A^3.

Q.2 If A, B are symmetric matrixes of same order than AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

Q.3 The product of n matrices

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.......\begin{bmatrix} 1 & n \\ 0 & n \end{bmatrix}$ is equal to matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$. Find n

Q.4 Find the transpose of matrix $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

Q.5 If α and β are roots of the equation

$$[1\quad 25]\begin{bmatrix}0 & \frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}^5\begin{bmatrix}1 & -1\\ 2 & 0\end{bmatrix}^{10}\begin{bmatrix}0 & \frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}^5\begin{bmatrix}x^2-5x+20\\ x+2\end{bmatrix}=[40]$$

then find the value of $(1-\alpha)(1-\beta)$.

Q.6 Using elementary transformation, find the inverse of the matrix $A=\begin{bmatrix}a & b\\ c & \left(\frac{1+bc}{a}\right)\end{bmatrix}$.

Q.7 If $A=\begin{bmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{bmatrix}$, then find the value of $|A||\text{adj } A|$.

Q.8 Matrices A and B satisfy $AB=B^{-1}$, where $B=\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}$.

Find Without finding B^{-1}, the value of K for which $KA=2B^{-1}+I=0$

ANSWERS

(1) 123 **(2)** (A) **(3)** 27

(4) $\begin{bmatrix}-1 & \sqrt{3} & 2\\ 5 & 5 & 3\\ 6 & 6 & -1\end{bmatrix}$ **(5)** 51 **(6)** $\begin{bmatrix}\frac{1+bc}{a} & -b\\ -c & a\end{bmatrix}$

(7) a^9 **(8)** 2

DETERMINANTS

HISTORICAL DEVELOPMENT

Development of determinants took place while mathematicians were trying to solve a system of simultaneous linear equations.

e.g. $\left.\begin{array}{l}a_1x+b_1y=c_1\\ a_2x+b_2y=c_2\end{array}\right]\Rightarrow x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$ and

$$y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$$

Mathematicians defined the symbol $\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}$ as determinant of order 2 and the four numbers arranged in row and column were called its elements. Its value was taken as $a_1b_2-a_2b_1$ which is the same as denominator. This kind of definition helped then to state the solution of the simultaneous equation as

$$x=\frac{D_1}{D}\text{ and } y=\frac{D_2}{D}\text{ where}$$

$$D=\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix};D_1=\begin{vmatrix}c_1 & b_1\\ c_2 & b_2\end{vmatrix};D_2=\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}$$

NOTE : A determinant of order 1 is the number itself.

The symbol $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ is called the determinant of order 3. Its value can be found as

$$D=a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-b_1\begin{vmatrix}a_2 & c_2\\ a_3 & c_3\end{vmatrix}+c_1\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}$$

$$=a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & c_1\\ b_3 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}$$

In the way we can expand a determinant in 6 ways using elements of $R_1, R_2, R_3, C_1, C_2, C_3$.

Example 13 :

Find the value of $\begin{vmatrix}1+\cos\theta & \sin\theta\\ \sin\theta & 1-\cos\theta\end{vmatrix}$

Sol. $\begin{vmatrix}1+\cos\theta & \sin\theta\\ \sin\theta & 1-\cos\theta\end{vmatrix}=(1+\cos\theta)(1-\cos\theta)-(\sin\theta)(\sin\theta)$

$$=1-\cos^2\theta-\sin^2\theta=0$$

Example 14 :

Find the value of $\begin{vmatrix}1 & 2 & 3\\ -4 & 3 & 6\\ 2 & -7 & 9\end{vmatrix}$.

Sol. $\begin{vmatrix}1 & 2 & 3\\ -4 & 3 & 6\\ 2 & -7 & 9\end{vmatrix}=1\begin{vmatrix}3 & 6\\ -7 & 9\end{vmatrix}-2\begin{vmatrix}-4 & 6\\ 2 & 9\end{vmatrix}+3\begin{vmatrix}-4 & 3\\ 2 & -7\end{vmatrix}$

$=1(3\times 9-6(-7))-2(-4\times 9-2\times 6)+3((-4)(-7)-3\times 2)$
$=(27+42)-2(-36-12)+3(28-6)=231$

MINOR & COFACTOR

Minor : The Determinant that is left by cancelling the row and column intersecting at a particular element is called the minor of that element.

If $\Delta=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ then Minor of a_{11} is

$$M_{11}=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}, \text{ Similarly } M_{12}=\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}$$

Using this concept the value of Determinant can be

$\Delta = a_{11}M_{11} - a_{12} M_{12} + a_{13} M_{13}$

or $\Delta = - a_{21}M_{21} + a_{22} M_{22} - a_{23} M_{23}$

or $\Delta = a_{31}M_{31} - a_{32} M_{32} + a_{33} M_{33}$

Cofactor : The cofactor of an element a_{ij} is denoted by F_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M is a minor of element a_{ij}

if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $F_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$F_{12} = (-1)^{1+2} M_{12} = - M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

NOTE :

(i) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e. $\Delta = a_{11}F_{11} + a_{12} F_{12} + a_{13}F_{13}$

(ii) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero i.e. $a_{11}F_{21} + a_{12} F_{22} + a_{13}F_{23} = 0$

(iii) If order of a determinant (Δ) is 'n' then the value of the determinant formed by replacing every element by its cofactor is Δ^{n-1}.

Example 15 :

Find the cofactor element 0 in Determinant $\begin{vmatrix} -1 & 2 & 1 \\ -2 & 3 & -3 \\ 4 & 0 & -4 \end{vmatrix}$

Sol. $F_{32} = (-1)^{3+2} \begin{vmatrix} -1 & 1 \\ -2 & -3 \end{vmatrix} = -[(-1)(-3) - (-2)(1)]$

$= -[3+2] = -5$

PROPERTIES OF DETERMINANT

P-1 The value of Determinant remains unchanged, if the rows and the column are interchanged.

This is always denoted by ' and is also called transpose

Ex. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Then D' = D, D and D' are transpose of each other

Note: Since the Determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'Columns'

P- 2 If any two rows (or columns) of a determinant be inter changed, the determinant is unaltered in numerical value, but is changed in sign only.

Ex. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then D' = – D

P- 3 If a Determinant has two rows (or columns) identical, then its value is zero.

Ex. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ then, D = 0

P- 4 If all the elements of any row (or column) be multiplied by the same number, then the value of Determinant is multiplied by that number.

Ex. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then D' = k D.

P- 5 : If each elements of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the Determinants

Ex. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P- 6 : The value of a Determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

Ex. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3-na_1 & b_3-nb_1 & c_3-nc_1 \end{vmatrix}$

then D' = D

Note: It should be noted that while applying P-6 at least one row (or column) must remain unchanged

P- 7 : If Δ = f(x) and f(a) = 0 then (x-a) is a factor of Δ.

Ex. $D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

If we replace a by b then $D = \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^2 & b^2 & c^2 \end{vmatrix} = 0$

⇒ (a – b) is a factor of D

P-8 : In a determinant the sum of the products of the elements of any row (column) with their corresponding cofactors is equal to the value of determinant.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let A_i, B_i, C_i be the cofactors of the elements a_i, b_i, c_i $(i = 1, 2, 3)$

Then, $a_1A_1 + b_1B_1 + c_1C_1 = D$

$a_2A_2 + b_2B_2 + c_2C_2 = D$

Similarly, in a determinant the sum of the products of the elements of any row (column) with the cofactors of corresponding elements of any other row (column) is zero.

i.e., $a_1A_2 + b_1B_2 + c_1C_2 = 0$

or $a_2A_1 + b_2B_1 + c_2C_1 = 0$

SOME IMPORTANT DETERMINANTS TO REMEMBER :

(1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Proof : Let $D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$\Rightarrow D = \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

$D = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

$D = (x-y)(y-z)(z-x)$. Hence proved.

(2) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

Proof : Let $D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$

Apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$. Given

$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$

$D = (x-y)(y-z)[y^2+yz+z^2-x^2-xy-y^2]$

$D = (x-y)(y-z)[y(z-x)+z^2-x^2]$

$= (x-y)(y-z)(z-x)(x+y+z)$.

(3) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Proof : Let $D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & y^4 \end{vmatrix}$

Apply $R_1 \to xR_1$; $R_2 \to yR_2, R_3 \to zR_3$ divide by xyz balancing.

$D = \dfrac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} = \dfrac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.

$= \begin{vmatrix} 0 & x^2-y^2 & x^3-y^3 \\ 0 & y^2-z^2 & y^3-z^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$= (x-y)(z-x)(y-z)(xy+yz+zy)$

(4) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) < 0$ if a, b, c are different and positive

Proof : $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a[bc-a^2]-[b^2-ac]+c(ab-c^2)$

$= 3abc-(a^3+b^3+c^3)$.

Example 16 :

If a, b, c are pth, qth and rth, terms of a G.P., then find

$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$

Sol. If A be the first term and R be the c.r. of G.P., then

$a = AR^{p-1}$, $b=AR^{q-1}$, $c=AR^{r-1}$

$\log a = \log A + (p-1)\log R$

$\therefore \Delta = \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$

$= 0 + \log R\begin{vmatrix} p-1 & p-1 & 1 \\ q-1 & q-1 & 1 \\ r-1 & r-1 & 1 \end{vmatrix} = 0$ [by $C_2 - C_1$]

Example 17 :

Find determinant $\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix}$

Sol. Applying $C_1 + (C_2 + C_3)$ and taking n(a+b+c) common from C_1, we get

$$\Delta = n(a+b+c)\begin{vmatrix} 1 & (n-1)a & (n-1)b \\ 1 & b+c+na & (n-1)b \\ 1 & (n-1)a & c+a+nb \end{vmatrix}$$

$$= n(a+b+c)\begin{vmatrix} 1 & (n-1)a & (n-1)b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= n(a+b+c)^3 \quad [\text{By } R_2 - R_1, R_3 - R_1]$$

Example 18 :

Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$

Sol. $\text{L.H.S.} = \begin{vmatrix} 0 & c & b \\ -2a & c+a & a \\ -2a & a & a+b \end{vmatrix} \quad [C_1 \to C_1 - (C_2 + C_3)]$

$$= -2a\begin{vmatrix} 0 & c & b \\ 1 & c+a & a \\ 1 & a & a+b \end{vmatrix} = -2a\begin{vmatrix} 0 & c & b \\ 0 & c & -b \\ 1 & a & a+b \end{vmatrix} [R_2 \to R_2 - R_3]$$

$$= -2a\begin{vmatrix} c & b \\ c & -b \end{vmatrix} \quad [\text{expanding along } C_1]$$

$$= -(-2a)(-2bc) = 4abc = \text{R.H.S.}$$

Example 19 :

Show that $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$

Sol. We have

$$\text{L.H.S.} = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} \quad [C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix} \quad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (a+b+c)\begin{vmatrix} c-a & b-a \\ c-b & c-a \end{vmatrix} [\text{expending along } C_1]$$

$$= (a+b+c)[(c-a)^2 - (c-b)(b-a)]$$
$$= (a+b+c)[(c^2 + a^2 - 2ac)^2 - (cb - ca - b^2 + ab)]$$
$$= (a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$$
$$= a^3 + b^3 + c^3 - 3abc = \text{R.H.S.}$$

Example 20 :

Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

Sol. $\text{L.H.S.} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$[R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$\begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$[\text{expanding along } C_1]$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3 = \text{R.H.S.}$$

Example 21 :

Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$

Sol. $\text{L.H.S.} = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix} \quad \begin{bmatrix} C_1 \to aC_1 \\ C_2 \to bC_2 \\ C_3 \to cC_3 \end{bmatrix}$

$$=\frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$\begin{bmatrix} \text{taking } a, b, c \text{ common from} \\ R_1, R_2, R_3 \text{ respectively} \end{bmatrix}$

$$=\begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$[C_1 \to C_1+C_2+C_3]$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \begin{bmatrix} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{bmatrix}$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad [\text{expanding along } C_1]$$

$= 1+a^2+b^2+c^2 =$ R.H.S.

Example 22 :

If A, B, C are the angle of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}=0$$

prove that ΔABC must be isosceles

Sol. Let $\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

$$=\begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$[C_2 \to C_2-C_1, C_3 \to C_3-C_1]$

$= (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B)$

Now, since Δ is given to be zero, therefore we have

$(\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B) = 0$

i.e. $\sin B - \sin A = 0$ or $\sin C - \sin A = 0$ or $\sin C - \sin B = 0$

i.e. $\sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$

i.e. $B = A$ or $C = A$ or $C = B$

In all the three cases, the triangle will be isosceles.

Example 23 :

Find the coefficient of x in the determinant

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}=0.$$

Sol. If f(x) be a polynomial in x,

then coefficient of x^n in $f(x)=\frac{f^n(0)}{n\,!}$

Let the given determinant be denoted by f (x), then

$$f'(x)=\begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2-1} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2-1} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2-1} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

Thus, we have

$$f'(0)=\begin{vmatrix} a_1b_1 & 1 & 1 \\ a_2b_1 & 1 & 1 \\ a_3b_1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & a_1b_2 & 1 \\ 1 & a_2b_2 & 1 \\ 1 & a_3b_2 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & a_1b_3 \\ 1 & 1 & a_2b_3 \\ 1 & 1 & a_3b_3 \end{vmatrix}=0$$

Hence, we have

Coeff. of x in $f(x)=\frac{f'(0)}{1\,!}=0$

SYMMETRIC & SKEW SYMMETRIC DETERMINANT

Symmetric determinant : A determinant is called symmetric Determinant if for its every element.

$a_{ij} = a_{ji} \quad \forall\ i, j$ Ex. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Skew Symmetric determinant : A determinant is called skew Symmetric determinant if for its every element $a_{ij} = -a_{ji} \ \forall\ i, j$

Ex. $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

NOTE

(i) Every diagonal element of a skew symmetric determinant is always zero.

(ii) The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

MULTIPLICATION OF TWO DETERMINANTS

Multiplication of two second order determinants is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Multiplication of two third order determinants is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Note: In above case the order of Determinant is same, if the order is different then for their multiplication first of all they should be expressed in the same order.

To express a determinants as a product of two determinants :

To express a determinant as product of two determinants one requires a lot of practice and this can be done only by inspection and trial. It can be understood by the following examples.

Example 24 :

Let $\Delta = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$, then Δ can be expressed as

(A) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$ (B) $\begin{vmatrix} c & b & a \\ a & b & c \\ c & a & b \end{vmatrix}^2$

(C) $\begin{vmatrix} a & b & c \\ c & b & a \\ c & a & b \end{vmatrix}^2$ (D) None

Sol. $\Delta = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

$$= \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\left(\because \text{by properties } \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \right)$$

SUMMATION OF DETERMINANTS

Let $\Delta = \begin{vmatrix} f(r) & a & \ell \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$ where a, b, c, ℓ, m and n are constants, independent of r. Then

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & \ell \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}.$$

Here function of r can be the elements of only one row or one column.

LIMIT OF A DETERMINANT

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\lim_{x\to a} \Delta(x) = \begin{vmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) & \lim\limits_{x\to a} h(x) \\ \lim\limits_{x\to a} \ell(x) & \lim\limits_{x\to a} m(x) & \lim\limits_{x\to a} n(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) & \lim\limits_{x\to a} w(x) \end{vmatrix},$$

provided each of nine limiting values exist finitely.

DIFFERENTIATION OF DETERMINANTS :

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\Delta'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Example 25 :

Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant.

Then find $\dfrac{d^3}{dx^3}[f(x)]$ at $x=0$.

Sol. $\dfrac{d}{dx}f(x)=\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}+0+0$

$$\frac{d^2}{dx^2}f(x)=\begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\frac{d^3}{dx^3}f(x)=\begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\frac{d^3}{dx^3}f(x) \text{ at } x=0 \text{ is } \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}=0$$

i.e. independent of p.

DETERMINANTS INVOLVING INTEGRATIONS

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ \ell & m & n \end{vmatrix}$ where a, b, c, ℓ, m and are constants.

$$\Rightarrow \int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ \ell & m & n \end{vmatrix}$$

Example 26 :

Let $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x\,\mathrm{cosec}\,x \\ \cos^2 x & \cos^2 x & \mathrm{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Prove that $\int_0^{\pi/2} f(x)\,dx=-\left(\dfrac{\pi}{4}+\dfrac{8}{15}\right)$.

Sol. Operate $R_1 \to R_1-\sec x\, R_3$

$$f(x)=\begin{vmatrix} 0 & 0 & \sec^2 x+\cot x\,\mathrm{cosec}\,x-\cos x \\ \cos^2 x & \cos^2 x & \mathrm{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$$=(\sec^2 x+\cot x\,\mathrm{cosec}\,x-\cos x)(\cos^4 x-\cos^2 x)$$

$$=\left(1+\frac{\cos^3 x}{\sin^2 x}-\cos^3 x\right)(\cos^2 x-1)$$

$$=-\sin^2 x\,\frac{\sin^2 x+\cos^3 x-\cos^3 x\sin^2 x}{\sin^2 x}$$

$$=-(\sin^2 x+\cos^5 x)$$

$$\int_0^{\pi/2} f(x)\,dx=-\int_0^{\pi/2}(\sin^2 x+\cos^5 x)\,dx$$

$$=-\left(\frac{1}{2}.\frac{\pi}{2}+\frac{4.2}{5.3.1}\right)=-\left(\frac{\pi}{4}+\frac{8}{15}\right)$$

APPLICATIONS OF DETERMINANT

1. Area of triangle : The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression

$$\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

Now this expression can be written in the form of a determinant as $\Delta=\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ .... (1)

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

2. System of linear equations :

Definition-1 :

A system of linear equations in n unknowns $x_1, x_2, x_3, \ldots\ldots x_n$ is of the form :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots\ldots(A)$$

If $b_1, b_2, \ldots\ldots, b_n$ are all zero, the system is called **homogeneous** and non-homogeneous if at least one b_i is non-zero.

Definition-2 :

The solution set of the system (A) is an n type $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ of real numbers (or complex numbers if the coefficients are complex) which satisfy each of the equations of the system.

Definition-3 :

A system of equations is called **consistent** if it has at least one solution; **inconsistent** if it does not have any solution; **determinate** if it has a unique solution; **indeterminate** if it has more than one solution.

(A) Non-homogeneous Equations in two unknowns :

Consider the system of equations

$$\left\{\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right\} \quad \ldots(i)$$

We consider the following cases.

(1) a_i, b_i, c_i (i = 1, 2) are all zero :

Then any pair of numbers (x, y) is a solution of the system (i) since in this case equation reduces to an identity.

So, in this case equations are always **consistent and indeterminate.**

(2) a_i, b_i **(i = 1, 2)** are all zero, but at least one c_1 **and** c_2 is non-zero. Then the system has solution i.e. the equation are **inconsistent.**

(3) Al least one of a_ib_i (i = 1, 2) is non-zero

Suppose $b_2 \neq 0$. Then system (i), is equivalent to the system.

$$\left\{\begin{array}{l} a_1x + b_1y = c_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{array}\right\} \quad \ldots(ii)$$

i.e., if the pair (x_0, y_0) is a solution of system (i) then it is also a solution of system (ii), and vice-versa.

Multiplying the second equation of system (ii) by b_1 and subtracting from first, we get

$$\left(a_1 - \frac{a_2}{b_2}b_1\right)x = c_1 - \frac{c_2}{b_2}\cdot b_1 \quad \ldots(iii)$$

Now replacing the first equation of system (ii) by equation (iii), we obtain the system

$$\left\{\begin{array}{l} \left(a_1 - \dfrac{a_2}{b_2}b_1\right)x = c_1 - \dfrac{c_2}{b_2}\cdot b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{array}\right\} \quad \ldots(iv)$$

(a) If $a_1 - \dfrac{a_2}{b_2}b_1 \neq 0$ i.e., if $a_1b_2 - a_2b_1 \neq 0$.

then we find from the first equation of system (iv) that

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \ldots(v)$$

Substituting this value of x into the second equation of system (iv), we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

For convenience, we write

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \ \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \ \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \ \ldots(vi)$$

[Note that Δ_x and Δ_y are obtained by replacing the first and second columns in Δ by the column of c_1 and c_2 respectively]. Then (v) and (vi) can be written as

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta} \quad \ldots(vii)$$

This is known as **Cramer's rule**. If $a_1b_2 - a_2b_1 \neq 0$ then the system (iv) or system (i) has the unique solution given by (vii). Hence in this case, the equations are **consistent and determinate.**

(b) Now let $\Delta = a_1b_2 - a_2b_1 = 0$.

Then the system (iv) has the form

$$\left\{\begin{array}{l} 0.x = c_1b_2 - c_2b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{array}\right\} \quad \ldots(viii)$$

Obviously this system has no solution if

$c_1b_2 - c_2b_1 = \Delta_x \neq 0$

thus in this case, the equations are inconsistent.

But if $\Delta_x = 0$, then any pair of numbers (x, y),

where $y = \dfrac{c_2}{b_2} - \dfrac{a_2}{b_2}x$, $x \in R$, is a solution of system (viii).

In this case, the equations are consistent and indeterminate.

We **summarize** the whole discussion given in (A) as follows:

(i) If $\Delta \neq 0$, then the system is consistent and determinant and its solution is given by

$$x = \frac{\Delta_x}{\Delta}, \ y = \frac{\Delta_y}{\Delta} \quad \text{(i.e., unique solution)}$$

(ii) If $\Delta = 0$, but at least one of the numbers Δ_x, Δ_y is non-zero, then the system is inconsistent i.e., it has no solution.

(iii) If $\Delta = 0$, and $\Delta_x = \Delta_y = 0$ but at least one of the numbers a_1, b_1, a_2, b_2 is non-zero, then the system has infinite number of solutions and hence it is consistent and indeterminante.

(iv) If $a_i = b_i = c_i = 0$ (i = 1, 2), then system has infinite number of solutions and so it is consistent and indeterminante.

(B) Homogenous linear equations in two unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{Bmatrix} \quad ......(ix)$$

The system always has the solution x = 0, y = 0. If follows from the discussion in part (A) that if $\Delta \neq 0$, then the system (ix) has the unique solution x = 0, y = 0.

And if $\Delta = 0$, and at least one of a_1, a_2, b_1, b_2 is non-zero then system (i) reduced to the single equation so that any pair of numbers (x, y) is a solution.

Thus system (ix) is always consistent.

(C) Non-homogeneous linear equations in three unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{Bmatrix} \quad ....(1)$$

Let us introduce the following notations

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Without going into details, we give the following rule for testing the consistency of the system (1).

(1) Let $a_i = b_i = c_i = d_i = 0$, i = 1, 2, 3
In this case any triplet (x, y, z) is a solution of the system. Hence equations are consistent and indeterminate.

(2) If $a_i = b_i = c_i = 0$, i = 1, 2, 3 and at least one d_i (i = 1, 2, 3) is non-zero, then the system has no solution, i.e., the equations in this case are inconsistent.

(3) Let $\Delta \neq 0$. In this case the system (1) has the unique

solution $x = \dfrac{\Delta_x}{\Delta}, y = \dfrac{\Delta_y}{\Delta}, z = \dfrac{\Delta_z}{\Delta}$ ....(2)

This is known as **Crammer's rule**. So equations in this case are consistent and determinate.

(4) If $\Delta = 0$, $\Delta_x \neq 0$ (or $\Delta_y \neq 0$ or $\Delta_z \neq 0$), then the system has no solution so the equations are inconsistent.

(5) If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ and at least one of the cofactors of Δ is non-zero, then the system will have an infinite number of solutions. In this case, any one of the variables can be given arbitrary value and other variables can be expressed in terms of that variable.

In such cases, the three equations reduce to two equations If all the cofactors $\Delta, \Delta_x, \Delta_y, \Delta_z$ are zero but elements of Δ are not all zero, then in this case the system will reduced to single equation and any two variables can be given arbitrary values. So equations are consistent and indeterminate.

(D) Homogeneous linear equations :

If in (1), we take $d_i = 0$ (i = 1, 2, 3) then the system is called the homogenous system of equations.

For such a system if $\Delta \neq 0$, then it has the unique solution x = 0, y = 0, z = 0. **(Trivial)**

So such system of equations is always consistent.

(1) Three equations in two unknowns :

Consider the equations

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_2 \end{Bmatrix} \quad ....(3)$$

The system (3) will be consistent if the solutions set of any satisfies the third equations, i.e., if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Note : The factors of the following two determinants be remembered.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

$= (a - b)(b - c)(c - a)(a + b + c).$

(2) Gist of discussion in simple language :

(i) Consistent : Solution exists whether unique infinite number of solutions.

(ii) Inconsistent : Solution does not exist.

(iii) Homogeneous Equations : constant terms zero.

(iv) Trivial solution : All variables zero i.e., x = 0, y = 0, z = 0.

(v) Non-trivial solution : Infinite number of solutions.
For example
$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_1 \text{ or } \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

$$\Delta_2 \text{ or } \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(3) Case-I : Intersecting lines

$2x + 3y = 10$ and $x + y = 4$

$\therefore \quad x = 2, y = 2$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \Delta \neq 0.$$

(4) Case II : $2x + 3y = 10$

$4x + 6y = 20$

Here $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0,$

but $\Delta_1 = \begin{vmatrix} 10 & 2 \\ 20 & 4 \end{vmatrix} = 0, \Delta_2 = 0$

As a matter of fact on division by 2 the second equation reduces to first. Thus we have got only one line $2x + 3y = 10$ on which lie infinite number of points. Thus there are infinite number of solutions and the system is consistent. $\left(k, \dfrac{10-3k}{2}\right)$ are infinite number of solutions by giving different values to k.

Case-III $\begin{matrix} 2x + 3y = 10 \\ 4x + 16y = 15 \end{matrix}$ or $\begin{matrix} 2x + 3y = 10 \\ 2x + 3y = 15/2 \end{matrix}$

i.e. parallel lines which we know do not intersect and hence no solution.

i.e. inconsistent. Here $\Delta = 0$ but $\Delta_1 \neq 0, \Delta_2 \neq 0$

NOTE

(i) $\Delta \neq 0$ Unique (Intersecting lines) Consistent

(ii) $\Delta = 0, \Delta_1 = 0, \Delta_2 = 0$ (Identical lines) Consistent, Infinite solution.

(iii) $\Delta = 0, \Delta_1 \neq 0$ (Parallel lines)

Inconsistent. No solution.

Homogeneous : $a_1x + b_1y = 0$

$a_2x + b_2y = 0$

$\Delta \neq 0$, Unique $x = 0, y = 0$, Trivial.

$\Delta = 0$, Identical line through origin, Non-trivial solution.

(5) Concurrent lines : Two variable, three equations :

$a_1x + b_1y = c, a_2x + b_2y = c_2, a_3x + b_3y = c_3$

The point of intersection of any two lines should satisfy the third.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

is the required condition.

Example 27 :

For what value of λ the equations

$2x + 3y = 8, 7x - 5y + 3 = 0$ and $4x - 6y + \lambda = 0$

are consistent ? Also find the solution of the system of equations for the values of λ.

Sol. Here the equations are linear. We have 3 equations in 2 unknowns.

$$\therefore \text{ they are consistent if } \begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$$

or $2(-5\lambda + 18) - 3(7y - 12) - 8(-42 + 20) = 0$

or $-10\lambda + 36 - 21\lambda + 36 + 176 = 0$

or $-31\lambda + 248 = 0$; $\therefore \lambda = 8$

$\therefore$ for $\lambda = 8$ the system has a solution which can be obtained by solving any two of the three equations.

Solving $2x + 3y - 8 = 0$

$7x - 5y + 3 = 0$ by Cramer's rule,

$$\frac{x}{\begin{vmatrix} 3 & -8 \\ -5 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -8 \\ 7 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 7 & -5 \end{vmatrix}}$$

or $\dfrac{x}{9-40} = \dfrac{-y}{6+56} = \dfrac{1}{-10-21}$

or $\dfrac{x}{-31} = \dfrac{-y}{62} = \dfrac{1}{-31}, \qquad \therefore x = 1, y = 2$

Example 28 :

For what values of p and q the system of equations

$2x + py + 6z = 8$

$x + 2y + qz = 5$

$x + y + 3z = 4$

has (i) unique solution (ii) no solution

(iii) infinite number of solutions?

Sol. Here the system of linear equations in x, y, z are

$2x + py + 6z - 8 = 0$

$x + 2y + qz - 5 = 0$

$x + y + 3z - 4 = 0$

$$\therefore \quad \Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & p-2 & 0 \\ 1 & 1 & q-3 \\ 1 & 0 & 0 \end{vmatrix},$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - 3 \times C_1$

$$= \begin{vmatrix} p-2 & 0 \\ 1 & q-3 \end{vmatrix} = (p-2)(q-3)$$

$\therefore$ If $p \neq 2, q \neq 3$ then $D \neq 0$

and so the system will have unique solution, i.e., the system will be independent/solvable/consistent.

If $p = 2$ or $q = 3$ then $\Delta = 0$.

and so the system cannot have unique solution.

When $p = 2$,

$$\Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0$$

$(\because R_1 \equiv R_3)$

$$\Delta_y = \begin{vmatrix} 2 & 6 & -8 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0 \quad (\because R_1 \equiv R_3)$$

$$\Delta_z = \begin{vmatrix} 2 & p & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & -4 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 0$$

$(\because R_1 \equiv R_3)$

$\therefore$ when $p = 2$, $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z$.

$\therefore$ the system of equations will have infinite number of solutions (the system of equations will be dependent) for $p = 2$ and any real value of q.

When $q = 3$,

$$\Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} p & 6 & -8 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} p-2 & 0 & 0 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix},$$

$= (p-2)3 \qquad R_1 \to R_1 - 2R_3$

$\therefore$ $p \neq 2$, $\Delta x \neq 0$ and so the system of equations will have no solutions, i.e., the system is solvable/inconsistent when $q = 3$ but $p \neq 2$.

Thus we find that the system of equations will have

(i) unique solution if $p \neq 2$ and $q \neq 3$

(ii) no solution if $p \neq 2$ and $q = 3$

(iii) infinite number of solutions if $p = 2$.

Example 29 :

Find values of k so that the following system of equations has non-trivial solution

$x + ky + 3z = 0$; $kx + 2y + 2z = 0$; $2x + 3y + 4z = 0$

Sol. Here $\Delta = 0 \Rightarrow \begin{vmatrix} 1 & k & 3 \\ k & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\Rightarrow 8 + 9k + 4k - 12 - 4k^2 - 6 = 0 \Rightarrow 4k^2 - 13k + 10 = 0$

$\therefore k = 2, 5/4$

Example 30 :

The system of equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has unique solution if

(1) $k = 0$ (2) $k \neq 0$

(3) $-1 < k < 1$ (4) $-2 < k < 2$

Sol. **(2).** Given system will have unique solution, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow k \neq 0$$

TRY IT YOURSELF-2

Q.1 Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

Q.2 If a, b, c are in A.P., then the determinants

$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is –

(A) 0 (B) 1
(C) x (D) 2x

Q.3 If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is equal to –

(A) –3 (B) 2
(C) 1 (D) None of these

Q.4 Without expanding the determinant at any stage show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$$

Q.5 If α, β are the roots of $ax^2 + bx + c = 0$, then the value of the determinant $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} =$

(A) a + b (B) 0
(C) a – b (D) a + b + c

Q.6 For what value of λ the equations $2x + 3y = 8$, $7x - 5y + 3 = 0$ and $4x - 6y + \lambda = 0$ are consistent? Also find the solution of the system of equations for the values of λ.

Q.7 If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x-1)x(x-1) \end{vmatrix}$

then f (100) is equal to –

(A) 0 (B) 1
(C) 100 (D) –100

ANSWERS

(2) (A) **(3)** (D)
(5) (B) **(6)** $\lambda = 8, x = 1, y = 2$
(7) (A)

USEFUL TIPS

Some important determinants to remember :

1. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

2. $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

3. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+a^3-3abc) < 0$

if a, b, c are different and positive.

ADDITIONAL EXAMPLES

Example 1 :

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ then find adj (adj A)

Sol. We know adj (adj. A) = $|A|^{n-2} A$

Now if n = 3 then adj (adj A) = |A| A

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} A = \{1(6-1)-2(4-3)+3(2-9)\} A$$

$$= (5-2-21) A = -18 A$$

Example 2 :

If $M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $M(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$

then find $[M(\alpha)\, M(\beta)]^{-1}$.

Sol. $[M(\alpha)\, M(\beta)]^{-1} = M(\beta)^{-1}\, M(\alpha)^{-1}]$

Now $M(\alpha)^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = M(-\alpha)$$

$$M(\beta)^{-1} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix} = M(-\beta)$$

$[M(\alpha)\, M(\beta)]^{-1} = M(-\beta)\, M(-\alpha)$

Example 3 :

If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

then find B in terms of I and J.

Sol. Here $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix}$

$$= \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = I\cos\theta + J\sin\theta$$

Example 4 :

If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then find A^{-n}.

Sol. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$A^{-2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$; $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

Example 5 :

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix}$ then find adj A.

Sol. Here $[A_{ij}] = \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 2 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -24 & -27 & 30 \\ 4 & 1 & -2 \\ 8 & 11 & -10 \end{bmatrix}$

Hence transposing $[A_{ij}]$ we get adj $A = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$

Example 6 :

If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then find $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$

Sol. We know that if any row of a determinant is multiplied by k, then the value of the determinant is also multiplied by k, Here all the three rows are multiplied by k, therefore the value of new determinant will be $k^3 \Delta$.

Example 7 :

Find $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

Sol. Applying $R_1 - (R_2 + R_3)$, we get

$$\text{Det.} = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix} \quad (\text{by } R_2 + R_1, R_3 + R_1)$$

$= 2\,(a^2b^2c^2 + a^2b^2c^2) = 4a^2b^2c^2$

Example 8 :

The determinant $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$ is equal to

(A) $2(3abc - a^3 - b^3 - c^3)$ (B) $2(a^3 + b^3 + c^3 - 3abc)$

(C) $2\,(a^3 + b^3 + c^3 + 3abc)$ (D) $3abc - \sum a^3$

Sol. **(A).** $C_1 : C_1 + C_2 + C_3$ gives,

$$D = \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

Taking $2(a + b + c)$ as common factor and then $R_2 : R_2 - R_1$ and $R_3 : R_3 - R_1$. gives

$$D = 2\,(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & a-b & b-c \\ 0 & a-c & b-a \end{vmatrix}$$

$= 2\,(a + b + c)\,[-(a-b)^2 - (b-c)\,(a-c)]$
$= -2(a+b+c)\,\{a^2 + b + + c^2 - ab - bc - ca\}$
$= -2(a^3 + b^3 + c^3 - 3\,abc)$

Example 9 :

If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then find the value of $|A'\,A^{-1}|$

Sol. $A' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix},$$

$$A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix} \Rightarrow |A'A^{-1}| = 1$$

Example 10 :

Find the number of positive integral solutions of the equation $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11.$

Sol. $$\text{LHS} = \begin{vmatrix} x^3 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} + \begin{vmatrix} 1 & x^2y & x^2z \\ 0 & y^3+1 & y^2z \\ 0 & yz^2 & z^3+1 \end{vmatrix}$$

$$= x\begin{vmatrix} x^2 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix} + (y^3+1)\,(z^3+1) - y^3 z^3 = x^3 + y^3 + z^3 + 1$$

As $10 = 2^3 + 1^3 + 1^3$, the solutions are (2,1,1), (1,2, 1), (1, 1, 2).

Example 11 :

Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if –

(A) The first column of M is the transpose of the second row of M.

(B) The product of entries in the main diagonal of M is not the square of an integer.

(C) M is a diagonal matrix with nonzero entries in the main diagonal.

(D) Both (B) and (C)

Sol. **(D).** Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where $a, b, c \in I$

For invertible matrix, $\det(M) \neq 0 \Rightarrow ac - b^2 \neq 0$
i.e. $ac \neq b^2$

Example 12 :

Let M and N be two 3 × 3 matrices such that MN = NM. $\neq N$ and $M^2 = N^4$, then

(A) Determinant of $(M^2 + MN^2)$ is 0

(B) There is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)\,U$ is the zero matrix.

(C) Determinant of $(M^2 + MN^2) \geq 1$

(D) Both (A) and (B)

Sol. **(D).** $M^2 - N^4 = 0 \Rightarrow (M - N^2)(M + N^2) = 0$

$M - N^2 = 0$ not Possible

$M + N^2 = 0\,;\ |M + N^2| = 0$

$M - N^2 \neq 0\,;\ |M - N^2| = 0$

In any case $|M + N^2| = 0$

(A) $(|M^2 + MN^2| = |M|\,|M + N^2| = 0$

(B) If $|A| = 0$ then $AU = 0$ will have solution.

Thus $(M^2 + MN^2)\,U = 0$ will have many 'U'.

Example 13 :

If α is a characteristic root of a non-singular matrix, then prove that $\left[\dfrac{A}{\alpha}\right]$ is a characteristic root of adj A.

Sol. Since a is a characteristic root of a non-singular matrix, therefore $a \neq 0$. Also a is a characteristic root of A implies that there exists a non-zero vector X such that

$AX = \alpha X$

$\Rightarrow$ (adj A) (AX) = (adj A) (αX)

$\Rightarrow$ [(adj A) A] X = α(adj A) X

$\Rightarrow$ |A| IX = α(adj A)X [$\because$ (adj) A] A = |A| I]

$$\Rightarrow |A|\,X = \alpha(\text{adj } A)X \Rightarrow \frac{|A|}{\alpha}X = (\text{adj } A)X$$

$$\Rightarrow (\text{adj } A)X = \frac{|A|}{\alpha}X$$

Since X is a non-zero vector, therefore $\left[\dfrac{A}{\alpha}\right]$ is a characteristic root of the matrix adj A.

Example 14 :

Solve the following system of equations, using matrix method : $x + 2y + z = 7,\ x + 3z = 11,\ 2x - 3y = 1$.

Sol. The given system of equation is

$x + 2y + z = 7, \quad x + 0y + 3z = 11,\ 2x - 3y + 0z = 1$

$$\text{or} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} \quad \text{or} \quad AX = B, \text{ where}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\text{Now,} \quad |A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 18$$

So, the given system of equation has a unique solution given by $X = A^{-1}B$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \quad X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 - 33 + 6 \\ 42 - 22 - 2 \\ -21 + 77 - 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \Rightarrow x = 2, y = 1, z = 3$$

Example 15 :

If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|A|\ \ |\text{adj } A|$.

Sol. $|A|\ |\text{adj } A| = |A \text{ adj } A| = |\,|A|\,I\,|$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3 = (a^3)^3 = a^9$$

Example 16 :

By the method of matrix inversion, solve the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

Sol.
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \ AX = B$ …(i)

Clearly $|A| = -4 \neq 0$. Therefore

$$\text{adj } A = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From equation (i),

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow x_1 = 1,\ x_2 = 3,\ x_3 = 5$ or $y_1 = -1,\ y_2 = 2,\ y_3 = 1$

Example 17 :

If A, B and C are $n \times n$ matrix and det(A) = 2, det(B) = 3 and det (C) = 5, then find the value of the det (A^2BC^{-1}).

Sol. Given that $|A| = 2,\ |B| = 3,\ |C| = 5.$

$$\det(A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\times 3}{5} = \frac{12}{5}$$

Example 18 :

Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$.

Find (i) without finding B^{-1}, the value of K for which $KA = 2B^{-1} + I = 0$, (ii) without finding A^{-1}, the matrix X satisfying $A^{-1}XA = B$.

Sol. (i) $AB = B^{-1} \Rightarrow AB^2 = I$

$KA - 2B^{-1} + I = O \Rightarrow KAB - 2B^{-1}B + IB = O$

$\Rightarrow KAB - 2I + B = O \Rightarrow KAB^2 - 2B + B^2 = O$

$\Rightarrow KI - 2B + B^2 = O$

$$\Rightarrow K\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} K-2 & 0 \\ 0 & K-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow K = 2$$

(ii) $A^{-1}XA = B$

$\Rightarrow AA^{-1}XA = AB \Rightarrow IXA = AB \Rightarrow XAB = AB^2$

$\Rightarrow XAB = I \Rightarrow XAB^2 = B \Rightarrow XI = B \Rightarrow X = B$

QUESTION BANK

CHAPTER 3 : MATRICES AND DETERMINANTS

EXERCISE - 1 [LEVEL-1]

PART - 1 - MATRICES

Q.1 If I is a unit matrix, then 3I will be
(A) A unit matrix (B) A triangular matrix
(C) A scalar matrix (D) None of these

Q.2 If A is a symmetric matrix, then matrix M'AM is
(A) Symmetric (B) Skew-symmetric
(C) Hermitian (D) Skew-Hermitian

Q.3 If A is a square matrix, then $A + A^T$ is
(A) Non singular matrix (B) Symmetric matrix
(C) Skew-symmetric matrix (D) Unit matrix

Q.4 If A is a square matrix satisfying the equation $A^2 - 4A - 5I = 0$ then A^{-1} is equal to –
(A) $A - 4I$ (B) $\frac{1}{3}(A - 4I)$
(C) $\frac{1}{4}(A - 4I)$ (D) $\frac{1}{5}(A - 4I)$

Q.5 If $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, then A^n (where $n \in N$) equals
(A) $\begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & n^2a \\ 0 & 1 \end{pmatrix}$
(C) $\begin{pmatrix} 1 & na \\ 0 & 0 \end{pmatrix}$ (D) $\begin{pmatrix} n & na \\ 0 & n \end{pmatrix}$

Q.6 If A and B are two square matrices of the same order such that AB = B and BA = A then $A^2 + B^2$ is always equal to
(A) I (B) A + B
(C) 2 AB (D) 2 BA

Q.7 Inverse of a diagonal non-singular matrix is –
(A) diagonal matrix
(B) scalar matrix
(C) skew symmetric matrix
(D) zero matrix

Q.8 If the multiplicative group of 2×2 matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, for $a \neq 0$ and $a \in R$, then the inverse of $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$
(A) $\begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$ (B) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$
(C) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (D) Does not exist

Q.9 If $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -2 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}$ then $P_{22} =$
(A) 40 (B) – 40
(C) – 20 (D) 20

Q.10 If $A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}, B = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}$ then
(A) AB = BA (B) AB ≠ BA
(C) AB = – BA (D) None of these

Q.11 If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and a and b are arbitrary constants then – $(aI + bA)^2 =$
(A) $a^2I + abA$ (B) $a^2I + 2abA$
(C) $a^2I + b^2A$ (D) None of these

Q.12 If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is square root of identity matrix of order 2 then –
(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 + \alpha^2 - \beta\gamma = 0$
(C) $1 - \alpha^2 + \beta\gamma = 0$ (D) $\alpha^2 + \beta\gamma = 1$

Q.13 If $A = \begin{bmatrix} 2 & 2 \\ a & b \end{bmatrix}$ and $A^2 = O$, then (a, b) =
(A) (–2, –2) (B) (2, –2)
(C) (–2, 2) (D) (2, 2)

Q.14 If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then AA' =
(A) 14 (B) $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$
(C) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (D) None

Q.15 If $A = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 17 \\ 0 & -10 \end{bmatrix}$ then |AB| is equal to
(A) 80 (B) 100
(C) –110 (D) 92

Q.16 If A is non singular matrix, then –
(A) $|A^{-1}| = |A|$ (B) $|A^{-1}| = A^{-1}$
(C) $|A^{-1}| = 0$ (D) $|A^{-1}| = 1/|A|$

Q.17 If A is a 3×3 nonsingular matrix and if |A| = 3, then $|(2A)^{-1}| =$
(A) 24 (B) 3
(C) 1/3 (D) 1/24

Q.18 If $\begin{bmatrix} 1 & 2 & -1 \\ 1 & x-2 & 1 \\ x & 1 & 1 \end{bmatrix}$ is singular, then the value of x is –

(A) 0 (B) 1
(C) 3 (D) 2

Q.19 If A and B are symmetric matrices of the same order, then which one of the following is NOT true?
(A) AB – BA is symmetric (B) AB + BA is symmetric
(C) A – B is symmetric (D) A + B is symmetric

Q.20 If A and B are square matrices of order 'n' such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be true?
(A) Either of A or B is zero matrix
(B) A = B
(C) AB = BA
(D) Either of A or B is an identity matrix

Q.21 If the matrix $\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} = A + B$, where A is symmetric and B is skew symmetric, then B =

(A) $\begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Q.22 If A is 3×4 matrix and B is a matrix such that A'B and BA' are both defined, then B is of the type
(A) 4×4 (B) 3×4
(C) 4×3 (D) 3×3

Q.23 The symmetric part of the matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{pmatrix}$ is

(A) $\begin{pmatrix} 0 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 8 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

Q.24 If A is a matrix of order 3, such that A (adj A) = 10 I, then | adj A | =
(A) 1 (B) 10
(C) 100 (D) 10 I

Q.25 The inverse of the matrix is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is

(A) $\frac{1}{24}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(C) $\frac{1}{24}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

PART - 2 - DETERMINANTS

Q.26 If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, then find the value of | A |

(A) 2 (B) 3
(C) 1 (D) 10

Q.27 Solve the equation $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$

for value of x
(A) –11/97 (B) 10/97
(C) –8/97 (D) –3/97

Q.28 If $A = \begin{vmatrix} \sin(\theta+\alpha) & \cos(\theta+\alpha) & 1 \\ \sin(\theta+\beta) & \cos(\theta+\beta) & 1 \\ \sin(\theta+\gamma) & \cos(\theta+\gamma) & 1 \end{vmatrix}$, then

(A) A = 0 for all θ
(B) A is an odd Function of θ
(C) A = 0 for $\theta = \alpha + \beta + \gamma$
(D) A is independent of θ

Q.29 The parameter on which the value of the determinant

$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend upon

(A) a (B) p
(C) d (D) x

Q.30 For all values of A, B, C and P, Q, R the value of

$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$ is

(A) 0 (B) cos A cos B cos C
(C) sin A sin B sin C (D) cos P cos Q cos R

Q.31 If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$

then f (100) is equal to
(A) 0 (B) 1
(C) 100 (D) – 100

Q.32 Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$,

then $\Delta_1 \times \Delta_2$ can be expressed as the sum of how many determinants

(A) 9 (B) 3
(C) 27 (D) 2

Q.33 If $C = 2\cos\theta$, then the value of the determinant

$$\Delta = \begin{vmatrix} C & 1 & 0 \\ 1 & C & 1 \\ 6 & 1 & C \end{vmatrix} \text{ is}$$

(A) $\frac{\sin 4\theta}{\sin\theta}$ (B) $\frac{2\sin^2 2\theta}{\sin\theta}$

(C) $4\cos^2\theta(2\cos\theta - 1)$ (D) None of these

Q.34 If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

(A) $\Delta_1 = 3(\Delta_2)^2$ (B) $\frac{d}{dx}(\Delta_1) = 3\Delta_2$

(C) $\frac{d}{dx}(\Delta_1) = 2(\Delta_2)^2$ (D) $\Delta_1 = 3\Delta_2^{3/2}$

Q.35 Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b+c-a & c+a-b & a+b-c \end{vmatrix}$

(A) 2 (B) 3
(C) 0 (D) 4

Q.36 Find the value of the determinant $\begin{vmatrix} 19 & 6 & 7 \\ 21 & 3 & 15 \\ 28 & 11 & 6 \end{vmatrix}$.

(A) 2 (B) 3
(C) 1 (D) 0

Q.37 If x, y, z are unequal and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then find the value of xyz.

(A) 2 (B) –1
(C) 4 (D) – 4

Q.38 If in the multiplication of $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ and $\begin{vmatrix} c & d \\ -d & c \end{vmatrix}$, A, B are the elements of the first row then the elements of the second row will be

(A) – B, A (B) A, B
(C) B, A (D) – B, – A

Q.39 If $\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = 125$, then find the value of $\begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix}$

(A) 12 (B) 22
(C) 10 (D) 25

Q.40 If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & x \end{vmatrix}$, then find $f(2x) - f(x)$ equals –

(A) $a(2a+3x)$ (B) $ax(2x+3a)$
(C) $ax(2a+3x)$ (D) $x(2a+x)$

Q.41 If $f(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $(f(\alpha)^{-1}) =$

(A) $f(\alpha)$ (B) $f(-\alpha)$
(C) $f(0)$ (D) None of these

Q.42 If $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0, (a \neq 0)$ then –

(A) a is one of cube root of unity
(B) b is one of cube root of unity
(C) (a/b) is one of cube root of unity
(D) (b/a) is one of cube root of –1

Q.43 $\begin{vmatrix} 1+i & 1-i & i \\ 1-i & i & 1+i \\ i & 1+i & 1-i \end{vmatrix} =$

(A) $-4-7i$ (B) $4+7i$
(C) $3+7i$ (D) $7+4i$

Q.44 $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} =$

(A) 2 (B) – 2
(C) $x^2 - 2$ (D) None of these

Q.45 If a, b, c are unequal what is the condition that the value of the following determinant is zero

$$\Delta = \begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix}$$

(A) $1 + abc = 0$ (B) $a + b + c + 1 = 0$
(C) $(a-b)(b-c)(c-a) = 0$ (D) None of these

Q.46 If $p\lambda^4+q\lambda^3+r\lambda^2+s\lambda+t=\begin{vmatrix}\lambda^2+3\lambda & \lambda-1 & \lambda+3\\ \lambda+1 & 2-\lambda & \lambda-4\\ \lambda-3 & \lambda+4 & 3\lambda\end{vmatrix}$, the value of t is
(A) 16 (B) 18
(C) 17 (D) 19

Q.47 If $\begin{vmatrix}x^2+x & x+1 & x-2\\ 2x^2+3x-1 & 3x & 3x-3\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}=Ax-12$, then the value of A is
(A) 12 (B) 24
(C) –12 (D) – 24

Q.48 $\begin{vmatrix}a^2 & b^2 & c^2\\ (a+1)^2 & (b+1)^2 & (c+1)^2\\ (a-1)^2 & (b-1)^2 & (c-1)^2\end{vmatrix}=$

(A) $4\begin{vmatrix}a^2 & b^2 & c^2\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}$ (B) $3\begin{vmatrix}a^2 & b^2 & c^2\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}$

(C) $2\begin{vmatrix}a^2 & b^2 & c^2\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}$ (D) None of these

Q.49 $2\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2-bc & b^2-ac & c^2-ab\end{vmatrix}=$
(A) 0 (B) 1
(C) 2 (D) 3abc

Q.50 If $\begin{vmatrix}-a^2 & ab & ac\\ ab & -b^2 & bc\\ ac & bc & -c^2\end{vmatrix}=Ka^2b^2c^2$, then K =
(A) – 4 (B) 2
(C) 4 (D) 8

Q.51 If $a\ne 6, b, c$ satisfy $\begin{vmatrix}a & 2b & 2c\\ 3 & b & c\\ 4 & a & b\end{vmatrix}=0$, then abc =
(A) a + b + c (B) 0
(C) b^3 (D) ab + bc

Q.52 If $\begin{vmatrix}a & b & a+b\\ b & c & b+c\\ a+b & b+c & 0\end{vmatrix}=0$; then a, b, c are in
(A) A. P. (B) G. P.
(C) H. P. (D) None of these

Q.53 $\begin{vmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{vmatrix}=$
(A) $3\sqrt{3}i$ (B) $-3\sqrt{3}i$
(C) $i\sqrt{3}$ (D) 3

Q.54 The value of $\begin{vmatrix}1 & 1 & 1\\ bc & ca & ab\\ b+c & c+a & a+b\end{vmatrix}$ is
(A) 1 (B) 0
(C) (a – b) (b – c) (c – a) (D) (a + b) (b + c) (c + a)

Q.55 If $\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=k(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$, then k =
(A) 1 (B) 2
(C) – 1 (D) – 2

Q.56 Evaluate $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\ \sin 75^\circ & \cos 75^\circ\end{vmatrix}$
(A) 1 (B) 0
(C) 2 (D) 3

Q.57 Let $D_1=\begin{vmatrix}a & b & a+b\\ c & d & c+d\\ a & b & a-b\end{vmatrix}$ and $D_2=\begin{vmatrix}a & c & a+c\\ b & d & b+d\\ a & c & a+b+c\end{vmatrix}$ then the value of $\frac{D_1}{D_2}$ where $b\ne 0$ and $ad\ne bc$, is
(A) – 2 (B) 0
(C) – 2b (D) 2b

Q.58 The value of the determinant $\begin{vmatrix}1 & 0 & -1\\ x & 1 & 1-x\\ y & x & 1+x-y\end{vmatrix}$ depends on
(A) only x (B) only y
(C) both x and y (D) neither x nor y

Q.59 If ω is an imaginary cube root of unity, then the value of $\begin{vmatrix}1 & \omega^2 & 1-\omega^4\\ \omega & 1 & 1+\omega^5\\ 1 & \omega & \omega^2\end{vmatrix}$ is –

(A) 4 (B) ω^2
(C) $\omega^2 - 4$ (D) -4

Q.60 If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then $\alpha =$

(A) ± 1 (B) ± 2
(C) ± 3 (D) ± 5

Q.61 If the determinant of the adjoint of a (real) matrix of order 3 is 25, then the determinant of the inverse of the matrix is
(A) 0.2 (B) ± 5
(C) $\frac{1}{\sqrt[5]{625}}$ (D) ± 0.2

Q.62 Consider the following statements:
(a) If any two rows or columns of a determinant are identical, then the value of the determinant is zero.
(b) If the corresponding rows and columns of a determinant are interchanged, then the value of determinant does not change.
(c) If any two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign.
Which of these are correct?
(A) (a) and (c) (B) (a) and (b)
(C) (a), (b) and (c) (D) (b) and (c)

PART - 3 - APPLICATION OF DETERMINANTS

Q.63 If the system of equation : $x + 2ay + az = 0; x + 3by + bz = 0; x + 4cy + cz = 0$ has non-zero solution then a, b, c are in–
(A) A.P. (B) G.P.
(C) H.P. (D) Satisfy at $a + 2b + 3c = 0$

Q.64 The system of equations $\alpha x + y + z = \alpha - 1$, $x + \alpha y + z = \alpha - 1$, $x + y + \alpha z = \alpha - 1$ has no solution if $\alpha =$
(A) -2 (B) $\alpha \neq -2$
(C) either -2 or 1 (D) $\alpha = 1$

Q.65 If $x + y - z = 0, 3x - \alpha y - 3z = 0, x - 3y + z = 0$ has non zero solution, then $\alpha =$
(A) -1 (B) 0
(C) 1 (D) -3

Q.66 The value of k for which the set of equations
$x + ky + 3z = 0,\ 3x + ky - 2z = 0,\ 2x + 3y - 4z = 0$
has a non trivial solution over the set of rationals is
(A) 15 (B) 31/2
(C) 16 (D) 33/2

Q.67 The equation $x + 2y + 3z = 1, 2x + y + 3z = 2, 5x + 5y + 9z = 4$ have –
(A) Unique solution
(B) Infinitely many solutions
(C) Inconsistent
(D) None of these

EXERCISE - 2 [LEVEL-2]

Q.1 If $f(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and if α, β, γ, are angle of a triangle, then $f(\alpha). f(\beta). f(\gamma)$ equals
(A) I_2 (B) $-I_2$
(C) 0 (D) None of these

Q.2 If $k \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is an orthogonal matrix then $k =$
(A) 1/3 (B) 1/2
(C) 1/4 (D) 1/16

Q.3 If $A = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}$ then which statement is true ?
(A) $AA^T = I$ (B) $BB^T = I$
(C) $AB \neq BA$ (D) $(AB)^T = I$

Q.4 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then $A^2 - 4A =$
(A) 3I (B) 4I
(C) 5I (D) None of these

Q.5 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ then which of the following statements is true –
(A) $AB = BA$ (B) $A^2 = B$
(C) $(AB)^T = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$ (D) None of these

Q.6 $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$ is equal to -
(A) $xyz(x - y)(y - z)(z - x)$ (B) $\frac{xyz}{6}(x - y)(y - z)(z - x)$
(C) $\frac{xyz}{12}(x - y)(y - z)(z - x)$ (D) None of these

Q.7 If $\Delta_r = \begin{vmatrix} r-1 & n & 6 \\ (r-1)^2 & 2n^2 & 4n-2 \\ (r-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$ then find $\sum_{r=1}^{n} \Delta_r$.
(A) 0 (B) 3
(C) 1 (D) 4

Q.8 If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then $k =$
(A) 2 (B) 3
(C) 1 (D) 4

Q.9 $f(x)=\begin{vmatrix} 5+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 5+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 5+4\sin 2x \end{vmatrix}$ then –

(A) domain of function $f(x) \in (0, \infty)$
(B) domain of function $f(x) \in (-\infty, 0)$
(C) Range of function $f(x)$ is $[50, 100]$
(D) Period of function $f(x)$ is π

Q.10 If α, β & γ are real numbers, then

$$D=\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} =$$

(A) -1
(B) $\cos\alpha\ \cos\beta\ \cos\gamma$
(C) $\cos\alpha+\cos\beta+\cos\gamma$
(D) zero

Q.11 Find the number of real roots of the equation

$$\begin{vmatrix} x^2-12 & -18 & -5 \\ 10 & x^2+2 & 1 \\ -2 & 12 & x^2 \end{vmatrix} = 0$$

(A) 2
(B) 3
(C) 1
(D) 4

Q.12 If $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $\lim\limits_{n\to\infty} \frac{1}{n}A^n$ is –

(A) a null matrix
(B) an identity matrix
(C) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(D) None of these

Q.13 If $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then the value of $|A'\ A^{-1}|$

(A) 2
(B) 1
(C) 4
(C) 3

Q.14 If a_1, a_2, a_3, ... are positive numbers in G.P. then the value of $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+1} & \log a_{n+2} & \log a_{n+3} \\ \log a_{n+2} & \log a_{n+3} & \log a_{n+4} \end{vmatrix}$ is –

(A) 1
(B) 4
(C) 3
(D) 0

Q.15 If $\Delta_r=\begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$ then $\sum\limits_{r=0}^{m} \Delta_r =$

(A) 0
(B) 4
(C) 3
(D) 1

Q.16 If $\begin{vmatrix} y+z & x-z & x-y \\ y-z & z-x & y-x \\ z-y & z-x & x+y \end{vmatrix} = k\,xyz$, then the value of k is

(A) 2
(B) 4
(C) 6
(D) 8

Q.17 If $A=\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ and $A^n = O$, minimum value of n is

(A) 2
(B) 3
(C) 4
(D) 5

Q.18 If $A=\begin{bmatrix} i & 0 \\ 0 & i/2 \end{bmatrix}$ $(i=\sqrt{-1})$, then $A^{-1}=$

(A) $\begin{bmatrix} i & 0 \\ 0 & i/2 \end{bmatrix}$
(B) $\begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}$
(C) $\begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}$
(D) $\begin{bmatrix} 0 & i \\ 2i & 0 \end{bmatrix}$

Q.19 If $A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}$, then the value of $|\text{adj } A|$ is

(A) 36
(B) 72
(C) 144
(D) None of these

Q.20 If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the equation $x^2-(a+d)x+k=0$, then

(A) $k = bc$
(B) $k = ad$
(C) $k = a^2+b^2+c^2+d^2$
(D) $k = ad-bc$

Q.21 Given $A=\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$; $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $A-\lambda I$ is a singular matrix then

(A) $\lambda \in \phi$
(B) $\lambda^2-3\lambda-4=0$
(C) $\lambda^2+3\lambda+4=0$
(D) $\lambda^2-3\lambda-6=0$

Q.22 Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta < 2\pi$, then

(A) Det (A) = 0
(B) Det A $\in (0, \infty)$
(C) Det (A) $\in [2, 4]$
(D) Det A $\in [2, \infty)$

Q.23 If $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1}=\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then

(A) $a = 1, c = -1$
(B) $a = 2, c = -1/2$
(C) $a = -1, c = 1$
(D) $a = 1/2, c = 1/2$

Q.24 If $f(r)=\begin{vmatrix} n(2n+1) & 2n+1 & 6n(n+1)r^2 \\ n+1 & 2n+2 & 2n(n+1)r \\ n & 2n+1 & 4r^3 \end{vmatrix}$

find value of $\sum\limits_{1}^{n} f(r)$.

(A) $2n^3(n+1)^2(2n+1)$
(B) $2n^3(n+1)^2(2n-1)$
(C) $n^3(n+1)^2(2n+1)$
(D) $2n^3(n-1)^2(2n-1)$

Q.25 If a, b, c are real then the value of determinant

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 \text{ if}$$

(A) $a+b+c=0$ (B) $a+b+c=1$
(C) $a+b+c=-1$ (D) $a=b=c=0$

Q.26 Let $a = \lim_{x\to 1} \frac{x}{\ln x} - \frac{1}{x\ln x}$; $b = \lim_{x\to 0} \frac{x^3-16x}{4x+x^2}$;

$c = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x}$ & $d = \lim_{x\to -1} \frac{(x+1)^3}{3(\sin(x+1)-(x+1))}$

then the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

(A) Idempotent (B) Involutary
(C) Non singular (D) Nilpotent

Q.27 Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ and suppose that det.(A) = 2 then

the det.(B) equals, where $B = \begin{bmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{bmatrix}$

(A) –2 (B) – 8
(C) –16 (D) 8

Q.28 The characteristic equation of a matrix A is $\lambda^3-5\lambda^2-3\lambda+2=0$ then $|\text{adj}(A)|$
(A) 4 (B) 9
(C) 25 (D) 21

Q.29 If $A = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$ and $B = \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix}$, then $\frac{dA}{dx} =$

(A) 3B + 1 (B) 3B
(C) –3B (D) 1 – 3B

Q.30 If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$ then the solution of the equation

$$\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix} = 0, \text{ is}$$

(A) $\frac{a+b+c}{3}$ (B) 1
(C) $\frac{a+b+c}{2}$ (D) $\sqrt[3]{abc}$

Q.31 Suppose a_1, a_2, ... real numbers, with $a_1 \neq 0$.
If a_1, a_2, a_3, ...are in A.P. then

(A) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular

(B) The system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has infinite number of solutions

(C) $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non singular ; where $i = \sqrt{-1}$

(D) All of these

Q.32 If a determinant of order 3 × 3 is formed by using the numbers of 1 or – 1 then minimum value of determinant is
(A) –2 (B) – 4
(C) 0 (D) – 8

Q.33 If A is a square matrix of order 3 such that |A| = 2 then $|(\text{adj } A^{-1})^{-1}|$ is –
(A) 1 (B) 2
(C) 3 (D) 4

Q.34 If $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ is the sum of a symmetric matrix B and skew symmetric matrix C, then B is –

(A) $\begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 & -2 \\ -2 & 5 & -2 \\ 2 & 2 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 6 & 7 \\ -6 & 2 & -5 \\ -7 & 5 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 6 & -2 \\ 2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

Q.35 If $\begin{vmatrix} (1+x) & (1+x)^2 & (1+x)^3 \\ (1+x)^4 & (1+x)^5 & (1+x)^6 \\ (1+x)^7 & (1+x)^8 & (1+x)^9 \end{vmatrix} = a_0 + a_1x + a_2x^2 + ...$

then a_1 is equal to –
(A) 1 (B) 2
(C) 3 (D) 0

Q.36 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ then $(A(\text{adj } A)A^{-1})A =$

(A) $2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1/6 & -1/6 \\ 2/6 & 1/6 & 3/6 \\ 3/6 & 2/6 & 1/6 \end{bmatrix}$ (D) None of these

Q.37 The value of $\Delta = \begin{vmatrix} 1 & 1+ac & 1+bc \\ 1 & 1+ad & 1+bd \\ 1 & 1+ae & 1+be \end{vmatrix}$ is –

(A) 1 (B) 0
(C) 3 (D) a + b + c

Q.38 If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax - 12$, then the value of A is –

(A) 12 (B) 24
(C) – 12 (D) – 24

Q.39 If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ & B = (adjA) and C = 5A then $\frac{|\text{adj } B|}{|C|} =$

(A) 5 (B) 25
(C) –1 (D) 1

Q.40 For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

then $\left(\frac{D}{(n!)^3} - 4\right)$ is divisible by –

(A) n^2 (B) n
(C) n^3 (D) 3n

Q.41 If $A = \begin{bmatrix} a^i & b^i \\ b^i & a^i \end{bmatrix}$ and if | a | < 1, | b | < 1 then

$\sum_{i=1}^{\infty} \det(A_i)$ is equal to –

(A) $\frac{a^2}{(1-a^2)} - \frac{b^2}{(1-b^2)}$ (B) $\frac{a^2-b^2}{(1-a^2)(1-b^2)}$

(C) $\frac{a^2}{(1-a^2)} + \frac{b^2}{(1-b^2)}$ (D) $\frac{a}{1+a} - \frac{b}{1+b}$

Q.42 If A, B are symmetric matrices of the same order then (AB – BA) is –

(A) symmetric matrix (B) skew-symmetric matrix
(C) Diagonal matrix (D) Unit matrix

Q.43 If $P = \begin{pmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} -i & i \\ 0 & 0 \\ i & -i \end{pmatrix}$ then PQ =

(A) $\begin{pmatrix} 2 & -2 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$ (B) $\begin{pmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$

(C) $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Q.44 The value of determinant

$$\begin{vmatrix} \sin^2 13° & \sin^2 77° & \tan 135° \\ \sin^2 77° & \tan 135° & \sin^2 13° \\ \tan 135° & \sin^2 13° & \sin^2 77° \end{vmatrix}$$ is –

(A) –1 (B) 0
(C) 1 (D) 2

Directions : Assertion-Reason type questions.

(A) Statement- 1 is True, Statement-2 is true, statement-2 is a correct explanation for Statement -1.
(B) Statement -1 is True, Statement-2 is true; statement-2 is NOT a correct explanation for Statement - 1.
(C) Statement - 1 is True, Statement- 2 is False.
(D) Statement -1 is False, Statement -2 is True.

Q.45 **Statement 1 :** For a singular square matrix A, if $AB = AC \Rightarrow B = C$.
Statement 2: If | A | = 0 then A^{-1} does not exist.

Q.46 **Statement 1 :** If $f_1(x), f_2(x)$ ...., $f_9(x)$ are polynomials whose degree ≥ 1, where $f_1(\alpha) = f_2(\alpha)$ .... $= f_9(\alpha) = 0$

and $A(x) = \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_4(x) & f_5(x) & f_6(x) \\ f_7(x) & f_8(x) & f_9(x) \end{bmatrix}$ and $\frac{A(x)}{x-\alpha}$ is also a matrix of 3 × 3 whose entries are also polynomials.
Statement 2 : $x - \alpha$ is a factor of polynomial f (x) if $f(\alpha) = 0$.

Q.47 Let x, y, z are three integers lying between 1 and 9 such that x 51, y 41, z 31 are three digit numbers.
Statement 1 : The value of determinant

$$\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$$ is zero.

Statement 2 : The value of determinant is zero. If the entries any two rows (or columns) of the determinants are correspondingly proportional.

Q.48 Let $A = \begin{bmatrix} \sin\alpha & -\cos\alpha & 1 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Statement–1 : $A^{-1} = \text{adj}(A)$
Statement–2 : |A| = 1

Passage (Q.49-Q.51) :

Consider the determinant $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

M_{ij} = Minor of the element if i^{th} row and j^{th} column

C_{ij} = Cofactor of the element if i^{th} row and j^{th} column

Q.49 Value of $b_1.C_{31} + b_2.C_{32} + b_3.C_{33}$ is

(A) 0 (B) Δ
(C) 2Δ (D) Δ^2

Q.50 If all the elements of the determinants are multiplied by 2, then the value of new determinant is –

(A) 0 (B) 8Δ
(C) 2Δ (D) $2^9.\Delta$

Q.51 $a_3M_{13} - b_3.M_{23} + d_3M_{33}$ is equal to –

(A) 0 (B) 4Δ
(C) 2Δ (D) Δ

Passage (Q.52-Q.53) :

Let A and B are two matrices of same order 3 × 3 where

$A = \begin{bmatrix} 1 & -2 & 2 \\ 5 & k & 6 \\ 3 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 4 & 2 \\ 3 & 5 & 2 \end{bmatrix}$

Q.52 If matrix (A + 2B) is singular, then the value of K is –

(A) $\frac{7}{12}$ (B) $\frac{22}{13}$ (C) $\frac{35}{13}$ (D) $\frac{-35}{13}$

Q.53 If C = A – B and Tr (C) = 2, then K is equal to –

(A) 11 (B) 9
(C) 10 (D) 5

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.54 If the system of the equations : $x + y + 2z = 6$ .... (1), $x + 3y + 3z = 10$ .... (2) $x + 2y + \lambda z = \mu$ .... (3) has infinite number of solutions, then find the value of $4(\lambda + \mu)$.

Q.55 If α, β, γ are different from 1 and are the roots of $ax^3 + bx^2 + cx + d = 0$ and $(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta) = 25/2$ then

$-\frac{2(a+b+c+d)\Delta}{d}$ = ..., where

$\Delta = \begin{vmatrix} \frac{\alpha}{1-\alpha} & \frac{\beta}{1-\beta} & \frac{\gamma}{1-\gamma} \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$.

Q.56 Let the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ be a zero divisor of the polynomial $f(x) = x^2 - 4x - 5$. Find the sum of all the elements in the matrix A^3.

Q.57 A is an involutary matrix given by

$A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ then the inverse of $\frac{A}{2}$ is xA. Find the value of x.

Q.58 If $a^2 + b^2 + c^2 = -2$ and

$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$

then f (x) is a polynomial of degree

Q.59 For a non - zero, real a, b and c,

$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = \alpha\, abc,$

then the values of α is

Q.60 The number of positive integral solutions of the equation

$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$ is

Q.61 Let three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ then

$t_r(A) + t_r\left(\frac{ABC}{2}\right) + t_r\left(\frac{A(BC)^2}{4}\right) + t_r\left(\frac{A(BC)^3}{8}\right) + ... + \infty =$

Q.62 A is a 2 × 2 matrix such that

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The sum of the elements of A, is

Q.63 A matrix has 12 elements. Find the possible number of orders it can have.

Q.64 If matrices A and B satisfy AB = A, BA = B, $A^2 = kA$, $B^2 = \ell B$ and $(A + B)^3 = m(A + B)$, then find the value of $k + \ell + m$.

Q.65 Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$ is equal to

Q.66 Let k be a positive real number and let

$$A=\begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det (adj B) = 10^6, then [k] is equal to
[Note : adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

Q.67 Let M be a 3 × 3 matrix satisfying

$$M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\2\end{bmatrix}, M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ and } M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}.$$

Then the sum of the diagonal entries of M is

Q.68 The total number of distinct x ∈ R for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix}=10 \text{ is} -$$

Q.69 Let $P=\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If Q = [q_{ij}] is a matrix such that P^{50} – Q = I, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals:

Q.70 For a real number α, if the system $\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$ of linear equations, has infinitely many solutions, then $1+\alpha+\alpha^2=$

Q.71 How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is 5 ?

Q.72 If a, b, c are in A.P., then the determinants

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \text{ is} -$$

EXERCISE - 3 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 If $A=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, then $A^2 = B^2 = C^2 =$ **[AIEEE 2002]**

(A) I^2 (B) I
(C) – I (D) 2 I

Q.2 If $A=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$, $B=\begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$ then AB =

(A) I_3 (B) $2I_3$ **[AIEEE 2002]**
(C) $4I_3$ (D) $18I_3$

Q.3 If p^{th}, q^{th}, r^{th} term of a GP are ℓ, m, n then the value of $\begin{vmatrix} \log \ell & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ is equal to- **[AIEEE 2002]**

(A) 0 (B) 1
(C) $\ell + m + n$ (D) None of these

Q.4 If 1, ω, ω^2 are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$ is equal to – **[AIEEE 2003]**

(A) ω^2 (B) 0
(C) 1 (D) ω

Q.5 If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2)$, $(1, b, b^2)$ & $(1, c, c^2)$ are non-coplanar, then the product abc equals-

(A) 0 (B) 2 **[AIEEE 2003]**
(C) – 1 (D) 1

Q.6 If $A=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2=\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then **[AIEEE 2003]**

(A) $\alpha = 2ab, \beta = a^2 + b^2$
(B) $\alpha = a^2 + b^2, \beta = ab$
(C) $\alpha = a^2 + b^2, \beta = 2ab$
(D) $\alpha = a^2 + b^2, \beta = a^2 - b^2$

Q.7 Let $A=\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The only correct statement about the matrix A is- **[AIEEE 2004]**

(A) A is a zero matrix
(B) A = (– 1) I, where I is a unit matrix
(C) A^{-1} does not exist
(D) $A^2 = I$

Q.8 Let $A=\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and $(10)B=\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$. If B is the inverse of matrix A, then α is- **[AIEEE 2004]**

(A) – 2 (B) – 1
(C) 2 (D) 5

Q.9 If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in G.P., then the value of the determinant $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$, is-

(A) 0 (B) 1 **[AIEEE 2005]**
(C) 2 (D) – 2

Q.10 The system equations $\alpha x + y + z = \alpha - 1$, $x + \alpha y + z = \alpha - 1$, $x + y + \alpha z = \alpha - 1$ has no solution, if α is - **[AIEEE 2005]**

(A) –2 (B) either –2 or 1
(C) not – 2 (D) 1

Q.11 If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then f(x) is a polynomial of degree - **[AIEEE 2005]**

(A) 1 (B) 0
(C) 3 (D) 2

Q.12 If $A^2 - A + I = 0$, then the inverse of A is - **[AIEEE-2005]**

(A) A + I (B) A
(C) A – I (D) I – A

Q.13 If $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds for all n ≥ 1, by the principle of mathematical induction - **[AIEEE-2005]**

(A) $A^n = nA - (n-1)I$ (B) $A^n = 2^{n-1}A - (n-1)I$
(C) $A^n = nA + (n-1)I$ (D) $A^n = 2^{n-1}A + (n-1)I$

Q.14 If A and B are square matrices of size n × n such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true– **[AIEEE 2006]**

(A) AB = BA
(B) Either of A or B is a zero matrix
(C) Either of A or B is an identity matrix
(D) A = B

Q.15 Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, a, b ∈ N. Then – **[AIEEE 2006]**

(A) there exist more than one but finite number of B's such that AB = BA
(B) there exist exactly one B such that AB = BA
(C) there exist infinitely many B's such that AB = BA
(D) there cannot exist any B such that AB = BA

Q.16 Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$ If $|A^2| = 25$, then $|\alpha|$ equals-

(A) 5^2 (B) 1 **[AIEEE 2007]**
(C) 1/5 (D) 5

Q.17 If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0, y \neq 0$ then D is

(A) divisible by neither x nor y **[AIEEE 2007]**
(B) divisible by both x and y
(C) divisible by x but not y
(D) divisible by y but not x

Q.18 Let A be a square matrix all of whose entries are integers. Then which one of the following is true ? **[AIEEE 2008]**
(A) If det A ≠ ± 1, then A^{-1} exists and all its entries are non-integers.
(B) If det A = ± 1, then A^{-1} exists and all its entries are integers.
(C) If det A = ± 1, then A^{-1} need not exist.
(D) If det A = ± 1, then A^{-1} exists but all its entries are not necessarily integers.

Q.19 Let A be a 2 × 2 matrix with real entries. Let I be the 2 × 2 identity matrix. Denote by tr (A), the sum of diagonal entries of A, Assume that $A^2 = I$. **[AIEEE 2008]**
Statement- 1: If A ≠ I and A ≠ – I, then det A = – 1
Statement -2 : If A ≠ I and A ≠ –I, then tr (A) ≠ 0
(A) Statement-1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is true, Statement -2 is true; Statement-2 is not a correct explanation for Statement-1.
(C) Statement-1 is true, Statement -2 is false.
(D) Statement-1 is false, Statement-2 is true.

Q.20 Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that x = cy + bz, y = az + cx, and z = bx + ay. Then $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE 2008]**
(A) –1 (B) 0
(C) 1 (D) 2

Q.21 Let A be a 2 × 2 matrix **[AIEEE 2009]**
Statement- 1: adj (adj A) = A
Statement -2 : |adj A| = |A|
(A) Statement-1 is true, Statement -2 is true; Statement-2 is not a correct explanation for Statement-1.
(B) Statement-1 is true, Statement -2 is false.
(C) Statement-1 is false, Statement -2 is true.
(D) Statement-1 is true, Statement-2 is true Statement-2 is a correct explanation for Statement-1.

Q.22 Let a, b, c be such that b(a+c) ≠ 0. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} =$$

0, then the value of n is : **[AIEEE 2009]**
(A) any even integer (B) any odd integer
(C) any integer (D) zero

Q.23 The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is – **[AIEEE 2010]**
(A) 5 (B) 6
(C) at least 7 (D) less than 4

Q.24 Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix. Define Tr(A) = sum of diagonal elements of A and |A| = determinant of matrix A. **[AIEEE 2010]**
Statement-1: Tr (A) = 0
Statement-2: |A| = 1
(A) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is false
(C) Statement-1 is false, Statement-2 is true
(D) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1.

Q.25 Consider the system of linear equations:
$x_1 + 2x_2 + x_3 = 3$; $2x_1 + 3x_2 + x_3 = 3$; $3x_1 + 5x_2 + 2x_3 = 1$
The system has **[AIEEE 2010]**
(A) exactly 3 solutions
(B) a unique solution
(C) no solution
(D) infinite number of solutions

Q.26 Let A and B be two symmetric matrices of order 3. **[AIEEE 2011]**
Statement-1 : A (BA) and (AB) A are symmetric matrices.
Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative.
(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is true; Statement-2 is true; Statement-2 is not a correct explanation for S-1.
(C) Statement-1 is true, Statement-2 is false.
(D) Statement-1 is false, Statement-2 is true.

Q.27 The number of values of k for which the linear equations
4x + ky + 2z = 0 ; kx + 4y + z = 0 ; 2x + 2y + z = 0
posses a non-zero solution is : **[AIEEE 2011]**
(A) 3 (B) 2
(C) 1 (D) zero

Q.28 Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to – **[AIEEE 2012]**

(A) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Q.29 Let P and Q be 3 × 3 matrices $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to : **[AIEEE 2012]**

(A) – 2 (B) 1
(C) 0 (D) – 1

Q.30 The number of values of k, for which the system of equations : $(k + 1)x + 8y = 4k$; $kx + (k + 3)y = 3k – 1$ has no solution, is – **[JEE MAIN 2013]**

(A) infinite (B) 1
(C) 2 (D) 3

Q.31 If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3 × 3 matrix A and $|A| = 4$, then α is equal to – **[JEE MAIN 2013]**

(A) 4 (B) 11
(C) 5 (D) 0

Q.32 If A is an 3 × 3 non-singular matrix such that AA' = A'A and $B = A^{-1}A'$, then BB' equals – **[JEE MAIN 2014]**

(A) I + B (B) I
(C) B^{-1} (D) $(B^{-1})'$

Q.33 If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$= K(1 – \alpha)^2 (1 – \beta)^2 (\alpha – \beta)^2$, then K is equal to – **[JEE MAIN 2014]**

(A) $\alpha\beta$ (B) $1/\alpha\beta$
(C) 1 (D) –1

Q.34 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3 × 3 identity matrix, then the ordered pair (a, b) is equal to **[JEE MAIN 2015]**

(A) (–2, 1) (B) (2, 1)
(C) (–2, –1) (D) (2, –1)

Q.35 The set of all values of λ for which the system of linear equations $2x_1 – 2x_2 + x_3 = \lambda x_1$; $2x_1 – 3x_2 + 2x_3 = \lambda x_2$; $–x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution

(A) Is a singleton **[JEE MAIN 2015]**
(B) Contains two elements
(C) Contains more than two elements
(D) Is an empty set

Q.36 The system of linear equations,
$x + \lambda y – z = 0$; $\lambda x – y – z = 0$; $x + y – \lambda z = 0$
has a non-trivial solution for : **[JEE MAIN 2016]**

(A) exactly one value of λ
(B) exactly two values of λ
(C) exactly three values of λ
(D) infinitely many values of λ

Q.37 If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj A = AA^T, then 5a + b =

(A) 5 (B) 4 **[JEE MAIN 2016]**
(C) 13 (D) –1

Q.38 If S is the set of distinct values of 'b' for which the following system of linear equations
$x + y + z = 1$; $x + ay + z = 1$; $ax + by + z = 0$
has no solution, then S is : **[JEE MAIN 2017]**

(A) a finite set containing two or more elements
(B) a singleton
(C) an empty set
(D) an infinite set

Q.39 If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to – **[JEE MAIN 2017]**

(A) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (B) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (C) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (D) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

Q.40 If the system of linear equations
$x + ky + 3z = 0$; $3x + ky – 2z = 0$; $2x + 4y – 3z = 0$
has a non-zero solution (x, y, z) , then $\frac{xz}{y^2}$ is equal to:

(A) –30 (B) 30 **[JEE MAIN 2018]**
(C) –10 (D) 10

Q.41 If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x – A)^2$, then the ordered pair (A, B) is equal to: **[JEE MAIN 2018]**

(A) (– 4, 5) (B) (4, 5)
(C) (–4, –5) (D) (–4, 3)

Q.142 The system of linear equations: **[JEE MAIN 2019 (Jan)]**
$x + y + z = 2$; $2x + 3y + 2z = 5$
$2x + 3y + (a^2 – 1)z = a + 1$

(A) has infinitely many solutions for a = 4.
(B) is inconsistent when $|a| = \sqrt{3}$.
(C) is inconsistent when a = 4.
(D) has a unique solution for $|a| = \sqrt{3}$.

Q.43 If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then the matrix A^{-50} when $\theta=\pi/12$, is equal to : **[JEE MAIN 2019 (Jan)]**

(A) $\begin{bmatrix}\sqrt{3}/2 & 1/2\\ -1/2 & \sqrt{3}/2\end{bmatrix}$ (B) $\begin{bmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}$

(C) $\begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}$ (D) $\begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$

Q.44 Let $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$, $(\alpha\in R)$ such that $A^{32}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then a value of α is **[JEE MAIN 2019 (April)]**

(A) $\pi/16$ (B) 0
(C) $\pi/32$ (D) $\pi/64$

Q.45 Let the number 2, b, c be in an A.P. and $A=\begin{bmatrix}1 & 1 & 1\\ 2 & b & c\\ 4 & b^2 & c^2\end{bmatrix}$. If det (A) $\in$ [2, 16], then c lies in the interval : **[JEE MAIN 2019 (April)]**

(A) [2,3) (B) $(2+2^{3/4}, 4)$
(C) $[3, 2+2^{3/4}]$ (D) [4,6]

Q.46 Let α and β be the roots of the equation $x^2+x+1=0$. Then for $y\neq 0$ in R,

$\begin{vmatrix}y+1 & \alpha & \beta\\ \alpha & y+\beta & 1\\ \beta & 1 & y+\alpha\end{vmatrix}=$ **[JEE MAIN 2019 (April)]**

(A) y^3 (B) y^3-1
(C) $y(y^2-1)$ (D) $y(y^2-3)$

Q.47 If $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & n-1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 78\\ 0 & 1\end{bmatrix}$, then the inverse of $\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ is **[JEE MAIN 2019 (April)]**

(A) $\begin{bmatrix}1 & -13\\ 0 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & 0\\ 12 & 1\end{bmatrix}$

(C) $\begin{bmatrix}1 & -12\\ 0 & 1\end{bmatrix}$ (D) $\begin{bmatrix}1 & 0\\ 13 & 1\end{bmatrix}$

Q.48 If α is a roots of equation $x^2+x+1=0$ and $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix}$ then $A^{31}=$ **[JEE MAIN 2020 (Jan)]**

(A) A (B) A^2
(C) A^3 (D) A^4

Q.49 If system of equations : $2x+2ay+az=0$; $2x+3by+bz=0$; $2x+4cy+cz=0$ have non-trivial solution then **[JEE MAIN 2020 (Jan)]**

(A) $a+b+c=0$ (B) a,b,c are in A.P.
(C) $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P. (D) a, b, c in G.P.

Q.50 Let $A=[a_{ij}]$, $B=[b_{ij}]$ are two 3×3 matrices such that $b_{ij}=\lambda^{i+j-2}a_{ij}$ and $|B|=81$. Find $|A|$ if $\lambda=3$. **[JEE MAIN 2020 (Jan)]**

(A) 1/9 (B) 3
(C) 1/81 (D) 1/27

Q.51 The number of all 3×3 matrices A, with enteries from the set $\{-1,0,1\}$ such that the sum of the diagonal elements of AA^T is 3, is **[JEE MAIN 2020 (JAN)]**

Q.52 Let ABC is a triangle whose vertices are A(1, –1), B (0, 2), C (x', y')and area of Δ ABC is 5 and C (x', y') lie on $3x+y-4\lambda=0$, then **[JEE MAIN 2020 (JAN)]**

(A) $\lambda=3$ (B) $\lambda=-3$
(C) $\lambda=4$ (D) $\lambda=2$

Q.53 The system of equation $3x+4y+5z=\mu$; $x+2y+3z=1$; $4x+4y+4z=\delta$ is inconsistent, then (δ,μ) can be **[JEE MAIN 2020 (JAN)]**

(A) (4, 6) (B) (3, 4)
(C) (4, 3) (D) (1, 0)

Q.54 If the matrices $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 3 & 4\\ 1 & -1 & 3\end{bmatrix}$, B = adj A and $C=3A$, then $\frac{|\text{adj } B|}{|C|}$ is equal to : **[JEE MAIN 2020 (JAN)]**

(A) 72 (B) 2
(C) 8 (D) 16

Q.55 If for some a and b in R, the intersection of the following three places $x+4y-2z=1$; $x+7y-5z=\beta$; $x+5y+\alpha z=5$ is a line in R^3, then $\alpha+\beta$ is equal to : **[JEE MAIN 2020 (JAN)]**

(A) 10 (B) –10
(C) 2 (D) 0

Q.56 The following system of linear equations
$7x+6y-2z=0$
$3x+4y+2z=0$
$x-2y-6z=0$, has **[JEE MAIN 2020 (JAN)]**

(A) infinitely many solutions, (x, y, z) satisfying $x=2z$
(B) no solution
(C) only the trivial solution
(D) infinitely many solutions, (x, y, z) satisfying $y=2z$

Q.57 Let $a-2b+c=1$. If $f(x)=\begin{vmatrix}x+a & x+2 & x+1\\ x+b & x+3 & x+2\\ x+c & x+4 & x+3\end{vmatrix}$, then **[JEE MAIN 2020 (JAN)]**

(A) $f(-50)=501$ (B) $f(-50)=-1$
(C) $f(50)=1$ (D) $f(50)=-501$

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	A	B	D	A	B	A	D	A	A	B	D	A	C	B	D	D	D	A	C	D	B	D	C	D
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	D	A	D	B	A	A	C	D	B	C	D	B	A	D	C	B	D	B	B	A	B	B	A	A	C
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67								
A	C	B	A	C	C	B	A	D	C	C	D	C	C	A	D	D	A								

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
A	B	A	D	C	C	C	A	C	D	D	A	A	B	D	A	D	A	B	C	D	B	C	A	A	D	D	C
Q	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
A	A	B	A	D	B	D	A	D	A	B	B	D	B	B	B	B	B	D	A	D	A	A	C	D	C	A	42
Q	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72									
A	25	375	2	2	4	3	6	5	6	6	1	5	9	2	103	1	198	0									

EXERCISE - 3

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	D	A	B	C	C	D	D	A	A	D	D	A	A	C	C	B	B	C	C	A	B	C	B	C
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	B	B	D	C	B	B	B	C	C	B	C	A	B	D	D	A	B	A	D	D	A	A	C	C	A
Q	51	52	53	54	55	56	57																		
A	672	A	C	C	A	A	C																		

CHAPTER- 3: MATRICES AND DETERMINANTS

SOLUTIONS TO TRY IT YOURSELF

TRY IT YOURSELF-1

(1) $A^3 = A\,.\,A^2 = A(4A+5I) = 4A^2 + 5A$

$$= 4(4A+5I) + 5A = 21A + 20I = \begin{bmatrix} 21 & 42 & 42 \\ 42 & 21 & 21 \\ 42 & 42 & 21 \end{bmatrix}$$

Trace $A^3 = 123$

(2) **(A).** A and B are symmetric matrices of same order

$\therefore\ A' = A$ and $B' = B$

Now, $(AB - BA)' = (AB)' - (BA)'$ $[\because\ (A-B)' = A' - B']$

$= B'A' - A'B'$ $[\because\ (AB)' = B'A']$

$= BA - AB$ $[\because\ A' = A \text{ and } B' = B]$

$= -(AB - BA)$

This shows that AB – BA is skew symmetric matrix.

(3) $\begin{bmatrix} 1 & \frac{n(n+1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$; $\frac{n(n+1)}{2} = 378 \Rightarrow n = 27$

(4) $A = \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

(5) $[1 \quad 25]\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{10} \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

Here, $AB = BA = I$ $\quad\therefore A^5 B^{10} A^5 = I$

$$[1 \quad 25]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

$$\Rightarrow\ [1 \quad 25]\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

$x^2 - 5x + 20 + 25x + 50 = 40 \Rightarrow x^2 + 20x + 30 = 0$ (roots α, β)

$(1-\alpha)(1-\beta) = 1 - (\alpha+\beta) + \alpha\beta = 1 - (-20) + 30 = 51$

(6) $A = \begin{bmatrix} a & b \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix}$

We write, $\begin{bmatrix} a & b \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & b/a \\ c & \left(\frac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1 \end{bmatrix} A$ $\quad\left(R_1 \to \frac{R_1}{a}\right)$

or $\begin{bmatrix} 1 & b/a \\ 0 & 1/a \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ -c/a & 1 \end{bmatrix} A$ $\quad[R_2 \to R_2 - cR_1]$

or $\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ -c & a \end{bmatrix} A$ $\quad[R_2 \to aR_2]$

or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A$ $\quad\left(R_1 \to R_1 - \frac{b}{a}R_2\right)$

$\Rightarrow\ A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

(7) $|A|\,|\text{adj}\,A| = |A\,\text{adj}\,A| = |\,|A|\,I\,|$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3 = (a^3)^3 = a^9$$

(8) $AB = B^{-1} \Rightarrow AB^2 = I$

Now, $KA - 2B^{-1} + I = O \Rightarrow KAB - 2B^{-1}B + IB = O$

$\Rightarrow\ KAB - 2I + B = O$

$\Rightarrow\ KAB^2 - 2B + B^2 = O \Rightarrow KI - 2B + B^2 = O$

$\Rightarrow K\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} K-2 & 0 \\ 0 & K-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow K = 2$

TRY IT YOURSELF-2

(1) L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$[R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$\begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

[Expanding along C_1]

$= (a+b+c)(a+b+c)^2 = (a+b+c)^3 = \text{R.H.S.}$

(2) **(A).** Let $A = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$.

Operating $R_2 \to 2R_2 - R_1 - R_3$

$$= \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2b-a-c \\ x+4 & x+5 & x+2c \end{vmatrix}$$

But a, b, c are in A.P.,

$$\therefore\ 2b = a+c = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

(3) **(D).** We have, $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix}$

$[C_1 \to C_1 + C_2 + C_3]$

$= 0$ $[\because \alpha+\beta+\gamma = 0$ from the equation $x^3 - 3x + 2 = 0]$

(4) L.H.S. $= \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

$[R_1 \to R_1 + R_3 - R_2]$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2+2 & 3 & 3x+3 \\ x^2+4 & 0 & 2x-1 \end{vmatrix} \qquad \begin{bmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x-3 \\ 4 & 0 & 2x-1 \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - \frac{x^2}{2}R_1 \\ R_3 \to R_3 - \frac{x^2}{4}R_1 \end{bmatrix}$$

$$= x\begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} = xA + B = \text{R.H.S.}$$

(5) **(B).** We have, $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix}$

[expanding along R_1]

$= (1-\cos^2\beta) + \cos(\alpha-\beta)[\cos\alpha\cos\beta - \cos(\alpha-\beta)]$

$\quad + \cos\alpha[\cos(\alpha-\beta)\cos\beta - \cos\alpha]$

$= \sin^2\beta + \cos(\alpha-\beta)[2\cos\alpha\cos\beta - \cos(\alpha-\beta)]$

$= \sin^2\beta + \cos(\alpha-\beta)\cos(\alpha+\beta) - \cos^2\alpha$

$= \sin^2\beta + (\cos^2\alpha - \sin^2\beta) - \cos^2\alpha = 0$

(6) Here the equations are linear. We have 3 equations in 2 unknowns.

$\therefore$ They are consistent if $\begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$

or $2(-5\lambda+18) - 3(7\lambda-12) - 8(-42+20) = 0$

or $-10\lambda + 36 - 21\lambda + 36 + 176 = 0$

or $-31\lambda + 248 = 0$ $\quad\therefore\ \lambda = 8$

$\therefore$ for $\lambda = 8$ the system has a solution which can be obtained by solving any two of the three equations.

Solving, $2x + 3y - 8 = 0$

$7x - 5y + 3 = 0$ By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 3 & -8 \\ -5 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -8 \\ 7 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 7 & -5 \end{vmatrix}}$$

or $\dfrac{x}{9-40} = \dfrac{-y}{6+56} = \dfrac{1}{-10-21}$ or $\dfrac{x}{-31} = \dfrac{-y}{62} = \dfrac{1}{-31}$

$\therefore$ $x = 1, y = 2$

(7) **(A).** We have, $f(x) = x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix} = 0$

$[C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3]$

Hence, $f(100) = 0$

CHAPTER- 3 : MATRICES AND DETERMINANTS
EXERCISE-1

(1) **(C).** It is based on fundamental concept.

(2) **(A).** $(M'AM)' = M'A'M = M'AM$

{A is symmetric. Hence M'AM is a symmetric matrix).

(3) **(B).** $A + A^T$ is a square matrix.

$(A + A^T)^T = A^T + (A^T)^T = A^T + A$

Hence A is a symmetric matrix.

(4) **(D).** $A^2 - 4A - 5I = 0$; $A(A - 4I) = 5I$; $A^{-1} = \frac{1}{5}(A - 4I)$

(5) **(A).** We have $A^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$

$A^3 = A^2A = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$

In general by induction, $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}, \forall n \in N]$

(6) **(B).** $A^2 + B^2 = A.A + B.B = A(BA) + B(AB)$
$= (AB)A + (BA)B = BA + AB = A + B$

(7) **(A).** $A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Rightarrow A^{-1}\begin{pmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{pmatrix}$

(8) **(D).** Given, A multiplicative group of 2 × 2 matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$. Let $A = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$ since |A| = 0, therefore inverse of A does not exist.

(9) **(A).**

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3\times3}\begin{bmatrix} -1 & -2 \\ -2 & 0 \\ 0 & -4 \end{bmatrix}_{3\times2}\begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2\times3}$

$P = \begin{bmatrix} -3 & -14 \\ -8 & -20 \\ -11 & -26 \end{bmatrix}_{3\times2}\begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2\times3}$

$P = \begin{bmatrix} 12 & 15 & 4 \\ 32 & 40 & 28 \\ 44 & 55 & 40 \end{bmatrix}_{3\times3} \Rightarrow P_{22} = 40$

(10) **(A).** Here $AB = \begin{bmatrix} pr - qs & ps + qr \\ -qr - ps & -qs + pr \end{bmatrix}$

Also $BA = \begin{bmatrix} rp - qs & qr + sp \\ -sp - qr & -qs + pr \end{bmatrix}$ Clearly AB = BA

(11) **(B).** Here $aI + bA = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

$\therefore (aI + bA)^2 = \begin{pmatrix} a^2 + 0 & ab + ba \\ 0 + 0 & 0 + a^2 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}$

$= a^2I + 2abA$

(12) **(D).** $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \sqrt{I_2}$; $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \alpha^2 + \beta\gamma = 1$

(13) **(A).** $A^2 = \begin{bmatrix} 2 & 2 \\ a & b \end{bmatrix}\begin{bmatrix} 2 & 2 \\ a & b \end{bmatrix} = \begin{bmatrix} 4 + 2a & 4 + 2b \\ 2a + ab & 2a + b^2 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow 4 + 2a = 0, 4 + 2b = 0, \ 2a + 2b = 0$

$2a + b^2 = 0$ must be consistent.

$\Rightarrow a = -2, b = -2$

(14) **(C).** $A' = [1\ 2\ 3]$, therefore

$AA' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$

(15) **(B).** Since A and B are square matrix

$\therefore |AB| = |A||B|$; $|A| = -10; |B| = -10$

$\therefore |AB| = 100.$

(16) **(D).** Since $|A| \neq 0$ therefore A^{-1} exist such that $AA^{-1} = I = A^{-1}A$

(17) **(D).** $|(2A)^{-1}| = \frac{1}{|2A|} = \frac{1}{2^3.|A|} = \frac{1}{2^3.3} = \frac{1}{24}$

(18) **(D).** Expanding: $x - 2 - 1 - 2(1 - x) - 1(1 - x^2 + 2x) = 0$
$x - 3 - 2 + 2x - 1 + x^2 - 2x = 0$
$x^2 + x - 6 = 0.$ $x = 2$ satisfies the above equation

(19) **(A).** AB – BA is skew symmetric

(20) **(C).** $A^2 - B^2 = A^2 - BA + AB - B^2$
$\Rightarrow 0 = -BA + AB \Rightarrow AB = BA$

(21) **(D).** $B = \frac{1}{2}(A - A') = \frac{1}{2}\left[\begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}\right]$

$= \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(22) **(B).**

(23) **(D).** Symmetric part of $A = \frac{1}{2}(A + A')$

$= \frac{1}{2}\left\{\begin{pmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 2 \\ 2 & 8 & -2 \\ 4 & 2 & 7 \end{pmatrix}\right\} = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

(24) **(C).** We know $A . \text{Adj } A = |A| I$
Clearly $|A| = 10$
$|\text{Adj } A| = |A|^{3-1} = |A|^2 = 10^2 = 100$

(25) **(D).** If $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$

When $a \neq 0, b \neq 0, c \neq 0$

(26) **(D).** $|A| = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = (4 \times 3 - 1 \times 2) = 12 - 2 = 10$

$$\left(\because \text{if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{then } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21}) \right)$$

(27) **(A).** Determinant $= \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ -3 & -4 & 3 \end{vmatrix}$,

$R_3 \leftarrow R_3 + R_1 - 2R_2$

$$= \begin{vmatrix} 12x+1 & 14x+3 & 50x+10 \\ 18x+2 & 21x+3 & 75x+9 \\ 0 & -1 & 0 \end{vmatrix} \begin{matrix} C_1 \leftarrow C_3 + C_1 \\ C_2 \leftarrow C_3 + C_2 \\ C_3 \leftarrow 4C_3 + 3C_2 \end{matrix}$$

$= -97x - 11$. So that $x = \frac{-11}{97}$.

(28) **(D).** Given $A = \begin{vmatrix} \sin(\theta+\alpha) & \cos(\theta+\alpha) & 1 \\ \sin(\theta+\beta) & \cos(\theta+\beta) & 1 \\ \sin(\theta+\gamma) & \cos(\theta+\gamma) & 1 \end{vmatrix}$

Operate $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\therefore A = \{\cos(\theta+\gamma) - \cos(\theta+\alpha)\}$

$\{\sin(\theta+\beta) - \sin(\theta+\alpha)\} - \{\cos(\theta+\beta)$

$- \cos(\theta+\alpha)\}\{\sin(\theta+\gamma) - \sin(\theta+\alpha)\}$

$= \sin(\beta-\gamma) - \sin(\beta-\alpha) - \sin(\alpha-\gamma)$

which is independent of θ.

(29) **(B).** $C_1 \to C_1 + C_3 - 2C_2 \cos dx$ gives

$$\Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$= (1+a^2-2a\cos dx)\sin dx$, (which is independent of p).

(30) **(A).** The determinant can be expanded as

$$\begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos A\cos Q + \sin A\sin Q & \cos A\cos R + \sin A\sin R \\ \cos B\cos P + \sin B\sin P & \cos B\cos Q + \sin B\sin Q & \cos B\cos R + \sin B\sin R \\ \cos C\cos P + \sin C\sin P & \cos C\cos Q + \sin C\sin Q & \cos C\cos R + \sin C\sin R \end{vmatrix}$$

This determinant can be written as 8 determinants and the value of each of these 8 determinants is zero;

e.g., $\cos P\cos Q\cos R \begin{vmatrix} \cos A & \cos A & \cos A \\ \cos B & \cos B & \cos B \\ \cos C & \cos C & \cos C \end{vmatrix} = 0$

Similarly other determinants can be shown zero.

(31) **(A).** Given determinant

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

$$= x(x+1)\begin{vmatrix} 1 & x & 1 \\ 2x & x-1 & x \\ 3x(x-1) & (x-1)(x-2) & x(x-1) \end{vmatrix}$$

$$= x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Applying $C_1 - C_3$ and $C_2 - C_3$

$$= x(x+1)(x-1)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 2x & -2 & x \end{vmatrix}$$

$= x(x+1)(x-1)[-2x+2x] = 0$

$\therefore f(x) = 0 \Rightarrow f(100) = 0$

(32) **(C).** Each term in $\Delta_1 \times \Delta_2$ is the sum of three terms. So each entry in C_1 or C_2 or C_3 in $\Delta_1 \times \Delta_2$ is the sum of three terms. Hence, $\Delta_1 \times \Delta_2$ can be written as the sum of $3 \times 3 \times 3 = 27$ determinants.

(33) **(D).** $\Delta = \begin{vmatrix} C & 1 & 0 \\ 1 & C & 1 \\ 6 & 1 & C \end{vmatrix} = C[C^2-1] - 1[C-6]$ $\because C = 2\cos\theta$

$\Rightarrow \Delta = 2\cos\theta(4\cos^2\theta - 1) - (2\cos\theta - 6)$

$\Rightarrow \Delta = 8\cos^3\theta - 4\cos\theta + 6$

(34) **(B).** $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} = x^3 - 3abx \Rightarrow \frac{d}{dx}\Delta_1 = 3(x^2 - ab)$

$\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} = x^2 - ab \Rightarrow \frac{d}{dx}(\Delta_1) = 3(x^2 - ab) = 3\Delta_2$

(35) **(C).** Determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(a+b+c) & -(a+b+c) & -(a+b+c) \end{vmatrix}$$

Applying $[R_3 - 2R_2]$, We get

$$= -(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(36) **(D).** Applying $C_1 - (C_2 + C_3)$ we get

$$\text{Det} = \begin{vmatrix} 6 & 6 & 7 \\ 3 & 3 & 15 \\ 11 & 11 & 6 \end{vmatrix} = 0 \ (\therefore C_1 = C_2)$$

(37) **(B).** Writing the given determinant as the sum of two determinants, we have

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} (1+xyz) = 0$$

$\Rightarrow (x-y)(y-z)(z-x)(1+xyz) = 0$

$\Rightarrow 1 + xyz = 0 \qquad (\because x \neq y \neq z)$

$\Rightarrow xyz = -1$

(38) **(A).** $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} c & d \\ -d & c \end{vmatrix} = \begin{vmatrix} ac+bd & -ad+bc \\ -bc+ad & bd+ac \end{vmatrix}$

$$= \begin{vmatrix} ac+bd & -bc+ad \\ -(bc+ad) & bd+ac \end{vmatrix} = \begin{vmatrix} A & B \\ -B & A \end{vmatrix}$$

$\therefore$ Required elements are -B, A.

(39) **(D).** $\begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix} = \begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ c & z & 5 \end{vmatrix}$

[changing rows into columns]

$$= \frac{1}{3}\begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ 3c & 3z & 15 \end{vmatrix} = \frac{3}{3}\times\frac{1}{5}\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = \frac{1}{5}(125) = 25$$

(40) **(C).** Applying $R_2 - xR_1, R_3 - xR_2$ then

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a-x \end{vmatrix} = a(a+x)^2$$

$\therefore\ f(2x) - f(x) = a(a+2x)^2 - a(a+x)^2 = ax(2a+3x)$

(41) **(B).** $(f(\alpha))^{-1}$

$$= \begin{bmatrix} +\cos\alpha & -\sin\alpha & +0 \\ -(-\sin\alpha) & +\cos\alpha & -0 \\ +0 & -0 & +1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(-\alpha)$$

(42) **(D).** $a^3 + b^3 = 0 \quad (\because a \neq 0)$

$\therefore\ b^3 = -a^3\ ;\ \dfrac{b^3}{a^3} = -1 \Rightarrow \dfrac{b}{a} = (-1)^{1/3}$

(43) **(B).** $\Delta = (2+i)\begin{vmatrix} 1 & 1 & i \\ 1 & 1+2i & 1+i \\ 1 & 2 & 1-i \end{vmatrix}$

$$= (2+i)\begin{vmatrix} 0 & -2i & -1 \\ 0 & -1+2i & 2i \\ 1 & 2 & 1-i \end{vmatrix} \text{ by } \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$= (2+i)\{-4i^2 + (-1+2i)\} = (2+i)(4-1+2i)$

$= (2+i)(3+2i) = 4+7i$.

(44) **(B).** $\Delta = \begin{vmatrix} -1 & -2 & x+4 \\ -2 & -3 & x+8 \\ -3 & -4 & x+14 \end{vmatrix}$, by $\begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$

$$= \begin{bmatrix} -1 & -1 & x \\ -2 & -1 & x \\ -3 & -1 & x+2 \end{bmatrix}, \text{ by } \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 + 4C_1 \end{matrix}$$

$= -(-x-2+x) + 1.(-2x-4+3x) + x(2-3)$

$= 2 + x - 4 - x = -2.$

(45) **(A).** Splitting the determinant into two determinants, we get $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$= (1+abc)[(a-b)(b-c)(c-a)] = 0$

Because a, b, c are different, the second factor cannot be zero. Hence, option (A), $1 + abc = 0$, is correct.

(46) **(B).** Since it is an identity in λ so satisfied by every value of λ. Now put $\lambda = 0$ in the given equation, we have

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 30 = 18.$$

(47) **(B).** Put $x = 1$, then we have

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = A - 12 \Rightarrow \begin{vmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \\ 5 & 1 & 1 \end{vmatrix} = A - 12$$

{Apply $C_1 \to C_1 - C_2$ }

$\Rightarrow -2+(-1)(-14) = A - 12 \Rightarrow A = 24$.

(48) **(A).** Apply $R_2 - R_3$ and note that

$(x+y)^2 - (x-y)^2 = 4xy$

$$\therefore \Delta = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \{\text{Applying } R_3 - (R_1 - 2R_2)\} .$$

(49) **(A).** We have $2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 - bc & b^2 - ac & c^2 - ab \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - \frac{2}{abc}\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ abc & abc & abc \end{vmatrix}$$

Applying $C_1(a), C_2(b), C_3(c)$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - \frac{2}{abc}(abc)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = 0 .$$

(50) **(C).** $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

$$= (abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = a^2b^2c^2(-1)(-4)$$

$= 4a^2b^2c^2 = Ka^2b^2c^2$, (given) $\Rightarrow K = 4$.

(51) **(C).** $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a-6 & 0 & 0 \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$

$[R_3 \to R_1 - 2R_2]$

$\Rightarrow (a-6)(b^2 - ac) = 0 \Rightarrow b^2 - ac = 0$ ($\because a \neq 6$)

$\therefore ac = b^2 \Rightarrow abc = b^3$.

(52) **(B).** Let a,b,c are in G.P. and assume $a = 1, b = 2, c = 4$

$$\therefore A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 0 \end{vmatrix} = 0 .$$

(53) **(A).** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = 3(\omega - \omega^2) = 3\left[\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right] = 3\sqrt{3}\,i$.

(54) **(C).** $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ c(b-a) & a(c-b) & ab \\ b-a & c+a & a+b \end{vmatrix}$

$\{C_1 \to C_1 - C_2, C_2 \to C_2 - C_3\}$

$$= (b-a)\,(c-b)\begin{vmatrix} 0 & 0 & 1 \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-a)\,(c-a)$$

$= (a-b)\,(b-c)\,(c-a)$

(55) **(C).** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc - a^2) - b(b^2 - ca) + c(ab - c^2)$

$= -a^3 - b^3 - c^3 + 3abc = -1\,[a^3 + b^3 + c^3 - 3abc]$

$= -[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)] \Rightarrow k = -1$.

(56) **(B).** $\Delta = \cos 15 \cos 75 - \sin 15 \sin 75$
$= \cos(15+75) = \cos 90$

(57) **(A).** Using $\to C_3 \to C_3 - (C_1 + C_2)$,

$D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$

$$\therefore \frac{D_1}{D_2} = \frac{-2b(ad-bc)}{b(ad-bc)} = -2$$

(58) **(D).** $C_1 \to C_1 + C_3$

$$D = \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 1-x \\ 1+x & x & 1+x-y \end{vmatrix} = -1[x-1-x] = 1$$

(59) **(C).** $1\,[\omega^2 - \omega - 1] - \omega^2\,[1 - 1 - \omega^2] + (1-\omega)\,[\omega^2 - 1]$
$= \omega^2 - \omega - 1 + \omega + \omega^2 - 1 - 1 + \omega$
$= \omega^2 - 3 + \omega + \omega^2 = \omega^2 - 4$

(60) **(C).** $|A^3| = |A|^3 = 125 = 5^3 \therefore |A| = 5 \Rightarrow a^2 - 4 = 5$
$\alpha^2 = 9 \Rightarrow \alpha = \pm 3$

(61) (D). $|\text{adj } A| = 25$; $x = 3$

We have $|\text{adj } A| = |A|^{n-1}$

$25 = |A|^2 \Rightarrow |A| = \pm 5 \quad \therefore |A^{-1}| = \frac{1}{|A|} = \pm\frac{1}{5} = \pm 0.2$

(62) (C).

(63) (C). $\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

$-bc + 2ac - ab = 0$; $2ac = ab + bc$

$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$; a, b, c are in H.P.

(64) (A). $\Delta = 0 \Rightarrow \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$

$\Rightarrow (\alpha - 1)^2 (\alpha + 2) = 0$; $\alpha = 1, -2$

But $\alpha \neq 1$ $(\because \Delta_1 = \Delta_2 = \Delta_3 = 0)$ $\therefore \alpha = -2$

(65) (D). The given system of homogeneous equations has a non-zero solution if, $\Delta = 0$

i.e., $\begin{vmatrix} 1 & 1 & -1 \\ 3 & -\alpha & -3 \\ 1 & -3 & 1 \end{vmatrix} = -2\alpha - 6 = 0$, i.e. if $\alpha = -3$.

(66) (D). Given set of equations will have a non trivial solution if the determinant of coefficient of x, y, z is zero

i.e., $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \Rightarrow 2k - 33 = 0$ or $k = \frac{33}{2}$.

(67) (A). Here $|A| \neq 0$. Hence unique solution.

EXERCISE-2

(1) (B). $f(\alpha) f(\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

Similarly $f(\alpha) f(\beta) f(\gamma)$

$= \begin{bmatrix} \cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) \\ -\sin(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) \end{bmatrix}$

$= \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix}$ as $\alpha + \beta + \gamma = \pi$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_2$

(2) (A). Let $A = k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \therefore A^T = k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Since A is orthogonal $\therefore AA^T = I$

$\Rightarrow k^2 \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$= k^2 \begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix}$

$= k^2 \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9k^2 I$; $k^2 = 9 \Rightarrow k^2 = 1/9 \Rightarrow k = \pm 1/3$

(3) (D). Here $AA^T = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(BB^T)_{11} = (D)^2 + (A)^2 \neq 1$

$(AB)_{11} = 8 - 7 = 1$, $(BA)_{11} = 8 - 7 = 1$

$\therefore$ AB ≠ BA may be not true

Now $AB = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}$

$= \begin{pmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $(AB)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(4) (C). Here $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$A^2 - 4A = \begin{bmatrix} 9-4 & 8-8 & 8-8 \\ 8-8 & 9-4 & 8-8 \\ 8-8 & 8-8 & 9-4 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5I$

(5) (C). We have $(AB)_{11} = 1.3 + 2.1 = 5$

$(BA)_{11} = 3.1 \ 4.3 = 15$

$\therefore AB \neq BA$ Again $(A^2)_{11} = 1.1 + 2.3 = 6 \neq 3 = (B)_{11}$

Also $(AB)^T = B^T A^T = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3+2 & 9+0 \\ 4+12 & 12+0 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$ is correct.

(6) **(C).** Det. $= \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix} = \frac{xyz}{12}$

$$\begin{vmatrix} x & x-1 & (x-1)(x-2) \\ y & y-1 & (y-1)(y-2) \\ z & z-1 & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \text{ (by } C_2+C_1, C_3+C_1+3C_2)$$

$$= \frac{xyz}{12}(x-y)(y-z)(z-x)$$

(7) **(A).** $\because \sum_{r=1}^{n}(r-1) = 1+2+..........+(n-1) = \frac{n(n-1)}{2}$

$$\sum_{r=1}^{n}(r-1)^2 = 1^2+2^2+.........+(n-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

$$\sum_{r=1}^{n}(r-1)^3 = 1^3+2^3+......+(n-1)^3 = \frac{n^2(n-1)^2}{4}$$

$$\therefore \sum_{r=1}^{n}\Delta_r$$

$$\begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{1}{6}n(n-1)(2n-1) & 2n^2 & 2(2n-1) \\ \frac{1}{4}n^2(n-1)^2 & 3n^3 & 3n(n-1) \end{vmatrix}$$

$$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 2(2n-1) & 2n^2 & 2(2n-1) \\ 3n(n-1) & 3n^3 & 3n(n-1) \end{vmatrix} = 0$$

(8) **(C).** Breaking the given determinant into two, determinants, we get

$$\begin{vmatrix} 3^2+k & 4^2 & 3^2+k \\ 4^2+k & 5^2 & 4^2+k \\ 5^2+k & 6^2 & 5^2+k \end{vmatrix} + \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 0 + \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0$$

[Applying $R_3 - R_2$ and $R_2 - R_1$ in second det.]

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 0 \qquad [\text{Applying } R_3 - R_2]$$

$$\Rightarrow \begin{vmatrix} 9+k & 7-k & 3 \\ 7 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 0 \qquad [\text{Applying } C_2 - C_1]$$

$\Rightarrow 2(7-k-6) = 0 \quad \Rightarrow k = 1$

(9) **(D).** Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$f(x) = \begin{vmatrix} 5 & -5 & 0 \\ 0 & 5 & -5 \\ \sin^2 x & \cos^2 x & 5+4\sin 2x \end{vmatrix}$$

After solving, $f(x) = 150 + 100\sin 2x$

Clearly, domain $\to (-\infty, \infty)$

Range $\to [50, 250]$; Period $\to \pi$

(10) **(D).** Write 1 as $\sin^2\alpha + \cos^2\alpha$ etc. to get

$$\begin{vmatrix} \sin^2\alpha+\cos^2\alpha & \cos\beta\cos\alpha+\sin\beta\sin\alpha & \cos\gamma\cos\alpha+\sin\gamma\sin\alpha \\ \cos\alpha\cos\beta+\sin\alpha\sin\beta & \cos^2\beta+\sin^2\beta & \cos\gamma\cos\beta+\sin\gamma\sin\beta \\ \cos\alpha\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma+\sin\beta\sin\gamma & \sin^2\gamma+\cos^2\gamma \end{vmatrix}$$

can be factorized into 2 determinant

$$\begin{vmatrix} \cos\alpha & \sin\alpha & x \\ \cos\beta & \sin\beta & x \\ \cos\gamma & \sin\gamma & x \end{vmatrix}\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ x & x & x \end{vmatrix} = 0$$

(11) **(A).** Observe that the sum of all the elements in a column is $x^2 - 4$. Therefore the determinant

$$= (x^2-4)\begin{vmatrix} 1 & 1 & 1 \\ 10 & x^2+2 & 1 \\ -2 & 0 & x^2 \end{vmatrix} = (x^2-4)\begin{vmatrix} 1 & 0 & 0 \\ 10 & x^2-8 & -9 \\ -2 & 14 & x^2+2 \end{vmatrix}$$

$= (x^2-4)(x^4-6x^2+110)$

$= x^4 - 6x^2 + 110 = (x^2-3)^2 + 101 > 0$

so that real roots are ± 2.

(12) **(A).** $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}; \frac{1}{n}A^n = \begin{bmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{bmatrix}$

But $-1 \le \cos n\theta \le 1$ and $-1 \le \sin n\theta \le 1$

$$\lim_{n\to\infty}\frac{\sin n\theta}{n} = 0, \quad \lim_{n\to\infty}\frac{\cos n\theta}{n} = 0$$

$$\lim_{n\to\infty}\frac{1}{n}A^n=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

(13) **(B).** $A'=\begin{bmatrix}1&-\tan x\\\tan x&1\end{bmatrix}$

$$A^{-1}=\frac{1}{1+\tan^2 x}\begin{bmatrix}1&-\tan x\\\tan x&1\end{bmatrix},$$

$$A'A^{-1}=\begin{bmatrix}\cos 2x&-\sin 2x\\\sin 2x&\cos 2x\end{bmatrix}\Rightarrow|A'A^{-1}|=1$$

(14) **(D).** If the G.P be a, ar, ar^2, ... then $a_n=ar^{n-1}$

$$D=\begin{vmatrix}\log a+(n-1)\log r&\log a+n\log r&\log a+(n+1)\log r\\\log a+n\log r&\log a+(n+1)\log r&\log a+(n+2)\log r\\\log a+(n+1)\log r&\log a+(n+2)\log r&\log a+(n+3)\log r\end{vmatrix}$$

$R_3 : R_3 - R_2$ and $R_2 : R_2 - R_1$ gives,

$$=\begin{vmatrix}\log a+(n-1)\log r&\log a+n\log r&\log a+(n+1)\log r\\\log r&\log r&\log r\\\log r&\log r&\log r\end{vmatrix}$$

$= 0$, since $R_2 = R_3$

(15) **(A).** $\Delta_r=\begin{vmatrix}2r-1&{}^mC_r&1\\m^2-1&2^m&m+1\\\sin^2(m^2)&\sin^2(m)&\sin^2(m+1)\end{vmatrix}$

$$\Rightarrow\sum_{r=0}^{m}\Delta_r=\begin{vmatrix}\sum_{r=0}^{m}(2r-1)&\sum_{r=0}^{m}{}^mC_r&\sum_{r=0}^{m}1\\m^2-1&2^m&m+1\\\sin^2(m^2)&\sin^2(m)&\sin^2(m+1)\end{vmatrix}$$

$$=\begin{vmatrix}m^2-1&2^m&m+1\\m^2-1&2^m&m+1\\\sin^2(m^2)&\sin^2(m)&\sin^2(m+1)\end{vmatrix}=0$$

(16) **(D).** $\begin{vmatrix}y+z&x-z&x-y\\y-z&z+x&y-x\\z-y&z-x&x+y\end{vmatrix}=\begin{vmatrix}y+z&x-z&x-y\\2y&2x&0\\2z&0&2x\end{vmatrix}$

$R_2\to R_2+R_1$ and $R_3\to R_3+R_1$

$$=4\begin{vmatrix}y+z&x-z&x-y\\y&x&0\\z&0&x\end{vmatrix}$$

$=4[(y+z)(x^2)-(x-z)(xy)\ +(x-y)(-zx)]$

$=4[x^2y+zx^2-x^2y+xyz-zx^2+xyz]\ =8xyz$

Hence, $k=8$.

(17) **(A).** $A^2=A.\ A=\begin{bmatrix}ab&b^2\\-a^2&-ab\end{bmatrix}\begin{bmatrix}ab&b^2\\-a^2&-ab\end{bmatrix}$

$$=\begin{bmatrix}a^2b^2-a^2b^2&ab^3-ab^3\\-a^3b+a^3b&-a^2b^2+a^2b^2\end{bmatrix}=O$$

$\Rightarrow A^3=A.A^2=0$ and $A^n=0$, for all $n\ge 2$.

(18) **(B).** For $A=\begin{bmatrix}i&0\\0&i/2\end{bmatrix}$, $\text{adj}(A)=\begin{bmatrix}i/2&0\\0&i\end{bmatrix}$ and

$|A|=-\frac{1}{2}$.

$$\therefore\ A^{-1}=\frac{1}{\Delta}(\text{adj}\,A)=\frac{1}{-1/2}\begin{bmatrix}i/2&0\\0&i\end{bmatrix}=\begin{bmatrix}-i&0\\0&-2i\end{bmatrix}.$$

(19) **(C).** $A=\begin{bmatrix}1&2&3\\1&4&9\\1&8&27\end{bmatrix}$

Let c_{ij} be co-factor of a_{ij} in A.
Then co-factor of elements of A are given by

$$C_{11}=\begin{vmatrix}4&9\\8&27\end{vmatrix}=36, C_{21}=\begin{vmatrix}2&3\\8&27\end{vmatrix}=-30,$$

$$C_{31}=\begin{vmatrix}2&3\\4&9\end{vmatrix}=6\ ;\ |\text{adj}\,A|=\begin{vmatrix}36&-30&6\\18&-24&6\\14&-6&-2\end{vmatrix}=144$$

(20) **(D).** $A^2=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{pmatrix}$

$$\therefore\ A^2-(a+d)A=\begin{pmatrix}bc-ad&0\\0&bc-da\end{pmatrix}=(bc-ad)\,I$$

As $A^2-(a+d)A+kI=0$, we get $(bc-ad)I+kI=0$
$\Rightarrow k=ad-bc$

(21) **(B).** $A-\lambda I=\begin{bmatrix}1&3\\2&2\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}1-\lambda&3\\2&2-\lambda\end{bmatrix}$

$=(1-\lambda)(2-\lambda)=\lambda^2-3\lambda+2=0$

i.e. for $A-\lambda I$ to be singular $\lambda^2-3\lambda+2=0$
since $A-\lambda I$ is singular $\Rightarrow$ det. $(A-\lambda I)=0$

hence $\begin{bmatrix}1-\lambda&3\\2&2-\lambda\end{bmatrix}=0\ \Rightarrow\ 2-\lambda-2\lambda+\lambda^2-6=0$

or $\lambda^2-3\lambda-4=0]$

(22) **(C).** $|A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1(1+\sin^2\theta) - \sin\theta\,(-\sin\theta + \sin\theta) + (1+\sin^2\theta)$

$= 2\,(1+\sin^2\theta)$

$|\sin\theta| \le 1 \Rightarrow -1 \le \sin\theta \le 1 \Rightarrow 0 \le \sin^2\theta \le 1$

$\Rightarrow 1 \le 1+\sin^2\theta \le 2 \Rightarrow 2 \le 2(1+\sin^2\theta) \le 4$

$\Rightarrow |A| \in [2, 4]$

(23) **(A).** $|A| = 2\,(a-2) \Rightarrow a \ne 2$

cofactor of 0 in $|A|$ is $2-3a$. According to value of A^{-1},

$\frac{2-3a}{|A|} = \frac{1}{2} \Rightarrow \frac{2-3a}{2(a-2)} = \frac{1}{2}$

$\Rightarrow 2-3a = a-2 \Rightarrow a = 1$

Again $c = \frac{\text{cofactor of } a \text{ in } |A|}{|A|} = \frac{\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}}{2(a-2)} = \frac{2}{2(1-2)}$

$= -1$

Alternative : $AA^{-1} = I$

(24) **(A).** Applying the result

$\begin{vmatrix} a & b_1 \\ c & d_1 \end{vmatrix} + \begin{vmatrix} a & b_2 \\ c & d_2 \end{vmatrix} = \begin{vmatrix} a & b_1+b_2 \\ c & d_1+d_2 \end{vmatrix}$ repeatdely

$$\sum_1^n f(r) = \begin{vmatrix} n(2n+1) & 2n+1 & 6n(n+1)\sum_1^n r^2 \\ n+1 & 2n+2 & 2n(n+1)\sum_1^n r \\ n & 2n+1 & 4\sum_1^n r^3 \end{vmatrix}$$

$$= \begin{vmatrix} n(2n+1) & 2n+1 & n^2(n+1)^2(2n+1) \\ n+1 & 2n+2 & n^2(n+1)^2 \\ n & 2n+1 & n^2(n+1)^2 \end{vmatrix}$$

$$= n^2(n+1)^2\,(2n+1) \begin{vmatrix} n & 1 & 1 \\ n+1 & 2n+2 & 1 \\ n & 2n+1 & 1 \end{vmatrix} = 2n^3\,(n+1)^2\,(2n+1)$$

(25) **(D).** Multiply R_1 by a, R_2 by b & R_3 by c & divide the determinant by abc. Now take a, b & c common from c_1, c_2 & c_3.

Now use $C_1 \to C_1 + C_2 + C_3$ to get

$$(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} = 1.$$

Now use $c_1 \to c_1 - c_2$ & $c_2 \to c_2 - c_3$

we get $1 + a^2 + b^2 + c^2 = 1 \Rightarrow a = b = c = 0$

(26) **(D).** $a = +2$; $b = -4$; $c = 1$; $d = -2$

Let $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

now $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = null matrix

hence A is nilpotent

note that any matrix of the form $\begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}$ is a nilpotent

(27) **(C).** $\det(B) = \begin{vmatrix} 4x & 2a & -p \\ 4y & 2b & -q \\ 4z & 2c & -r \end{vmatrix} = (4)(2)(-1)\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$

$$= -8\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = -8\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -8 \times 2 = -16$$

$|AA^{-1}| = |I| \Rightarrow |A|\,.\,|A^{-1}| = 1 \qquad \because |A^{-1}| = \frac{1}{|A|}$

(28) **(A).** If the char eqn is $\lambda^3 + a\lambda^2 + b\lambda + c = 0$,

$|A| = -c \qquad \therefore |A| = -2$

Order of $A = 3 \times 3 \qquad \therefore n = 3$

$|\text{adj } A| = |A|^{n-1} = (-2)^{3-1} = 4$

(29) **(B).** $A = x\,(x^2-1) - 1\,(x-1) + 1\,(1-x) = x^3 - x - x + 1 + 1 - x$

$A = x^3 - 3x + 2$; $\frac{dA}{dx} = 3x^2 - 3 \quad (B = x^2 - 1) = 3B$

(30) **(A).** Multiply column 1^{st} by $(x-a)$

Multiply column 2^{nd} by $(x-b)$

Multiply column 3^{rd} by $(x-c)$

$$\therefore \frac{1}{\prod(x-a)}\begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ \prod(x-a) & \prod(x-a) & \prod(x-a) \end{vmatrix} = 0$$

Take $\prod(x-a)$ out from 3^{rd} row

$$\therefore \begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \end{vmatrix} = 0$$

Using $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

$\therefore (b-a)(c-b)(a-c)[3x - (a+b+c)] = 0$

$\therefore x = \frac{a+b+c}{3}$

(31) **(D).** Let We have

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 3d & 3d & 3d \\ d & d & d \end{vmatrix} = 0$$

[Using $R_3 \to R_3 - R_2$, and $R_2 \to R_2 - R_1$]

$\Rightarrow$ A is singular

$\therefore$ The given system of homogeneous equations has infinite number of solutions.

Also $|B| = a_1^2 + a_2^2 \neq 0$. Thus B is non- singular.

(32) **(B).** Let $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$; $C_2 \to C_2 - \frac{a_{12}}{a_{11}} C_1$

$C_3 \to C_3 - \frac{a_{13}}{a_{11}} C_1$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} \times a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

so minimum value = – 4

(33) **(D).** $|(\text{adj } A^{-1}| = |A^{-1}|^2 = \frac{1}{|A|^2}$

$$|(\text{adj } A^{-1})^{-1}| = \frac{1}{|\text{adj } A^{-1}|} = |A|^2 = 2^2 = 4$$

(34) **(A).** We know that every square matrix A can be written as sum of a symmetric & skew-symmetric matrix

$$A = \frac{A + A^T}{2} - \frac{A - A^T}{2}$$

$$\Rightarrow B = \frac{A + A^T}{2} = \frac{\begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}}{2} = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$$

(35) **(D).** $(1+x)(1+x)^4(1+x)^7 \begin{vmatrix} 1 & (1+x) & (1+x)^2 \\ 1 & (1+x) & (1+x)^2 \\ 1 & (1+x) & (1+x)^2 \end{vmatrix}$

$= a_0 + a_1 x + a_2 x^2 + ...$

Since all the rows are identical so the value of determinant is zero. $\therefore a_1 = 0$

(36) **(A).** We have $|A| = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 6$

$(A(\text{adj } A)A^{-1})A = (A(\text{adj } A))(A^{-1}A) = (|A|I)I = |A|I$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(37) **(B).** We have, $\Delta = \begin{vmatrix} 1 & 1+ac & 1+bc \\ 1 & 1+ad & 1+bd \\ 1 & 1+ae & 1+be \end{vmatrix}$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & ac & bc \\ 1 & ad & bd \\ 1 & ae & be \end{vmatrix} = ab \begin{vmatrix} 1 & c & c \\ 1 & d & d \\ 1 & e & e \end{vmatrix}$$

(38) **(B).** $\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = Ax - 12$

$C_2 \to C_2 - C_3$

$$\begin{vmatrix} x^2 + x & 3 & x-2 \\ 2x^2 + 3x - 1 & 3 & 3x - 3 \\ x^2 + 2x + 3 & 0 & 2x - 1 \end{vmatrix} = Ax - 12$$

$R_2 \to R_2 - R_1$

$$\begin{vmatrix} x^2 + x & 3 & x-2 \\ x^2 + 2x - 1 & 0 & 2x - 1 \\ x^2 + 2x + 3 & 0 & 2x - 1 \end{vmatrix} = Ax - 12$$

$R_3 \to R_3 - R_2$

$$\begin{vmatrix} x^2 + x & 3 & x-2 \\ x^2 + 2x - 1 & 0 & 2x - 1 \\ 4 & 0 & 0 \end{vmatrix} = Ax - 12$$

$4(6x - 3 - 0) = Ax - 12 \Rightarrow 24x - 12 = Ax - 12$

$\Rightarrow A = 24$

(39) **(D).** Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = (\text{adj } A)$ and $C = 5A$

$|A| = 1(0+3) + 1(0+6) + 1(-4) = 5$

$$\frac{|\text{adj } B|}{|C|} = \frac{|\text{adj.adj } A|}{5^3 |A|} = \frac{|A|^{(n-1)^2}}{5^3 |A|} = \frac{(5)^{(3-1)^2}}{5^3 .5} = \frac{5^4}{5^4} = 1$$

(40) **(B).** $D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & (n+1) & (n+2)\,(n+1) \\ 1 & (n+2) & (n+3)\,(n+2) \\ 1 & (n+3) & (n+4)\,(n+3) \end{vmatrix}$

Applying $R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 0 & -1 & -2\,(n+2) \\ 0 & -1 & -2\,(n+3) \\ 0 & (n+3) & (n+4)\,(n+3) \end{vmatrix}$$

Now expanding along C_1

$D = 2n!(n+1)!\,(n+2)!$

So, $\dfrac{D}{(n!)^3} - 4 \Rightarrow \dfrac{2n!(n+1)!\,(n+2)!}{(n!)^3} - 4$

$= 2.(n+1)\,(n+2)\,(n+1) - 4$

$= n\,(2n^2 + 8n + 10)$. So divisible by n

(41) **(B).** $\sum_{i=1}^{\infty} \det(A_i) = \det(A_1) + \det(A_2) + ...\infty$

$$= \begin{vmatrix} a & b \\ b & a \end{vmatrix} + \begin{vmatrix} a^2 & b^2 \\ b^2 & a^2 \end{vmatrix} + \begin{vmatrix} a^3 & b^3 \\ b^3 & a^3 \end{vmatrix} + ...$$

$= (a^2 - b^2) + (a^4 - b^4) + (a^6 - b^6) + ...$

$\Rightarrow (a^2 + a^4 + a^6 + ...) - (b^2 + b^4 + b^6 + ...)$

$$\Rightarrow \frac{a^2}{1-a^2} - \frac{b^2}{1-b^2} \Rightarrow \frac{a^2 - b^2}{(1-a^2)\,(1-b^2)}$$

(42) **(B).** It is obvious form the properties of symmetric & skew symmetric matrices.

(43) **(B).** $PQ = \begin{pmatrix} -i^2 + 0 - i^2 & i^2 + 0 + i^2 \\ 0 + 0 + i^2 & 0 + 0 - i^2 \\ i^2 + 0 + 0 & -i^2 + 0 + 0 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$

(44) **(B).** Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} \sin^2 13° & \cos^2 13° & -1 \\ \cos^2 13° & -1 & \sin^2 13° \\ -1 & \sin^2 13° & \cos^2 13° \end{vmatrix} = \begin{vmatrix} 0 & \cos^2 13° & -1 \\ 0 & -1 & \sin^2 13° \\ 0 & \sin^2 13° & \cos^2 13° \end{vmatrix} = 0$$

(45) **(D).** A^{-1} exist only for non-singular matrix.

$AB = AC \Rightarrow B = C$ if A^{-1} exist.

If A^{-1} exist

(46) **(A).** $A(\alpha) = \begin{bmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ f_4(\alpha) & f_5(\alpha) & f_6(\alpha) \\ f_7(\alpha) & f_8(\alpha) & f_9(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x - \alpha$ is a factor of $f_1(x), f_2(x)$ ...., $f_9(x)$

$f(x) = (x - \alpha)\,\phi(x)$

$f(\alpha) = 0 \Rightarrow x - \alpha$ is a factor of $f(x)$

(47) **(D).** $\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 100x + 50 + 1 & 100y + 40 + 1 & 100z + 30 + 1 \\ x & y & z \end{vmatrix}$

$$= \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} (R_2 \to R_2 - 100\,R_3 - 10R_1) \neq 0$$

(48) **(A).** Since $|A| = 1$

We know $A^{-1} = 1/|A|\ \text{adj}\ A \Rightarrow A^{-1} = \text{adj}(A)$

(49) (A), (50) (C), (51) (D).

(i) $b_1.C_{31} + b_2.C_{32} + b_3.C_{33}$

$$= b_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - b_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + b_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

(ii) Value of new determinants $= 2^3\Delta = 8\Delta$

(iii) $a_3M_{13} - b_3.M_{23} + d_3.M_{33} = a_3C_{13} + b_3.C_{23} + d_3.C_{33} = \Delta$

By definition

(52) **(C).** Given matrix $A + 2B$ is singular $\Rightarrow |A + 2B| = 0$

$$A + 2B = \begin{bmatrix} 1 & -2 & 2 \\ 5 & K & 6 \\ 3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 2 \\ 8 & 8 & 4 \\ 6 & 10 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 13 & K+8 & 10 \\ 9 & 11 & 2 \end{bmatrix}$$

$$|A + 2B| = 0 \Rightarrow 2\begin{bmatrix} 5 & 4 & 2 \\ 13 & K+8 & 5 \\ 9 & 11 & 1 \end{bmatrix} = 0$$

$2\,[5\,(K + 8 - 55) - 4\,(13 - 45) + 2\,(143 - 9K - 72)] = 0$

$5\,(K - 47) - 4\,(-32) + 2\,(71 - 9K) = 0$

$5K - 235 + 128 + 142 - 18\,K = 0$

$-13K + 35 = 0 \Rightarrow K = \dfrac{35}{13}$

(53) **(A).** Given $C = A - B$ and $\text{Tr}(C) = 2$

$$C = \begin{bmatrix} -1 & -5 & 1 \\ 1 & K-4 & 4 \\ 0 & -4 & -4 \end{bmatrix}, \text{Tr}(C) = 2$$

$\Rightarrow -1 + K - 4 - 4 = 2 \Rightarrow K = 11$

(54) **42.** $\begin{bmatrix} 1 & 1 & 2 & 6 \\ 1 & 3 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & \lambda - 2 & \mu - 6 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 1/2 & 2 \\ 0 & 1 & \lambda - 2 & \mu - 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & \lambda - \frac{5}{2} & \mu - 8 \end{bmatrix}$$

$\therefore\ \lambda = \dfrac{5}{2}, \mu = 8 \qquad \therefore\ 4\,(\lambda + \mu) = 42$

(55) **25.** $\Delta = \alpha\beta\gamma \begin{vmatrix} \frac{1}{1-\alpha} & \frac{1}{1-\beta} & \frac{1}{1-\gamma} \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix}$

$= \alpha\beta\gamma \begin{vmatrix} \frac{1}{1-\alpha} & \frac{1}{1-\beta}-\frac{1}{1-\alpha} & \frac{1}{1-\gamma}-\frac{1}{1-\alpha} \\ 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \end{vmatrix}$

$= \frac{\alpha\beta\gamma\,(-1)\,(\beta-\alpha)\,(\gamma-\alpha)}{(1-\alpha)\,(1-\beta)\,(1-\gamma)} \begin{bmatrix} 1-\gamma & 1-\beta \\ 1 & 1 \end{bmatrix}$

$= \frac{\alpha\beta\gamma\,(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)}{(1-\alpha)\,(1-\beta)\,(1-\gamma)}$

Since, α, β, γ are the roots of $ax^3+bx^2+cx+d=0$

$\therefore ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)(x-\gamma)$ and

$\alpha\beta\gamma = -\frac{d}{a} \;\therefore\; \Delta = \frac{\left(-\frac{d}{a}\right)\left(\frac{25}{2}\right)}{\frac{(a+b+c+d)}{a}} = -\frac{25d}{2\,(a+b+c+d))}$

$\therefore$ Required value $= 25$

(56) **375.** Given $A^2-4A-5I=0$

$A^3 = A\cdot A^2 = A(4A+5I) = 4A^2+5A$

$= 4(4A+5I)+5A = 21A+20I$

$= \begin{bmatrix} 21 & 42 & 42 \\ 42 & 21 & 42 \\ 42 & 42 & 21 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 315+60 = 375$

(57) **2.** A is involutary $\Rightarrow A^2 = I \Rightarrow A = A^{-1}$

Also $(KA)^{-1} = \frac{1}{k}(A)^{-1}$; hence $\left(\frac{1}{2}A\right)^{-1} = 2(A)^{-1} \Rightarrow 2A$

(58) **2.** $C_1 \to C_1+C_2+C_3$

$\begin{vmatrix} 1+2x+x(a^2+b^2+c^2) & (1+b^2)x & (1+c^2)x \\ 1+2x+x(a^2+b^2+c^2) & 1+b^2x & (1+c^2)x \\ 1+2x+x(a^2+b^2+c^2) & (1+b^2)x & 1+c^2x \end{vmatrix}$

$\begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$\begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$

$f(x) = (1-x)^2 = 1-2x+x^2$

(59) **4.** $\frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$

use $R_1 \to R_1 - (R_2+R_3)$

$\frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$

$R_2 \to R_2 + 1/2R_1$ and $R_3 \to R_3 + 1/2\,R_1$

$\frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & c^2 & 0 \\ b^2 & 0 & c^2 \end{vmatrix}$

$= \frac{1}{abc}\,[\,2b^2(a^2c^2) - 2a^2(-b^2c^2)\,] = \frac{4a^2b^2c^2}{abc} = 4abc$

(60) **3.** $\begin{vmatrix} x^4+x & x^3y & x^3z \\ xy^3 & y^4+y & y^3z \\ xz^3 & yz^3 & z^4+z \end{vmatrix} = 11$

$\frac{1}{xyz}\begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$

use $R_1 \to R_1 + R_2 + R_3$

$D = (x^3+y^3+z^3+1)\begin{vmatrix} 1 & 1 & 1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$

hence $x^3+y^3+z^3 = 10$

$(2,1,1), (1,2,1), (1,1,2)$

(61) **6.** $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$t_r(A) + t_r\left(\frac{A}{2}\right) + t_r\left(\frac{A}{2^2}\right) + \ldots\ldots = t_r(A) + \frac{1}{2}t_r(A)$

$+ \frac{1}{2^2}t_r(A) + \ldots\ldots = \frac{t_r(A)}{1-(1/2)} = 2\,t_r(A) = 2(2+1) = 6$

(62) **5.** $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ....(1)

$A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ....(2)

Let A be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The first equation gives

$a-b=-1$ ....(3) and $c-d=2$ ....(4)

For second equation,

$$A^2\begin{bmatrix}1\\-1\end{bmatrix} = A\left(A\begin{bmatrix}1\\-1\end{bmatrix}\right) = A\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}.$$

This gives $-a+2b=1$ ....(5)

and $-c+2d=0$ ....(6)

$(3)+(5) \Rightarrow b=0$ and $a=-1$

$(4)+(6) \Rightarrow d=2$ and $c=4$

so the sum $a+b+c+d=5$.

(63) **6.** Possible orders $(1\times 12);(12\times 1);(2\times 6);(6\times 2);(3\times 4);(4\times 3)$

(64) **6.**

$A^2 = A.A = (AB).A = A(BA) = BA = A \quad \therefore k=1$

$B^2 = B.B = (BA)B = B(AB) = BA = B \quad \therefore \ell = 1$

$(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2)$

$= (A+A+B+B) = 2(A+B)$

$(A+B)^3 = (A+B)^2.(A+B) = 2(A+B)(A+B)$

$= 2^2(A+B)$

$\therefore k+\ell+m=6$

(65) **1.** $\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0;\ z\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z[(z+\omega^2)(z+\omega)-1-\omega(z+\omega-1) + \omega^2(1-z-\omega^2)]=0$

$\Rightarrow z^3=0$

$\Rightarrow z=0$ is only solution.

(66) **5.** $|A|=(2k+1)^3,\ |B|=0$

(Since B is a skew.symmetric matrix of order 3)

$\Rightarrow \det(\text{adj } A) = |A|^{n-1} = ((2k+1)^3)^2 = 10^6$

$\Rightarrow 2k+1=10 \Rightarrow 2k=9 \Rightarrow [k]=4$.

(67) **9.** Let $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $a_{12}=-1, a_{22}=2, a_{32}=3$

$a_{11}-a_{12}=1 \Rightarrow a_{11}=0,$

$a_{21}-a_{22}=1 \Rightarrow a_{21}=3,$

$a_{31}-a_{32}=1 \Rightarrow a_{31}=2,$

$a_{11}+a_{12}+a_{13}=0 \Rightarrow a_{13}=1$

$a_{21}+a_{22}+a_{23}=0 \Rightarrow a_{23}=-5$

$a_{31}+a_{32}+a_{33}=12 \Rightarrow a_{33}=7$

Hence sum of diagonal of M is $= a_{11}+a_{22}+a_{33}=0$

$\Rightarrow 0+2+7=9$

(68) **2.** $$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

$$\begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} \equiv 10$$

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 2x & 4x^2 \\ 1 & 3x & 9x^2 \end{vmatrix} + x\cdot 2x\cdot 3x\begin{vmatrix} 1 & x & x^2 \\ 1 & 2x & 4x^2 \\ 1 & 3x & 9x^2 \end{vmatrix} \equiv 10$$

$-x \times -x \times 2x(1+6x^3) \equiv 10$

$2x^3(1+6x^3) \equiv 10$

$x^3(1+6x^3) \equiv 5\ ;\ x^3=t$

$t+6t^2 \equiv 5\ ;\ 6t^2+t-5=0$

$6t^2+6t-5t-5=0$

$6t(t+1)-5(t+1)=0\ ;\ t\equiv -1, t=5/6$

$x^3=-1,\ x^3=5/6$

$x=-1$ and $x\equiv(5/6)^{1/3}$

(69) **103.** $P \equiv \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}, P^2 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix}$

$P^3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}, P^4 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 160 & 6 & 1 \end{bmatrix}$

Pattern of element P_{31} is 16 [1, 3, 6, 10,......]

$\therefore$ 50^{th} term is 16×1275

[By observing that T_n of $S = 1+3+6+10+\ldots$ is $\frac{n^2+n}{2}$]

$\therefore\ P^{50} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 16\times 275 & 200 & 1 \end{bmatrix}$

$Q = P^{50} - I = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 16\times 275 & 200 & 0 \end{bmatrix}$

$\therefore\ \frac{q_{31}+q_{32}}{q_{21}} = \frac{16\times 1275+200}{200} = 102+1=103$

(70) **1.** $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & 1 & \alpha \\ 1 & \alpha & 1 \end{vmatrix} = (\alpha^2-1)^2 = 0$

$\alpha^2=1\ ;\ \alpha=\pm 1$

$\alpha=1$ (two planes are parallel) (Rejected)

$\alpha=-1$ (two planes are coincident)

(71) **198.** $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$; $M^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$M^TM = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$T_r(M^TM) = (a_1^2+b_1^2+c_1^2)+(a_2^2+b_2^2+c_2^2)+(a_3^2+b_3^2+c_3^2) = 5$$

$$5 = 1^2+1^2+1^2+1^2+1^2+0^2+0^2+0^2+0^2$$

$\Rightarrow$ $5, 1's, 4, 0's \Rightarrow {}^9C_5$

$1^2+2^2+0^2+0^2+0^2+0^2+0^2+0^2+0^2$

$\Rightarrow$ $1 \to 1, 1 \to 2, 7 \to 0's \Rightarrow {}^9C_7 \times {}^2C_1$

$\Rightarrow$ $126 + \frac{9 \cdot 8}{2} \times 2 = 198$

(72) **0.** Let $A = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$.

Operating $R_2 \to 2R_2 - R_1 - R_3$

$$= \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2b-a-c \\ x+4 & x+5 & x+2c \end{vmatrix}$$

But a, b, c are in A.P., $\therefore$ $2b = a + c$

$$= \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

EXERCISE-3

(1) **(C).** $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

$C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \Rightarrow C^2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

(2) **(D).** $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$

$$\Rightarrow AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5+2+21 & 7-10+3 & 1+14-14 \\ -15+1+14 & 21-5+2 & 3+7-10 \\ -10+3+7 & 14-15+1 & 2+21-5 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 18I_3$$

(3) **(A).** ℓ, m & n are p^{th}, q^{th} & r^{th} terms of G.P.

Let first term of G.P. is a and common ratio is R

$\therefore$ $\ell = aR^{p-1}$

$m = aR^{q-1}$

$n = aR^{r-1}$

Now, $\begin{vmatrix} \log \ell & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log aR^{p-1} & p & 1 \\ \log aR^{q-1} & q & 1 \\ \log aR^{r-1} & r & 1 \end{vmatrix}$

$$= \begin{vmatrix} \log a + \log R^{p-1} & p & 1 \\ \log a + \log R^{q-1} & q & 1 \\ \log a + \log R^{r-1} & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log a & p & 1 \\ \log a & q & 1 \\ \log a & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log a & p & 1 \\ (q-1)\log a & q & 1 \\ (r-1)\log a & r & 1 \end{vmatrix}$$

$$= 0 + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$

$$\Rightarrow \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = \log R \times 0 = 0$$

(4) **(B).** 1, ω & ω^2 are the cube roots of unity,
$\therefore\ 1+\omega+\omega^2=0$ and $\omega^3=1 \Rightarrow \omega^{3n}=1$

$$\therefore\ \Delta=\begin{vmatrix}1 & \omega^n & \omega^{2n}\\ \omega^n & \omega^{2n} & 1\\ \omega^{2n} & 1 & \omega^n\end{vmatrix}$$

We write 1 as ω^{3n} in R_1 and ω^n common from R_1

$$=\begin{vmatrix}\omega^{3n} & \omega^n & \omega^{2n}\\ \omega^n & \omega^{2n} & 1\\ \omega^{2n} & 1 & \omega^n\end{vmatrix}=\omega^n\begin{vmatrix}\omega^{2n} & 1 & \omega^n\\ \omega^n & \omega^{2n} & 1\\ \omega^{2n} & 1 & \omega^n\end{vmatrix}=0$$

{$\because$ R_1, R_3 are identical}

(5) **(C).** $\begin{vmatrix}a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+c^3\end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}+\begin{vmatrix}a & a^2 & a^3\\ b & b^2 & b^3\\ c & c^2 & c^3\end{vmatrix}=0$$

$$\Rightarrow (a-b)(b-c)(c-a)+abc\begin{vmatrix}1 & a & a^3\\ 1 & b & b^3\\ 1 & c & c^3\end{vmatrix}=0$$

$\Rightarrow (a-b)(b-c)(c-a)+abc\,(a-b)(b-c)(c-a)=0$
$\Rightarrow (a-b)(b-c)(c-a)+[1+abc]=0$
$(1, a, a^2)$ $(1, b, b^2)$, $(1, c, c^2)$ are non coplanar

$$\therefore\ \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\neq 0 \Rightarrow (a-b)(b-c)(c-a)\neq 0$$

$\therefore\ 1+abc=0 \Rightarrow abc=-1$

(6) **(C).** $A=\begin{bmatrix}a & b\\ b & a\end{bmatrix}$ and $A^2=\begin{bmatrix}\alpha & \beta\\ \beta & \alpha\end{bmatrix}$ $\because$ $A=\begin{bmatrix}a & b\\ b & a\end{bmatrix}$

$$A^2=\begin{bmatrix}a & b\\ b & a\end{bmatrix}\begin{bmatrix}a & b\\ b & a\end{bmatrix}=\begin{bmatrix}a^2+b^2 & 2ab\\ 2ab & a^2+b^2\end{bmatrix}$$

but $A^2=\begin{bmatrix}\alpha & \beta\\ \beta & \alpha\end{bmatrix}$ (given)

$$\therefore\ \begin{bmatrix}a^2+b^2 & 2ab\\ 2ab & a^2+b^2\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \beta & \alpha\end{bmatrix}$$

$\Rightarrow \alpha=a^2+b^2$ & $\beta=2ab$

(7) **(D).** $A=\begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}$ .... (1)

Clearly from (1) $A\neq 0$ (first option cancelled)

$$|A|=\begin{vmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{vmatrix}=-1\,(0-1)=\neq 0$$

$\therefore\ A^{-1}$ exist (third option cancelled)

$$A\neq(-1)\,I\ \left\{\because\ -I=\begin{vmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{vmatrix}\right.$$

(IInd option cancelled)

and $\begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$

(8) **(D).** $A=\begin{pmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & -2 & 3\end{pmatrix}$ and $(10)\,B=\begin{pmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{pmatrix}$

$\because\ B=A^{-1}$
$BA=A^{-1}A=I$; $(10B)\,A=10I$

$$\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{bmatrix}$$

$$\begin{bmatrix}10 & 0 & 0\\ -5+\alpha & 5+\alpha & -5+\alpha\\ 0 & 0 & 10\end{bmatrix}=\begin{bmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{bmatrix}$$

On comparing, $-5+\alpha=0 \Rightarrow \alpha=5$

(9) **(A).** $\begin{vmatrix}\log a_n & \log a_{n+1} & \log a_{n+2}\\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5}\\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8}\end{vmatrix}$ .... (1)

$\because$ a_1, a_2, a_3 ....a_n are in G.P.
$\therefore\ a^2_{n+1}=a_n\,a_{n+2} \Rightarrow 2\log a_{n+1}=\log a_n+\log a_{n+2}$
again $a^2_{n+4}=a_{n+3}\,a_{n+5}$
$\Rightarrow 2\log a_{n+4}=\log a_{n+3}+\log a_{n+5}$ and $a^2_{n+7}=a_{n+6}\,a_{n+8}$
$\Rightarrow 2\log a_{n+7}=\log a_{n+6}+\log a_{n+8}$
Putting these in the second column of the given determinant (1) we get

$$\Delta=\frac{1}{2}\begin{vmatrix}\log a_n & \log a_n+\log a_{n+2} & \log a_{n+2}\\ \log a_{n+3} & \log a_{n+3}+\log a_{n+5} & \log a_{n+5}\\ \log a_{n+6} & \log a_{n+6}+\log a_{n+8} & \log a_{n+8}\end{vmatrix}$$

$=\frac{1}{2}(0)=0$ {$\because$ C_2 is the sum of two elements, first identical with C_1 and second with C_3}

(10) **(A).** $\alpha x + y + z = \alpha - 1$

$x + ay + z = \alpha - 1$; $x + y + \alpha z = \alpha - 1$

System of equation has no solution

$$\therefore \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0 \ ; C_1 \to C_1 + C_2 + C_3 \Rightarrow \begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$R_1 \to R_1 - R_3 \qquad R_2 \to R_2 - R_3$

$$\Rightarrow (\alpha+2)\begin{vmatrix} 0 & 1 & 1-\alpha \\ 0 & \alpha-1 & 1-\alpha \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$= (\alpha + 2)\,[(1-\alpha)\,[0 - (\alpha - 1)]] = 0$

$\Rightarrow (\alpha + 2)\,(1 - \alpha)^2 = 0 \Rightarrow \alpha = -2\ \&\ 1$

But $\alpha = 1$ makes three equation same

$\therefore$ the system of equation have infinite solution

$\therefore \alpha = -2$

(11) **(D).** $$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$f(x) = \begin{vmatrix} 1+a^2x+x+b^2x+x+c^2x & (1+b^2)x & (1+c^2)x \\ x+a^2x+1+b^2x+x+c^2x & 1+b^2x & (1+c^2)x \\ x+a^2x+x+b^2x+1+c^2x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= (1 + 2x + a^2x + b^2x + c^2x)\begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$= \{1 + 2x + a^2x + b^2x + c^2x)\}$

$$\begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} \quad \{\because\ a^2 + b^2 + c^2 = -2 \text{ (given)}\}$$

$= [1 + 2x + (-2)\,x]\,[1\,\{(1-x)^2 - 0\}] = 1.\,(1-x)^2$

which is polynomial of degree 2.

(12) **(D).** $\because\ A^2 - A + I = 0$

$\Rightarrow A^{-1}A^2 - A^{-1}A + A^{-1}I = 0$

$\Rightarrow A^{-1}A\,.\,A - I + A^{-1} = 0$

$\Rightarrow A - I + A^{-1} = 0 \Rightarrow A^{-1} = I - A$

(13) **(A).** $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Similarly, $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ .... (1)

Now, $nA = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix}$ .... (2)

and $(n-1)\,I = \begin{bmatrix} n-1 & 0 \\ n & n-1 \end{bmatrix}$ .... (3)

Now from eq. (1), (2) and (3)

$A^n = nA - (n-1)\,I$

(14) **(A).** $\because\ A^2 - B^2 = (A - B)\,(A + B)$

$A^2 - B^2 = A2 + AB - BA - B^2$

this will be value if $AB - BA = 0$

$\Rightarrow AB = BA$

(15) **(C).** $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Now, $AB = \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix}$ and $BA = \begin{pmatrix} a & 2a \\ 3b & 4b \end{pmatrix}$

According to option AB = BA

$\Rightarrow \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3b & 4b \end{pmatrix}$

On comparing, $2b = 2a \Rightarrow a = b$

$\therefore$ For AB = BA there are infinitely many value of B's are possible.

(16) **(C).** $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \qquad \because\ |A^2| = |A|^2$

$\therefore |A| = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} = 5\,(5\alpha - 0) = 25\alpha$

then $|A|^2 = (25\alpha)^2 = 25 \times 25\alpha^2$

but given $|A^2| = 25 = |A|^2 \quad \{\because\ |A|^n = |A^n|\}$

$\therefore\ 25 = 25 \times 25 \times \alpha^2 \Rightarrow \alpha^2 = \dfrac{1}{25} \Rightarrow |\alpha| = \dfrac{1}{5}$

(17) **(B).** $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

$R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$\therefore D = \begin{vmatrix} 0 & 0 & -y \\ 0 & x & -y \\ 1 & 1 & 1+y \end{vmatrix} = -y\,[0 - x] = xy$

$\therefore$ D is divisible by both x & y

(18) **(B).** All entries of matrix A are integers

$\therefore$ cofactors of matrix A will be integer and all the entries of adj A will be integer.

if det A = I

$\therefore\ A^{-1} = \dfrac{\text{adj } A}{\det A}\quad \{|A| = \pm 1\} = \pm(\text{adj } A)$

$\therefore A^{-1}$ exist and all its entries are integers.

(19) **(C).** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$$

According to question, $A^2 = I$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, $a^2 + bc = 1 = bc + d^2$ ..... (1)

and $ab + bd = ac + dc = 0$

$\Rightarrow b(a+d) = c(a+d) = 0$

$\Rightarrow a + d = 0 \Rightarrow a = -d$ .... (2)

If $A \neq I$ and $A \neq -I$ then trace (A) = a + d = 0 statement (2) is incorrect.

det. of A = ad – bc

$= -d^2 - bc$ {from (2)}

$= -(1 - bc) - bc = -1 + bc - bc = -1$

statement (1) is correct.

(20) **(C).** $x = cy + bz$ .... (1)

$y = az + cx$ .... (2)

$z = bx + ay$ .... (3)

Put value of x in (2) & (3) we get

$$y = az + c(cy + bz) \Rightarrow y = \frac{(a+bc)\,z}{1-c^2} \text{ .... (4)}$$

$$\text{and } z = b(cy + bz) + ay \Rightarrow y = \frac{1-b^2}{a+bc} \text{ .... (5)}$$

From eq. (4) & (5)

$$\frac{(a+bc)\,z}{1-c^2} = \left(\frac{1-b^2}{a+bc}\right) z$$

$\Rightarrow (a+bc)^2 = (1-b^2)(1-c^2)$

$\Rightarrow a^2 + a^2b^2 + 2abc = 1 - c^2 - b^2 + b^2c^2$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

(21) **(A).** Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow \text{adj}(\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Also | adj A | = | A | but this does not explain the S-1.

(22) **(B).** The given eqaution can be written as

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0$$

$\Rightarrow$ n has to be any odd integer.

(23) **(C).** First row with exactly one zero;

Total number of cases = 6

First row 2 zeros we get more cases

Total we get more than 7.

(24) **(B).** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, abcd $\neq 0$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$\Rightarrow a^2 + bc = 1,\ bc + d^2 = 1$

$ab + bd = ac + cd = 0$; $c \neq 0$ and $b \neq 0 \Rightarrow a + d = 0$

Trace A = a + d = 0

$|A| = ad - bc = -a^2 - bc = -1.$

(25) **(C).** $D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 0$; $D_1 = \begin{vmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & 2 \end{vmatrix} \neq 0$

$\Rightarrow$ Given system, does not have any solution.

$\Rightarrow$ No solution.

(26) **(B).** A' = A , B' = A ; P = A(BA) ; P' = (A(BA))'

= (BA)' A' = (A'B') A' = (AB) A = A(BA)

$\therefore$ A(BA) is symmetric. similarly (AB) A is symmetric

S-2 is correct but not correct explanation of S-1.

(27) **(B).** $\Delta = \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow 8 - k(k-2) - 2(2k-8) = 0$

$\Rightarrow 8 - k^2 + 2k - 4k + 16 = 0$

$\Rightarrow -k^2 - 2k + 24 = 0 \Rightarrow k^2 + 2k - 24 = 0$

$\Rightarrow (k+6)(k-4) = 0 \Rightarrow k = -6, 4$

Number of values of k is 2

(28) **(D).** $A(u_1 + u_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $|A| = 1$; $A^{-1} = \dfrac{1}{|A|}$ adj A

$$(u_1 + u_2) = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix};\quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(29) **(C).** Subtracting $P^3 - P^2Q = Q^3 - Q^2P$

$P^2(P - Q) + Q^2(P - Q) = 0$

$(P^2 + Q^2)(P - Q) = 0$

If $|P^2 + Q^2| \neq 0$ then $P^2 + Q^2$ is invertible

$\Rightarrow P - Q = 0$ contradiction

Hence $|P^2 + Q^2| = 0$

(30) **(B).** For no solution, $\dfrac{k+1}{k} = \dfrac{8}{k+3} \neq \dfrac{4k}{3k-1}$

$k^2 + 4k + 3 = 8k$; $k^2 - 4k + 3 = 0$; $k = 1, 3$

If $k = 1$ then $\frac{8}{1+3} = \frac{4.1}{3-1}$ False

And If $k = 3$ then $\frac{8}{6} \neq \frac{4.3}{9-1}$ True

Therefore, $k = 3$

Hence only one value of k.

(31) **(B).** $|P| = 1(12-12) - \alpha(4-6) + 3(4-6) = 2\alpha - 6$

$|P| = |A|^2 = 16$

$2\alpha - 6 = 16 \Rightarrow \alpha = 11.$

(32) **(B).** $B = A^{-1}A' \Rightarrow AB = A'$

$ABB' = A'B' = (BA)' = (A^{-1}A'A)' = (A^{-1}AA')' = A$

$\Rightarrow BB' = I$

(33) **(C).** $\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$= ((\alpha-1)(\beta^2-1) - (\beta-1)(\alpha^2-1))^2$

$= (\alpha-1)^2(\beta-1)^2(\alpha-\beta)^2$

$\Rightarrow k = 1$

(34) **(C).** $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$a + 4 + 2b = 0$; $2a + 2 - 2b = 0$; $a + 1 - b = 0$

$2a - 2b = -2$; $a + 2b = -4$

Solving, $b = -1, a = -2$; $(-2, -1)$

(35) **(B).** $x_1(2-\lambda) - 2x_2 + x_3 = 0$

$2x_1 + x_2(-\lambda-3) + 2x_3 = 0$; $-x_1 + 2x_2 - \lambda x_3 = 0$

$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -\lambda-3 & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$(2-\lambda)(\lambda^2 + 3\lambda - 4) + 2(-2\lambda + 2) + (4 - \lambda - 3) = 0$

$(\lambda-1)(\lambda+3)(\lambda-1) = 0 \Rightarrow \lambda = 1, 1, -3$ Two elements

(36) **(C).** $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$(\lambda+1) - \lambda(\lambda^2+1) - (\lambda+1) = 0$

$(\lambda+1)(1 + \lambda(\lambda-1) - 1) = 0$; $\lambda = -1$ or 0 or 1

(37) **(A).** $A(\text{adj } A) = |A| I_n = AA^T$ (Given) ; $|A| = 10a + 3b$

$A^T = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$; $AA^T \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix} = \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix} = \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix}$

$\Rightarrow 15a - 2b = 0 \Rightarrow a = 2b/15$ and $10a + 3b = 13$

$\Rightarrow a = \frac{13-3b}{10} \Rightarrow \frac{2b}{15} = \frac{13-3b}{10} \Rightarrow 4b = 39 - 9b$

$\Rightarrow 13b = 39 \Rightarrow b = 3 \Rightarrow a = \frac{2}{15} \times 3 = \frac{6}{15} = \frac{2}{5} \Rightarrow 5a = 2$

$\therefore 5a + b = 2 + 3 = 5$

(38) **(B).** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = (a-b) - (1-a) + (b-a^2)$

$= a - b - 1 + a + b - a^2 = -(a^2 - 2a + t) = -(a-1)^2 = 0$

$a = 1$

$x + y + z = 1$

$x + by + z = 0$

$\Rightarrow$ Two plane should be parallel, $b = 1$

(39) **(D).** $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$

$3A^2 = \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix}$; $12A = \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$

$3A^2 + 12A = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

$\text{Adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

(40) **(D).** $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0 \Rightarrow k = \frac{7}{2}$

$x + ky + 3z = 0$ .....(i) $3x + ky - 2z = 0$ .....(ii)

$2x + 4y - 3z = 0$ .....(iii)

On solving (i) and (ii), $2x - 5z = 0$ .....(iv)

On solving (iii) and (iv), $4y = -2z$

$$\frac{xz}{y^2} = \frac{\frac{5}{2} z \times z}{z^2/4} = 10$$

(41) **(A).** $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2,$

Put $x = 0$, $\begin{vmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{vmatrix} = A^3$; $A = -4$

Put $x=1$, $\begin{vmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{vmatrix} = (A+B)(1-A)^2$

$-3(9-4)-2(-6-4)+2(4+6)$
$-15+20+20=(-4+B)\,25$
$1=(-4+B)\,;\ B=5$

(42) **(B).** $D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & a^2-1 \end{vmatrix} = a^2-3$

$D_1=\begin{vmatrix} 2 & 1 & 1 \\ 5 & 3 & 2 \\ a+1 & 3 & a^2-1 \end{vmatrix} = a^2-a+1$

$D_2=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & a+1 & a^2-1 \end{vmatrix} = a^2-3$

$D_3=\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 2 & 3 & a+1 \end{vmatrix} = a-4$

$D=0$ at $|a|=\sqrt{3}$ but $D_3=\pm\sqrt{3}-4\neq 0$

So the system is inconsistent for $|a|=\sqrt{3}$

(43) **(A).** Here, $AA^T=I$

$\Rightarrow A^{-1}=A^T=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Also, $A^{-n}=\begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}$

$A^{-50}=\begin{bmatrix} \cos(50)\theta & \sin(50)\theta \\ -\sin(50)\theta & \cos(50)\theta \end{bmatrix}$

$=\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

(44) **(D).** $A=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A^2=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$=\begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$

$A^3=\begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$=\begin{bmatrix} \cos 3\alpha & -\sin 3\alpha \\ \sin 3\alpha & \cos 3\alpha \end{bmatrix}$

Similarly

$A^{32}=\begin{bmatrix} \cos 32\alpha & -\sin 32\alpha \\ \sin 32\alpha & \cos 32\alpha \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \cos 32\alpha=0\ \&\ \sin 32\alpha=1$

$\Rightarrow 32\alpha=(4n+1)\frac{\pi}{2},\ n\in I$

$\alpha=(4n+1)\frac{\pi}{64}, n\in I\ ;\quad \alpha=\frac{\pi}{64}$ for $n=0$

(45) **(D).** Put $b=\frac{2+c}{2}$ in determinant of A

$|A|=\frac{c^3-6c^2+12c-8}{4}\in[2,16]$

$\Rightarrow (c-2)^3\in[8,64]\Rightarrow c\in[4,6]$

(46) **(A).** Roots of the equation $x^2+x+1=0$ are
$\alpha=\omega$ and $\beta=\omega^2$
where ω, ω^2 are complex cube roots of unity

$\therefore\ \Delta=\begin{vmatrix} y+1 & \omega & \omega^2 \\ \omega & y+\omega^2 & 1 \\ \omega^2 & 1 & y+\omega \end{vmatrix}$ $R_1\to R_1+R_2+R_3$

$\Rightarrow \Delta=y\begin{vmatrix} 1 & 1 & 1 \\ \omega & y+\omega^2 & 1 \\ \omega^2 & 1 & y+\omega \end{vmatrix}$

Expanding along R_1, we get
$\Delta=y.y^2\Rightarrow D=y^3$

(47) **(A).** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 78 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1+2+3+\dots+n-1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

$\frac{n(n-2)}{2}=78\Rightarrow n=13,-12$ (reject)

We have to find inverse of $\begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix}\ \therefore \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

(48) **(C).** $A^2=\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow A^4=I\Rightarrow A^{30}=A^{28}\times A^3=A^3$

(49) **(C).** For non-trivial solution

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0 \; ; \; \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$(3bc - 4bc) - (2ac - 4ac) + (2ab - 3ab) = 0$
$-bc + 2ac - ab = 0$
$ab + bc = 2ac$
a, b, c in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

(50) **(A).** $|B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} 3^0 a_{11} & 3^1 a_{12} & 3^2 a_{13} \\ 3^1 a_{21} & 3^2 a_{22} & 3^3 a_{23} \\ 3^2 a_{31} & 3^3 a_{32} & 3^4 a_{33} \end{vmatrix}$

$\Rightarrow \; 81 = 3^3 . 3 . 3^2 |A| \Rightarrow 3^4 = 3^6 |A|$
$\Rightarrow \; |A| = 1/9$

(51) **672.00**

Trace $(AA^T) = \Sigma a_{ij}^2 = 3$

Hence, number of such matrices $= {}^9C_3 \times 2^3 = 672.00$

(52) **(A).** $D = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ x' & y' & 1 \end{vmatrix}$

$-2(1 - x') + (y' + x') = \pm 10$
$-2 + 2x' + y' + x' = \pm 10$
$3x' + y' = 12$ or $3x' + y' = -8$
$\lambda = 3, -2$

(53) **(C).** $D = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 4 & 4 \end{vmatrix} \; (R_3 \to R_3 - 2R_1 + 3R_2)$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Now let $P_3 \equiv 4x + 4y + 4z - \delta = 0$.
If the system has solutions it will have infinite solution, so $P_3 \equiv \alpha P_1 + \beta P_2$
Hence $3\alpha + \beta = 4$ & $4\alpha + 2\beta = 4 \Rightarrow \alpha = 2$ & $\beta = -2$
So for infinite solution $2\mu - 2 = \delta \Rightarrow 2\mu \neq \delta + 2$
System inconsistent

(54) **(C).** $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix} \Rightarrow |A| = 6$

$$\frac{|\text{adj } B|}{|c|} = \frac{|\text{adj (adjA)}|}{|9A|} = \frac{|A|^4}{3^3|A|} = \frac{|A|^3}{3^3} = \frac{(6)^3}{(3)^3} = 8$$

(55) **(A).** For planes to intersect on a line
There should be infinite solution of the given system of equations
For infinite solutions

$$\Delta = \begin{vmatrix} 1 & 4 & -2 \\ 1 & 7 & -5 \\ 1 & 5 & \alpha \end{vmatrix} = 0 \Rightarrow 3\alpha + 9 = 0 \Rightarrow \alpha = -3$$

$$\Delta_z = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 7 & \beta \\ 1 & 5 & 5 \end{vmatrix} = 0 \Rightarrow 13 - \beta = 0 \Rightarrow \beta = 13$$

$\therefore \; \alpha + \beta = -3 + 13 = 10$

(56) **(A).** $7x + 6y - 2z = 0$ .... (1)
$3x + 4y + 2z = 0$ .... (2)
$x - 2y - 6z = 0$ .... (3)

$$\Delta = \begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} = 0 \Rightarrow \text{Infinite solutions}$$

Now $(1) + (2) \Rightarrow y = -x$ put in (1), (2) & (3) all will lead to $x = 2z$

(57) **(C).** $R_1 \to R_1 + R_3 - 2R_2$

$$f(x) = \begin{vmatrix} a + c - 2b & 0 & 0 \\ x + b & x + 3 & x + 2 \\ x + c & x + 4 & x + 3 \end{vmatrix}$$

$= (a + c - 2b)((x + 3)^2 - (x + 2)(x + 4))$
$= x^2 + 6x + 9 - x^2 - 6x - 8 = 1$
$\Rightarrow \; f(x) = 1 \Rightarrow f(50) = 1$